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Baltimore County Public Schools: College Algebra and Trigonometry

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Contents

CHAPTER 1 Visual Representations of Functions

Chapter Outline

1.1 Graphical Transformations

Learning Objectives

Here you will learn how a graph changes when you change its equation by adding, subtracting, and multiplying by constants.

The basic functions are powerful, but they are extremely limited until you can change them to match any given situation. Transformation means that you can change the equation of a basic function by adding, subtracting, and/or multiplying by constants and thus cause a corresponding change in the graph. What are the effects of the following transformations?

1. $f(x) \to f(x+3)$
2. $h(x) \to h(x)-5$
3. $g(x) \to -g(2x)$
4. $j(x) \to j\left(-\frac{x}{2}\right)$

Transforming Functions

A **function** is a rule that takes any input x and gives a specific output. When you use letters like f, g, h, or j to describe the rule, this is called function notation. In order to interpret what effect the algebraic change in the equation will have on the graph, it is important to be able to read those changes in general function notation and then apply them to specific cases.

When **transforming** a function, you can transform the **argument** (the part inside the parentheses with the x), or the function itself. There are two ways to linearly transform the argument. You can multiply the x by a constant and/or add a constant to the x as shown below:

$$f(x) \to f(bx+c)$$

The function itself can also be linearly transformed in the same ways:

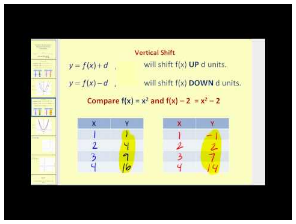
$$f(x) \to af(x)+d$$

Each of the letters a, b, c, and d corresponds to a very specific change. Some of these changes are straightforward, while others may be the opposite of what you might expect.

- a is a vertical stretch. If a is negative, there is also a reflection across the x axis.
- d is a vertical shift. If d is positive, then the shift is up. If d is negative, then the shift is down.

When transforming the argument of the function things are more complicated.

- $\frac{1}{b}$ is a horizontal stretch. If b is negative, there is also a reflection across the y axis.
- c is a horizontal shift. If c is positive, then the shift is to the left. If c is negative, then the shift is to the right. Notice that this is the opposite of what most people think at first.



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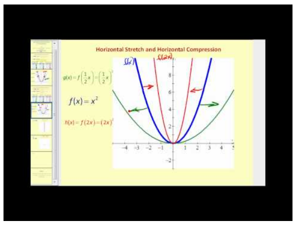

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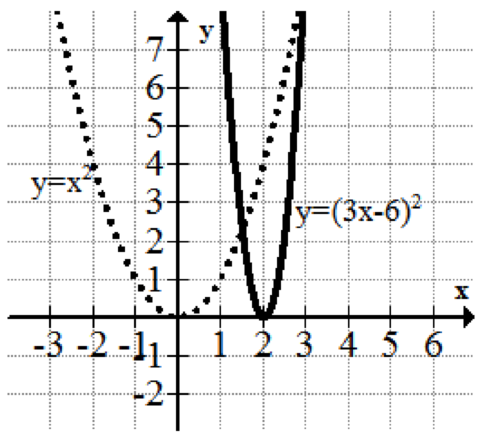
The trickiest part with transforming the argument of a function is the order in which you carry out the transformations. Often it makes sense to apply the transformation to a specific function that is known and then describe the transformation that you see.

The graph below shows the transformation $f(x) \rightarrow f(3x-6)$ applied to a simple parabola:



Clearly the graph is narrower and to the right, but in order to be specific you must look closer. First, notice that the transformation is entirely within the argument of the function. This affects only the horizontal values. This means while the graph seems like it was stretched vertically, you must keep your perspective focused on a horizontal compression.

Look carefully at the vertex of the parabola. It has moved to the right two units. This is because first the entire graph was shifted entirely to the right 6 units. Then the function was horizontally compressed by a factor of 3 which means the point (6, 0) became (2, 0) and the x value of every other point was also compressed by a factor of 3 towards the line $x = 0$. This method is counter-intuitive because it requires reading the transformations backwards (the opposite of the way the order of operations tells you to).

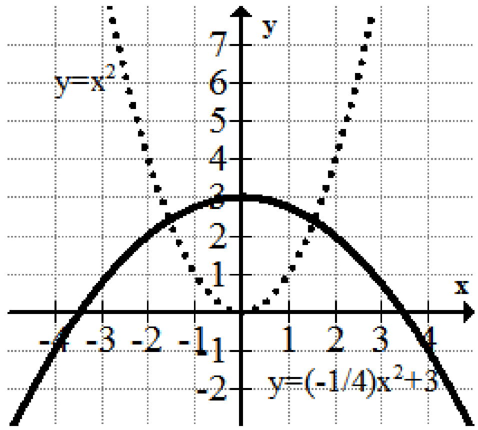
Alternatively, the argument can be factored and each component of the transformation will present itself.

$f(3(x-2))$

This time the stretch occurs from the center of the transformed graph, not the origin. This method is ultimately the preferred method.

Either way, this is a horizontal compression by a factor of 3 and a horizontal shift to the right by 2 units.

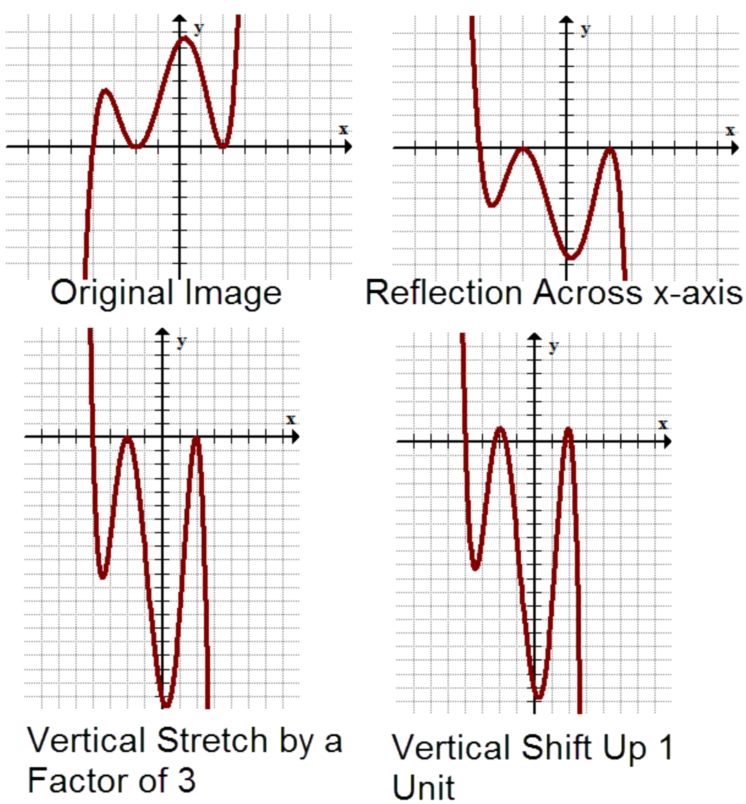
Now take the transformation $f(x) \rightarrow -\frac{1}{4}f(x)+3$. It describes a vertical stretch by a factor of $\frac{1}{4}$, a reflection over the x axis, and a vertical shift 3 units up. As opposed to what you saw above, the order of the transformations for anything outside of the argument is directly what the order of operations dictates.



First, the parabola is reflected over the x axis and compressed vertically so it appears wider. Then, every point is moved up 3 units.

The transformation $f(x) \rightarrow -3f\left(-\frac{1}{2}x-1\right)+1$ contains every possible transformation. The horizontal and the vertical components do not interact with each other and so your description of the transformation can begin with either component. Here, start by describing the vertical components of the transformation:

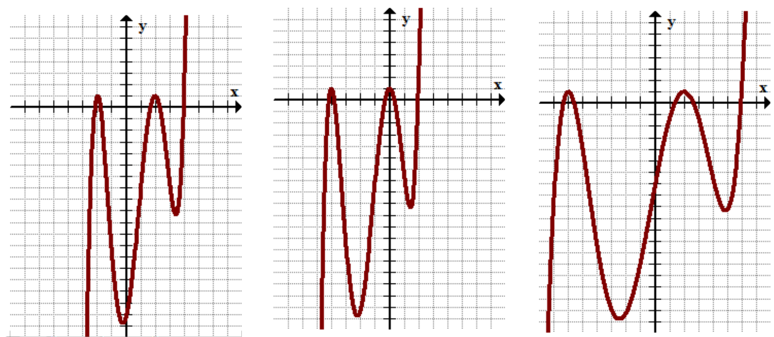
First, there is reflection across the x axis and a vertical stretch by a factor of 3. Then, there is a vertical shift up 1 unit. Below is an image of a non-specific function going through the vertical transformations.



In order to figure out the horizontal components of the transformation, start by factoring the inside of the parentheses (the argument):

$f\left(-\frac{1}{2}x-1\right)=f\left(-\frac{1}{2}(x+2)\right)$

Factoring reveals a reflection across the y axis, a horizontal shift left 2 units and a horizontal stretch by a factor of 2. Below is an image of the same function going through the horizontal transformations.



Reflection Across y-axis | Horizontal Shift Left 2 Units | Horizontal Stretch by a Factor of 2

Examples

Example 1

Earlier, you were given a problem about the effects of the following transformations:

$f(x)\rightarrow f(x+3)$

This transformation shifts the entire graph left 3 units. A common misconception is to shift right because the three is positive.

$h(x)\rightarrow h(x)-5$

This transformation shifts the entire graph down 5 units.

$g(x)\rightarrow -g(2x)$

This transformation is a vertical reflection across the x axis and a horizontal compression by a factor of 2.

$j(x)=j\left(-\frac{x}{2}\right)$

This transformation is a horizontal reflection across the y axis and a horizontal stretch by a factor of 2. A common misconception is to see the $\frac{1}{2}$ and believe that the x values will be half as big which is a horizontal compression. However, the x values need to be twice as big to counteract this factor of $\frac{1}{2}$.

Example 2

Describe the following transformation in words: $g(x)\rightarrow 2g(-x)$

Vertical stretch by a factor of 2 and a reflection across the y axis.

Example 3

Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 3.

- Vertical shift down 4 units.
- Horizontal shift right 5 units.

$\frac{1}{3}h(x-5)-4$

Example 4

Describe the transformation that would change $f(x)$ in the following ways:

- Horizontal stretch by a factor of 4 and a horizontal shift 3 units to the right.
- Vertical reflection across the x axis and a shift down 2 units.

$-f\left(\frac{1}{4}(x-3)\right)-2$ or $-f\left(\frac{1}{4}x-\frac{3}{4}\right)-2$

Review

Describe the following transformations in words.

1. $g(x) \rightarrow -g(-x)$
2. $f(x) \rightarrow -f(x+3)$
3. $h(x) \rightarrow h(x+1)-2$
4. $j(x) \rightarrow j(-x+3)$
5. $k(x) \rightarrow -k(2x)$
6. $f(x) \rightarrow 4f\left(\frac{1}{2}x+1\right)$
7. $g(x) \rightarrow -3g(x-2)-2$
8. $h(x) \rightarrow 5h(x+1)$
9. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical stretch by a factor of 2
- Vertical shift up 3 units.
- Horizontal shift right 2 units.

10. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical reflection across the x axis.
- Vertical shift down 1 unit.
- Horizontal shift left 2 units.

11. Describe the transformation that would change $g(x)$ in the following ways:

- Vertical compression by a factor of 4.
- Reflection across the y axis.

12. Describe the transformation that would change $j(x)$ in the following ways:

- Horizontal compression by a factor of 3.
- Vertical shift up 3 units.

- Horizontal shift right 2 units.

13. Describe the transformation that would change $k(x)$ in the following ways:

- Horizontal stretch by a factor of 4.
- Vertical shift up 3 units.
- Horizontal shift left 1 unit.

14. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 2.
- Horizontal shift right 3 units.
- Reflection across the y axis.

15. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical stretch by a factor of 5.
- Reflection across the x axis.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.2.

1.2 Domain and Range

Learning Objectives

Here you will refine your understanding of domain and range from Algebra 2 by exploring tables, basic functions and irregular graphs.

Analyze means to examine methodically and in detail. One way to analyze functions is by looking at possible inputs (domain) and possible outputs (range). Which of the basic functions have limited domains and why?

Domain and Range

Notation

Domain and range are described in interval notation. Parentheses, ( ), mean non-inclusive. Brackets, [ ], mean inclusive. The following descriptions of numbers into interval notation have been converted to interval notation.

1. All numbers.

   $(-\infty,\infty)$ Note: Parentheses are always used with infinity.

2. All negative numbers not including 0.

   $(-\infty,0)$

3. All positive numbers including 0.

   $[0,\infty)$

4. Every number between 1 and 4 including 1 and 4.

   $[1, 4]$

5. Every number between 5 and 6 not including 5 or 6.

   $(5, 6)$

6. The numbers 1 through 2 including 1 but not including 2 and the numbers 10 through 25 including both 10 and 25

   $[1,2)\cup[10,25]$

Note: The $\cup$ symbol means Union and refers to the fact that if some number x is in this union, then it is either in the first group or it is in the second group. This symbol is associated with the OR statement. While it is true that the Union symbol seems to bring one group and another group together, the symbol for AND is $\cap$ which means intersection. Intersection is different from union because intersection means all numbers that are in both the first group and second group at the same time.

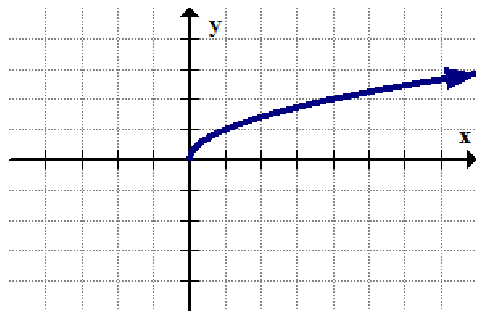
Domain and Restricted Domain

Domain is the possible inputs to a function. Many functions allow any kind of numbers to be inputted. This includes numbers that are positive, negative, zero, fractions or decimals. The squaring function $y = x^2$ is an example that has a domain of all possible real numbers. Three functions have very specific restrictions:

The square root function: $y = \sqrt{x}$

Domain restriction: $x \geq 0$

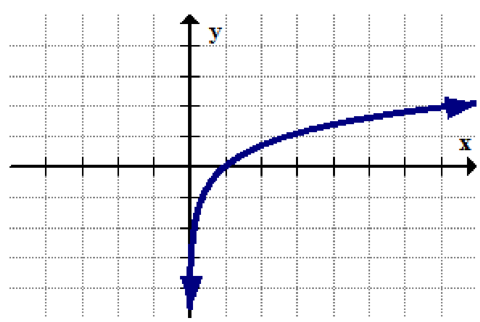
This is because the square root of a negative number is not a real number. This restriction can be observed in the graph because the curve ends at the point (0, 0) and is not defined anywhere where x is negative.



The logarithmic function: $y = \ln x$

Domain restriction: $x > 0$

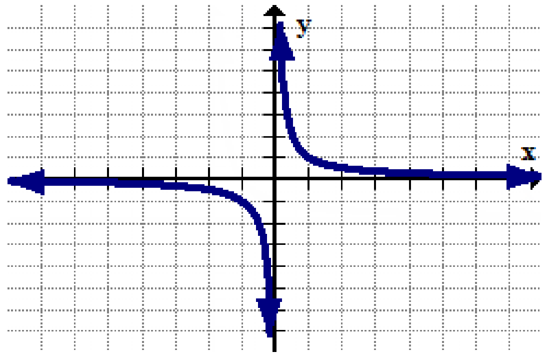
The log function is only defined on numbers that are strictly bigger than zero. This is because the logarithmic function is a different way of writing exponents. One property of exponents is that any positive number raised to any power will never produce a negative number or zero. The restriction can be observed in the graph by the way the log function approaches the vertical line $x = 0$ and shoots down to infinity.



The reciprocal function: $y = \frac{1}{x}$

Domain Restriction: $x \neq 0$

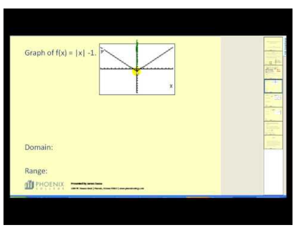
The reciprocal function is restricted because you cannot divide numbers by zero. Any x values that make the denominator of a function zero are outside of the domain. This restriction can be observed in the graph by the way the reciprocal function never touches the vertical line $x = 0$.



Range

Range is the possible outputs of a function. Just about any function can produce any output through the use of transformations and so determining the range of a function is significantly less procedural than determining the domain. Use what you know about the shape of each function and their equations to decide which y values are possible to produce and which y values are impossible to produce.

Finding Domain and Range



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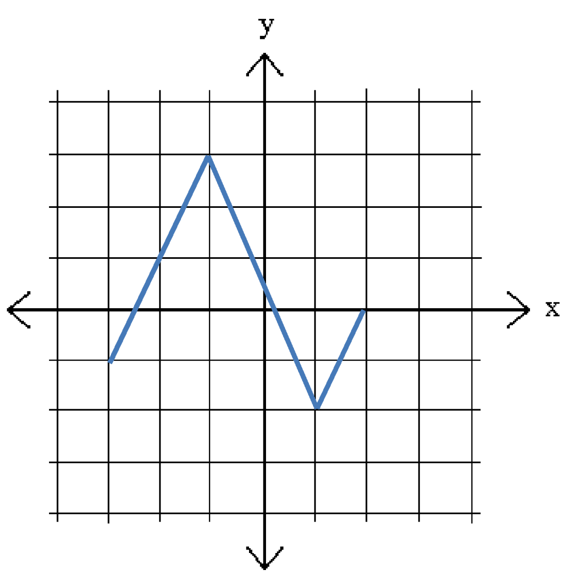


The domain and range for the graph above are:

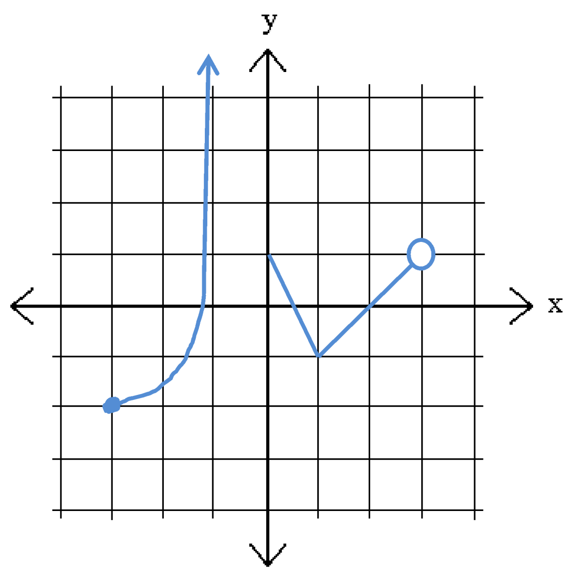
Domain: $x \in [-3, 2]$

Range: $y \in [-2, 3]$

Note that the $\in$ symbol means "is an element of" and means that the x or the y is in that interval and the numbers in the interval are always written in increasing order. [3, -2] is considered improper.

Note that even though the [-3, 2] may look similar to the ordered pair that represents the point where $x = -3$ and $y = 2$, this is not the case. Both the -3 and the 2 are x values. This misconception is why you should always write $x \in$ because it reminds you of this fact. Many people get very confused when they see something like $x \in (-2, 1)$ because they see the parenthesis and immediately see a point when they should see an interval on the x axis.

In the graph below, there are two different pieces of the function.



The domain and range for the graph above are:

Domain: $x \in [-3, -1) \cup [0, 3)$

Range: $y \in [-2, \infty)$

The function seems to approach the vertical line $x = -1$ without actually reaching it so an open bracket is used. Also, the empty hole at the point (3, 1) which is why the domain excludes the x value of 3.

Examples

Example 1

Earlier, you were asked which basic functions have limited domains. The three functions that have limited domains are the square root function, the log function and the reciprocal function. The square root function has a restricted domain because you cannot take square roots of negative numbers and produce real numbers. The log function is restricted because the log function is not defined to operate on non-positive numbers. The reciprocal function is restricted because numbers that are divided by zero are not defined.

Example 2

Identify the domain and range of the following function written in a table:

TABLE 1.1:

x	y
0	5
1	6
2	7
$\frac{1}{2}$	6
π	$\frac{\pi}{2}$

The specific equation of the function may be hidden, but from the table you can determine the domain and range directly from the x and y values. It may be tempting to guess that other values could potentially work in the table, especially if the pattern is obvious, but this is not the kind of question that asks what the function could be. Instead this question just asks what is the stated domain and range.

Domain: $x \in \{0, 1, 2, \frac{1}{2}, \pi\}$

Range: $y \in \{5, 6, 7, \frac{\pi}{2}\}$

Notice that the two 6's that appear in the table do not need to be written twice in the range.

Example 3

Identify the domain of the following three transformed functions.

1. $y = 10\sqrt{2-x} - 3$

The argument of the function must be greater than or equal to 0.

$$\begin{aligned} 2 - x &\geq 0 \\ -x &\geq -2 \\ x &\leq 2 \end{aligned}$$

Domain: $x \in (-\infty, 2]$

2. $y = \frac{3x}{x^2+7x+12}$

The denominator cannot be equal to 0. First find what values of x would make it equal to zero and then you can exclude those values.

$$\begin{aligned} x^2 + 7x + 12 &= 0 \\ (x+4)(x+3) &= 0 \\ x &= -4, -3 \end{aligned}$$

Domain: $x \in (-\infty, -4) \cup (-4, -3) \cup (-3, \infty)$

3. $y = -4\ln(3x-9) + 11$

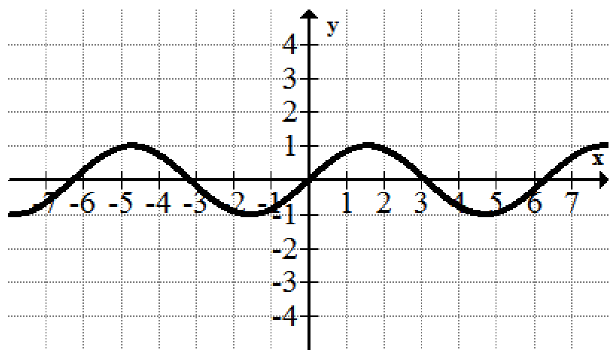
The argument must be strictly greater than 0.

$$\begin{aligned} 3x-9 &> 0 \\ 3x &> 9 \\ x &> 3 \end{aligned}$$

Domain: $x \in (3, \infty)$

Example 4

What is the domain and range of the sine wave?



Domain: $x \in (-\infty, \infty)$

Range: $y \in [-1, 1]$

Review

Convert the following descriptions of numbers into interval notation.

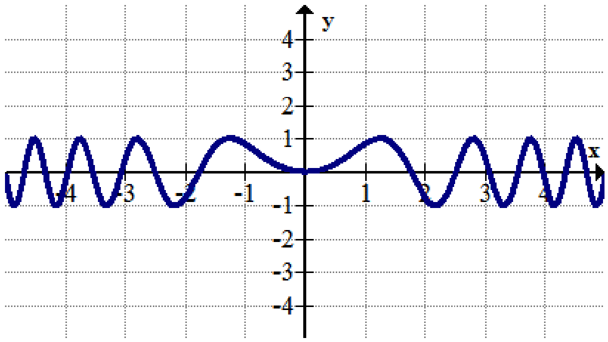
1. All positive numbers not including 0.
2. Every number between -1 and 1 including -1 but not 1.
3. Every number between 1 and 5 not including 2 or 3, but including 1 and 5.
4. Every number greater than 5, not including 5.
5. All real numbers except 1.

Translate the following inequalities into interval notation.

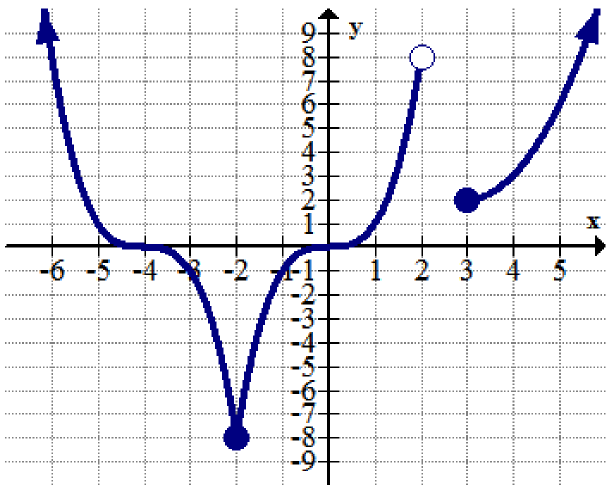
6. $-4 < x \leq 5$
7. $x > 0$
8. $-\infty < x \leq 4 \text{ or } 5 < x < \infty$

Find the domain and range of each graph below.

9.



10.



Given the stated domain and range, draw a possible graph.

11. Domain: $x \in [0, \infty)$ Range: $y \in (-2, 2]$

12. Domain: $x \in [-4, 1) \cup (1, \infty)$ Range: $y \in (-\infty, \infty)$

13. Given the table, find the domain and range.

TABLE 1.2:

x	y
-2	7
3	7
2	1
$\frac{3}{4}$	5
$\frac{\pi}{2}$	π

Find the domain for the following functions.

14. $y = -3\sqrt{x+4} - 1$

15. $y = \frac{7}{x+6} - 1$

16. $y = 5\ln(x^2 - 1) + 4$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.4.

1.3 Maximums and Minimums

Learning Objectives

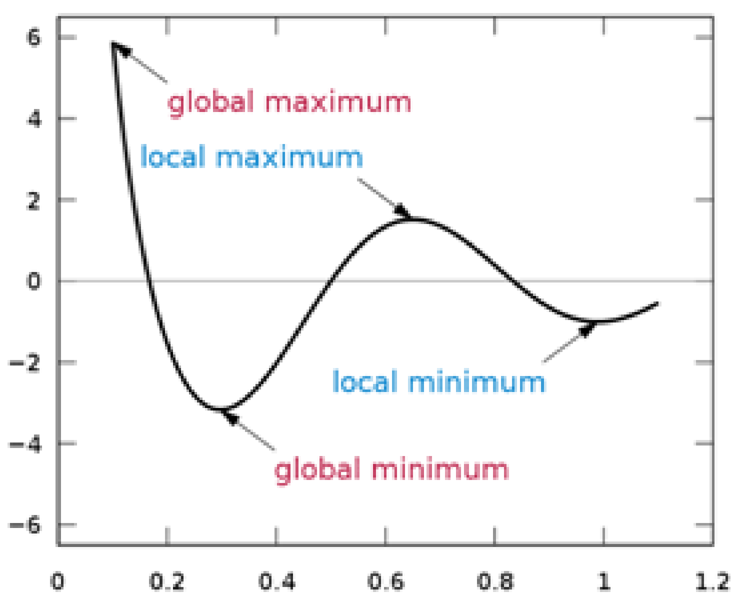
Here you will learn to identify the maximums and minimums in various graphs and be able to differentiate between global and relative extreme values.

When riding a roller coaster there is always one point that is the absolute highest off the ground. There are usually many other places that reach fairly high, just not as high as the first. There are also places where the roller coaster dips with one being the absolute lowest the roller coaster can go. How do you identify and distinguish between these different peaks and valleys in a precise way?

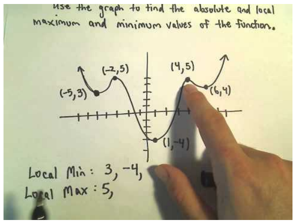
Finding Maximums and Minimums

A **global maximum** refers to the point with the largest y value possible on a function. A **global minimum** refers to the point with the smallest y value possible. Together these two values are referred to as **global extrema**. There can only be one global maximum and only one global minimum. Global refers to entire space where the function is defined. Global extrema are also called **absolute extrema**.

In addition to global maximums and global minimums, there are also **local extrema** or **relative maximums** and **relative minimums**. The word relative is used because in relation to some neighborhood, these values stand out as being the highest or the lowest.



Calculus uses advanced analytic tools to compute extreme values, but for the purposes of PreCalculus it is sufficient to be able to identify and categorize extreme values graphically or through the use of technology. For example, the TI-84 has a maximum finder when you select <2^{nd}>then <trace>.



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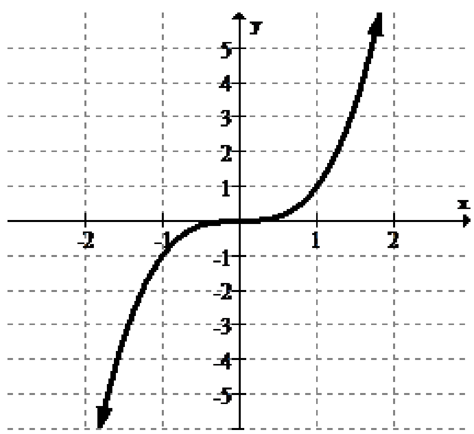
Examples

Example 1

Earlier, you were asked to identify and distinguish between different peaks and valleys on a graph. Maximums and minimums should be intuitive because they simply identify the highest points and the lowest points, or the peaks and the valleys, in a graph. There is a formal distinction about whether a maximum is the highest on some local open interval (does not matter how small), or whether it is simply the highest overall.

Example 2

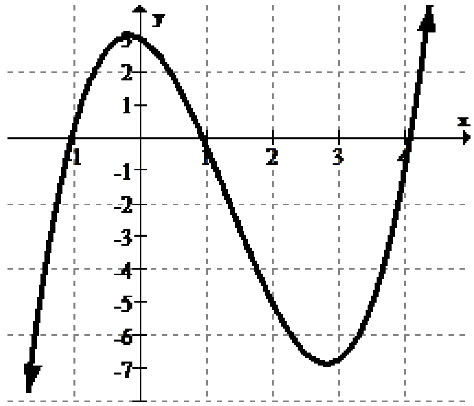
Identify and categorize all extrema.



There are no global or local maximums or minimums. The function flattens, but does not actually reach a peak or a valley.

Example 3

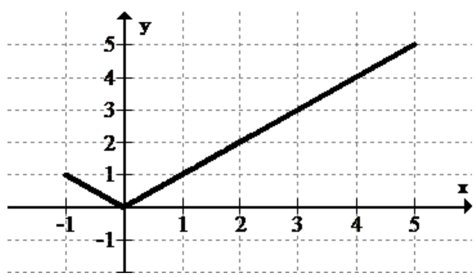
Identify and categorize all the extrema in the graph below:



Since the function appears from the arrows to increase and decrease beyond the display, there are no global extrema. There is a local maximum at approximately (0, 3) and a local minimum at approximately (2.8, -7).

Example 4

Identify and categorize all the extrema in the graph below:

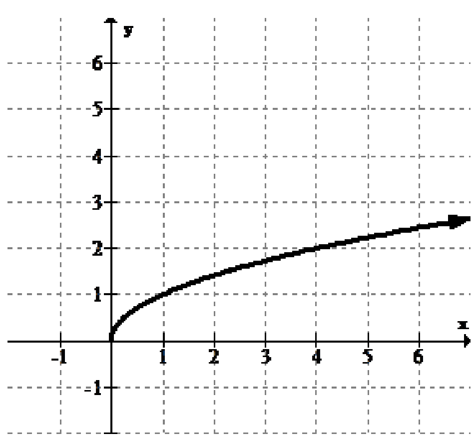


Since the function seems to abruptly end at the end points and does not go beyond the display, the endpoints are important.

There is a global minimum at (0, 0). There is a local maximum at (-1, 1) and a global maximum at (5, 5).

Example 5

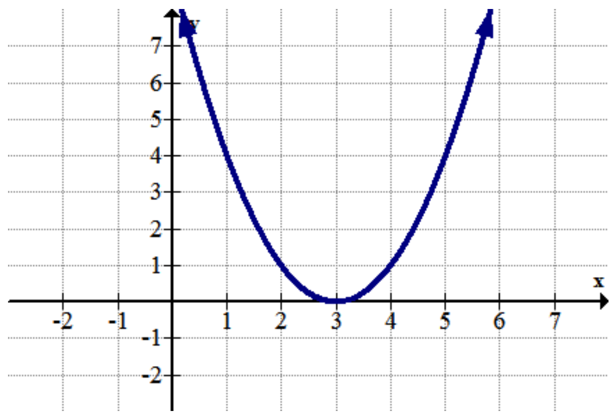
Identify and categorize all the extrema in the graph below:



Since this function appears to increase to the right as indicated by the arrow there is no global maximum. There are not any other high points either, so there are no local maximums. There is only the end point at (0, 0) which is a global minimum.

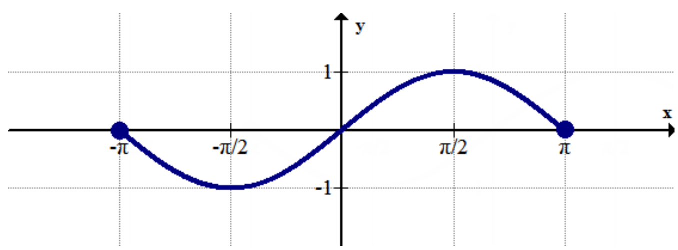
Review

Use the graph below for 1-2.



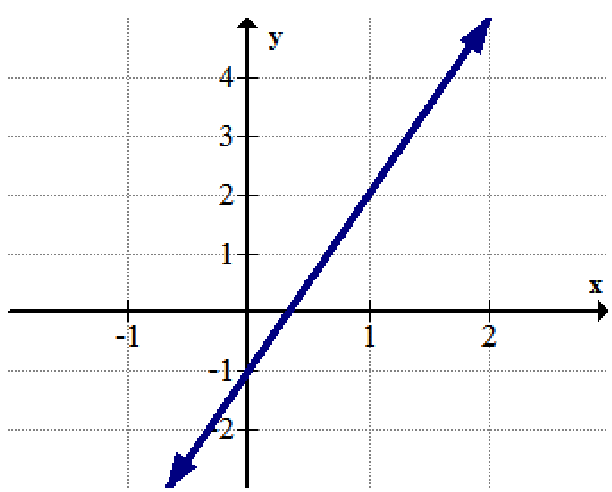
1. Identify any global extrema.
2. Identify any local extrema.

Use the graph below for 3-4.



3. Identify any global extrema.
4. Identify any local extrema.

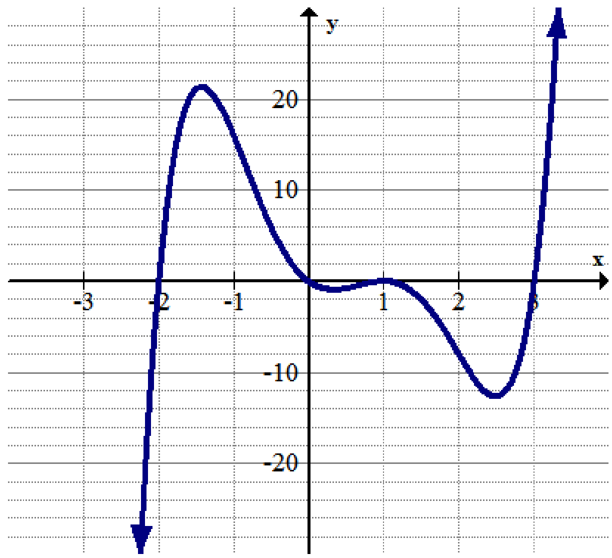
Use the graph below for 5-6.



5. Identify any global extrema.

6. Identify any local extrema.

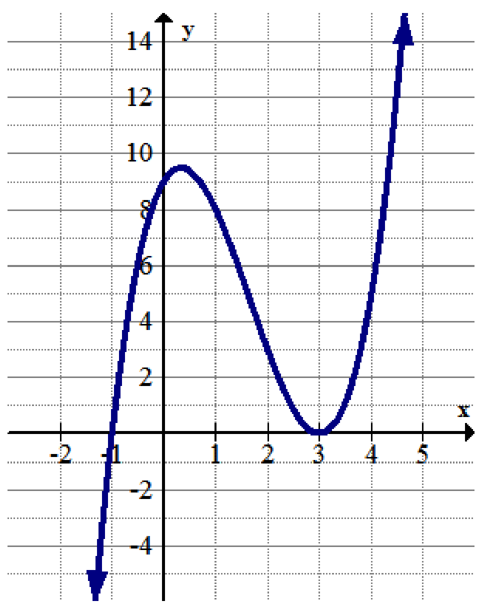
Use the graph below for 7-8.



7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10.



9. Identify any global extrema.

10. Identify any local extrema.

11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use your graphing calculator to identify and categorize the extrema of:
$f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24.$
15. Use your graphing calculator to identify and categorize the extrema of:
$g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3.$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.5.

1.4 Symmetry

Learning Objectives

Here you will review rotation and reflection symmetry as well as explore how algebra accomplishes both.

Some functions, like the sine function, the absolute value function and the squaring function, have reflection symmetry across the line $x = 0$. Other functions like the cubing function and the reciprocal function have rotational symmetry about the origin.

Why is the first group categorized as even functions while the second group is categorized as odd functions?

Even and Odd Functions

Even Functions

Functions symmetrical across the line $x = 0$ (the y axis) are called even. **Even functions** have the property that when a negative value is substituted for x, it produces the same value as when the positive value is substituted for the x. In other words, the equation $f(-x) = f(x)$ holds true for even functions.

To show that the function $f(x) = 3x^4 - 5x^2 + 1$ is even, show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= 3(-x)^4 - 5(-x)^2 + 1 \\ &= 3x^4 - 5x^2 + 1 \\ &= f(x) \end{aligned}$$

The property that both positive and negative numbers raised to an even power are always positive is the reason why the term even is used. It does not matter that the coefficients are even or odd, just the exponents.

Odd Functions

Functions that have rotational symmetry about the origin are called odd functions. **Odd functions** have the property that when a negative x value is substituted into the function, it produces a negative version of the function evaluated at a positive value. In other words, the equation $f(-x) = -f(x)$ holds true for odd functions.

This property becomes increasingly important in problems and proofs of Calculus and beyond, but for now it is sufficient to identify functions that are even, odd or neither and show why.

To show that $f(x) = 4x^3 - x$ is odd, show that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 + x \\ &= -(4x^3 + x) \\ &= -f(x) \end{aligned}$$

Just like even functions are named, odd functions are named because negative signs don't disappear and can always be factored out of odd functions.

Even and odd functions describe different types of symmetry, but both derive their name from the properties of exponents. A negative number raised to an even number will always be positive. A negative number raised to an odd number will always be negative.

Examples

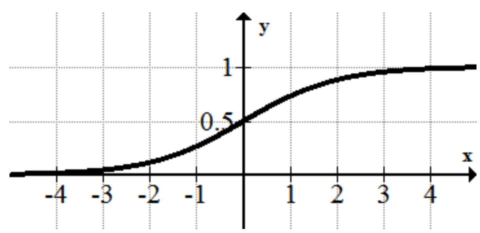
Example 1

Which of the basic functions are even, which are odd and which are neither?

Even Functions: The squaring function and the absolute value function.

Odd Functions: The identity function, the cubing function, the reciprocal function, the sine function.

Neither: The square root function, the exponential function and the log function. The logistic function is also neither because it is rotationally symmetric about the point $\left(0, \frac{1}{2}\right)$ as opposed to the origin.



Example 2

Suppose $h(x)$ is an even function and $g(x)$ is an odd function. $f(x) = h(x) + g(x)$. Is $f(x)$ even or odd? If $h(x)$ is even then $h(-x) = h(x)$. If $g(x)$ is odd then $g(-x) = -g(x)$.

Therefore: $f(-x) = h(-x) + g(-x) = h(x) - g(x)$

This does not match $f(x) = h(x) + g(x)$ nor does it match $-f(x) = -h(x) - g(x)$.

This is a proof that shows the sum of an even function and an odd function will never itself be even or odd.

Example 3

Determine whether the following function is even, odd, or neither.

$f(x) = x(x^2 - 1)(x^4 + 1)$

Identify whether the function is even, odd or neither and explain why.

$$\begin{aligned}
f(x) &= x(x^2 - 1)(x^4 + 1) \\
f(-x) &= (-x)((-x)^2 - 1)((-x)^4 + 1) \\
&= -x(x^2 - 1)(x^4 + 1) \\
&= -f(x)
\end{aligned}$$

The function is odd because $f(-x) = -f(x)$ holds true.

$f(x) = 4x^3 - |x|$

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 - x \end{aligned}$$

This does not seem to match either $f(x) = 4x^3 - |x|$ or $-f(x) = -4x^3 + |x|$. Therefore, this function is neither even nor odd.

Note: This function is a difference of an odd function and an even function. This should be a clue that the resulting function is neither even nor odd.

Review

Determine whether the following functions are even, odd, or neither.

1. $f(x) = -4x^2 + 1$
2. $g(x) = 5x^3 - 3x$
3. $h(x) = 2x^2 - x$
4. $j(x) = (x-4)(x-3)^3$
5. $k(x) = x(x^2-1)^2$
6. $f(x) = 2x^3 - 5x^2 - 2x + 1$
7. $g(x) = 2x^2 - 4x + 2$
8. $h(x) = -5x^4 + x^2 + 2$
9. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) - g(x)$ is neither even nor odd.
10. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = \frac{h(x)}{g(x)}$ is odd.
11. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) \cdot g(x)$ is odd.
12. Is the sum of two even functions always an even function? Explain.
13. Is the sum of two odd functions always an odd function? Explain.
14. Why are some functions neither even nor odd?
15. If you know that a function is even or odd, what does that tell you about the symmetry of the function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.6.

1.5 Increasing and Decreasing

Learning Objectives

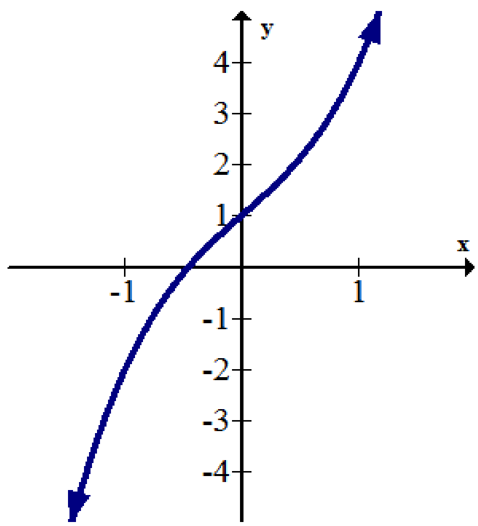
Here you will apply interval notation to identify when functions are increasing and decreasing.

It is important to be able to distinguish between when functions are increasing and when they are decreasing. In business this could mean the difference between making money and losing money. In physics it could mean the difference between speeding up and slowing down.

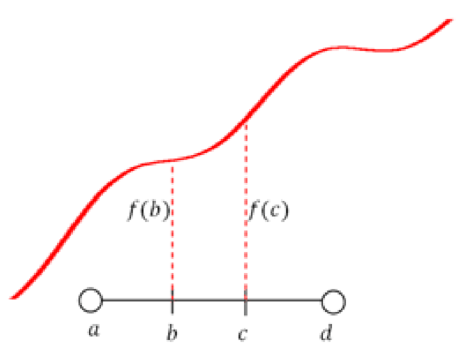
How do you decide when a function is increasing or decreasing?

Increasing and Decreasing Functions

Increasing means places on the graph where the slope is positive.

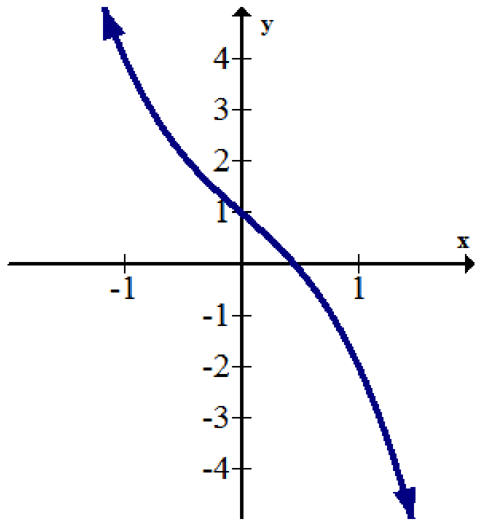


The formal definition of an increasing interval is: an open interval on the x axis of (a,d) where every $b,c \in (a,d)$ with $b < c$ has $f(b) \leq f(c)$.



A interval is said to be strictly increasing if $f(b) < f(c)$ is substituted into the definition.

Decreasing means places on the graph where the slope is negative. The formal definition of decreasing and strictly decreasing are identical to the definition of increasing with the inequality sign reversed.



A function is called **monotonic** if the function only goes in one direction and never switches between increasing and decreasing.

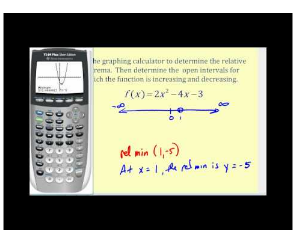
Out of the basic functions, the monotonically increasing functions are:

$f(x) = x, f(x) = x^3, f(x) = \sqrt{x}, f(x) = e^x, f(x) = \ln x, f(x) = \frac{1}{1+e^{-x}}$

The only basic functions that are not monotonically increasing are:

$f(x) = x^2, f(x) = |x|, f(x) = \frac{1}{x}, f(x) = \sin x$

Identifying analytically where functions are increasing and decreasing often requires Calculus. For PreCalculus, it will be sufficient to be able to identify intervals graphically and through your knowledge of what the parent functions look like.



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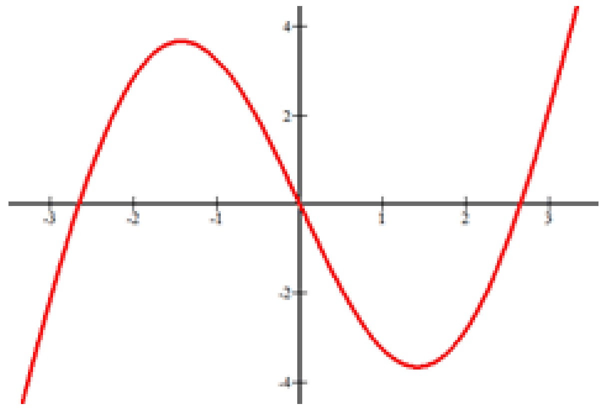
Examples

Example 1

Earlier, you were asked how to determine if a function is increasing or decreasing. Increasing is where the function has a positive slope and decreasing is where the function has a negative slope. A common misconception is to look at the squaring function and see two curves that symmetrically increase away from zero. Instead, you should always read functions from left to right and draw slope lines and decide if they are positive or negative.

Example 2

Estimate where the following function is increasing and decreasing.

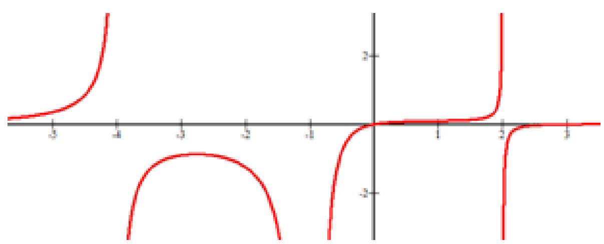


Increasing: $x \in (-\infty, -1.5) \cup (1.5, \infty)$.

Decreasing: $x \in (-1.5, 1.5)$

Example 3

Estimate where the following function is increasing and decreasing.

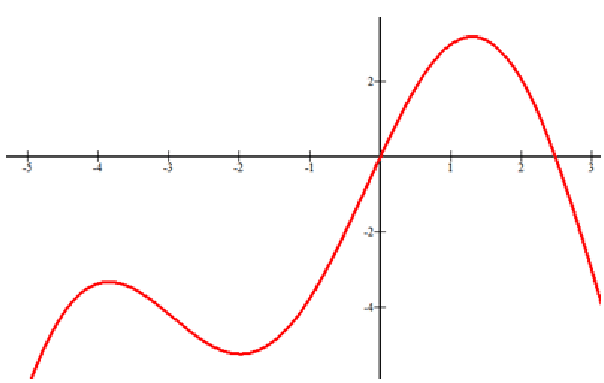


Increasing $x \in (-\infty, -4) \cup (-4, -2.7) \cup (-1, 2) \cup (2, \infty)$.

Decreasing $x \in (-2.7, -1)$

Example 4

Estimate the intervals where the function is increasing and decreasing.



Increasing: $x \in (-\infty, -4) \cup (-2, 1.5)$

Decreasing: $x \in (-4, -2) \cup (1.5, \infty)$

Notice that open intervals are used because at $x = -4, -2, 1.5$ the slope of the function is zero. This is where the slope transitions from being positive to negative. The reason why open parentheses are used is because the function is not actually increasing or decreasing at those specific points.

Example 5

A continuous function has a global maximum at the point (3, 2), a global minimum at (5, -12) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

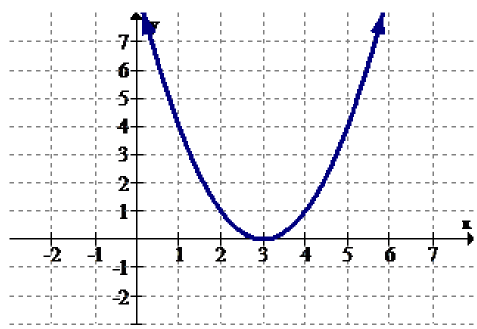
Increasing $x \in (-\infty, 3) \cup (5, \infty)$.

Decreasing $x \in (3, 5)$

Note: The y coordinates are not used in the intervals. A common mistake is to want to use the y coordinates.

Review

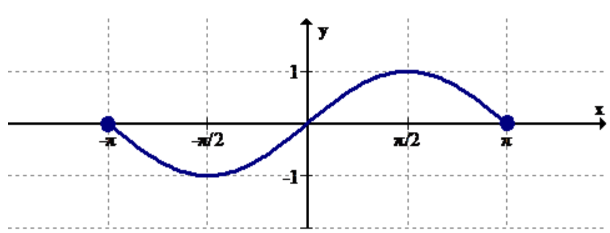
Use the graph below for 1-2.



1. Identify the intervals (if any) where the function is increasing.

2. Identify the intervals (if any) where the function is decreasing.

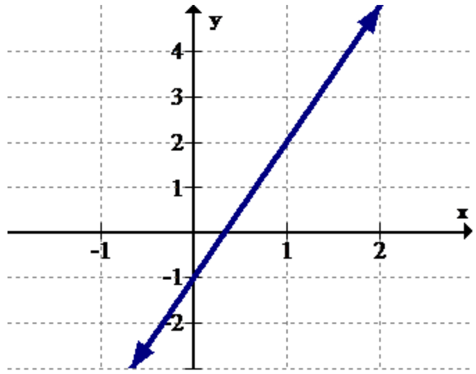
Use the graph below for 3-4.



3. Identify the intervals (if any) where the function is increasing.

4. Identify the intervals (if any) where the function is decreasing.

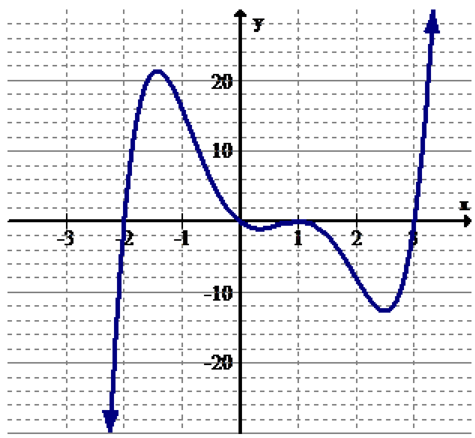
Use the graph below for 5-6.



5. Identify the intervals (if any) where the function is increasing.

6. Identify the intervals (if any) where the function is decreasing.

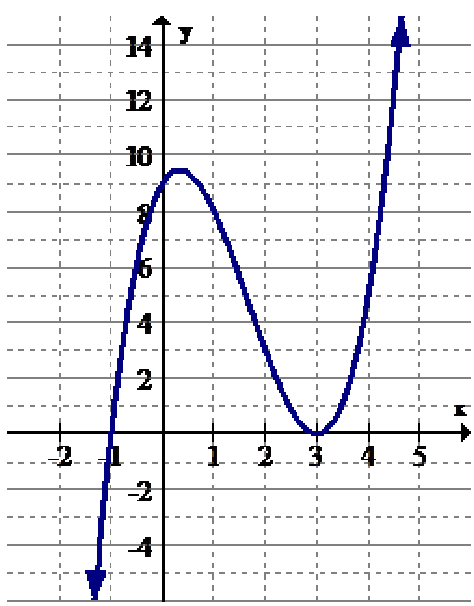
Use the graph below for 7-8.



7. Identify the intervals (if any) where the function is increasing.

8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10.



9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A continuous function has a global maximum at the point (1, 4), a global minimum at (3, -6) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

14. A continuous function has a global maximum at the point (1, 1) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

15. A continuous function has a global minimum at the point (5, -15) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

Review (Answers)

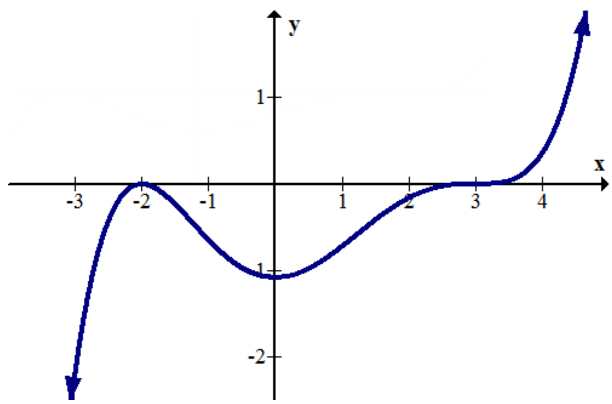
To see the Review answers, open this PDF file and look for section 1.7.

1.6 Zeroes and Intercepts of Functions

Learning Objectives

Here you will learn about x and y intercepts. You will learn to approximate them graphically and solve for them exactly using algebra.

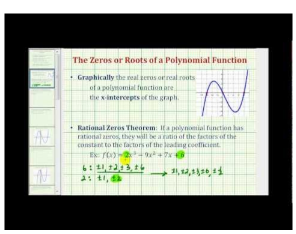
An **intercept** in mathematics is where a function crosses the x or y axis. What are the intercepts of this function?



X and Y Intercepts

The first type of intercept you may have learned is the y-intercept when you learned the slope intercept form of a line: $y = mx + b$. A **y-intercept** is the unique point where a function crosses the y axis. It can be found algebraically by setting $x = 0$ and solving for y.

x-intercepts are where functions cross the x axis and where the height of the function is zero. They are also called roots, solutions and zeroes of a function. They are found algebraically by setting $y = 0$ and solving for x. Watch the videos below for practice:



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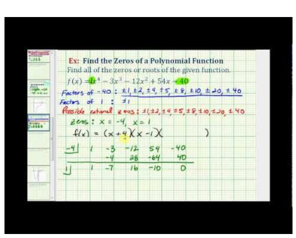

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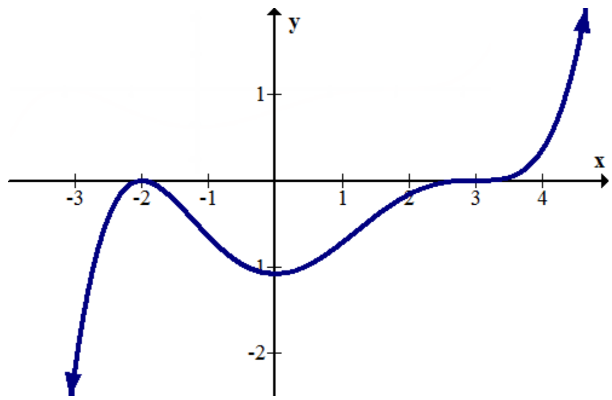
Examples

Example 1

Earlier, you were asked what the intercepts of the graph below are.



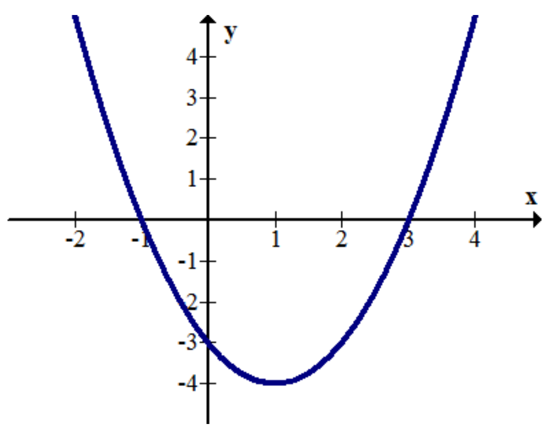
Graphically the function has zeroes at -2 and 3 with a y intercept at about -1.1.

Note: In order for a function to pass the vertical line test, it must only have one y-intercept, but it may have multiple x-intercepts.

Example 2

What are the zeroes and y-intercepts of the parabola $y = x^2 - 2x - 3$?

Using a graph:



The zeroes are at (-1, 0) and (3, 0). The y-intercept is at (0, -3).

Using Algebra:

Substitute 0 for y to find zeroes.

$0 = x^2 - 2x - 3 = (x-3)(x+1)$

$y = 0, x = 3, -1$

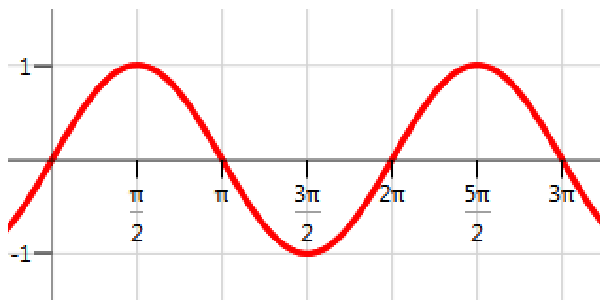
Substitute 0 for x to find the y-intercept.

$y=(0)^2-2(0)-3=-3$

$x=0, y=-3$

Example 3

Identify the zeroes and y-intercepts for the sine function.



The y-intercept is (0, 0). There are four zeroes visible on this portion of the graph. One thing you know about the sine graph is that it is periodic and repeats forever in both directions. In order to capture every x-intercept, you must identify a pattern instead of trying to write out every single one.

The visible x-intercepts are $0, \pi, 2\pi, 3\pi$. The pattern is that there is an x-intercept every multiple of π including negative multiples. In order to describe all of these values you should write:

The x-intercepts are $\pm n\pi$ where n is an integer $\{0, \pm 1, \pm 2, \ldots\}$.

Example 4

Identify the intercepts and zeroes of the function: $f(x)=\frac{1}{100}(x-3)^3(x+2)^2$.

To find the y-intercept, substitute 0 for x:

$y=\frac{1}{100}(0-3)^3(0+2)^2=\frac{1}{100}(-27)(4)=-\frac{108}{100}=-1.08$

To find the x-intercepts, substitute 0 for y:

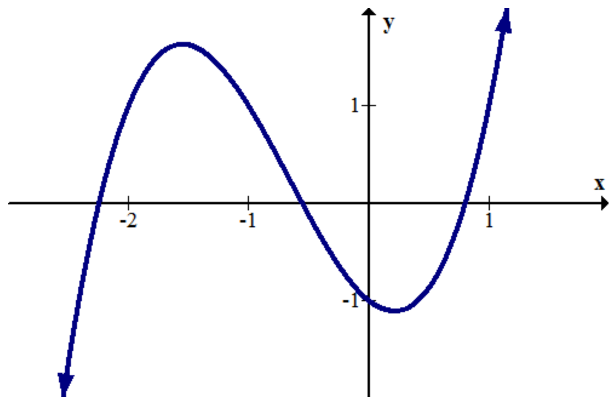
$0=\frac{1}{100}(x-3)^3(x+2)^2$

$x=3,-2$

Thus the y-intercept is (0, -1.08) and the x-intercepts are (3, 0) and (-2, 0).

Example 5

Determine the intercepts of the following function graphically.



The y-intercept is approximately (0, -1). The x-intercepts are approximately (-2.3, 0), (-0.4, 0) and (0.7, 0). When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

Review

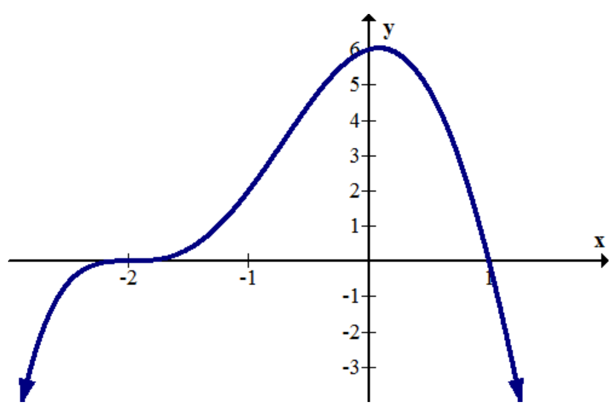
1. Determine the zeroes and y-intercept of the following function using algebra:

   $f(x) = (x+1)^3(x-4)$

2. Determine the roots and y-intercept of the following function using algebra or a graph:

   $g(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$

3. Determine the intercepts of the following function graphically:



Find the intercepts for each of the following functions.

4. $y = x^2$
5. $y = x^3$
6. $y = \ln(x)$
7. $y = \frac{1}{x}$
8. $y = e^x$
9. $y = \sqrt{x}$
10. Are there any functions without a y-intercept? Explain.
11. Are there any functions without an x-intercept? Explain.
12. Explain why it makes sense that an x-intercept of a function is also called a "zero" of the function.

Determine the intercepts of the following functions using algebra or a graph.

13. $h(x) = x^3 - 6x^2 + 3x + 10$
14. $j(x) = x^2 - 6x - 7$
15. $k(x) = 4x^4 - 20x^3 - 3x^2 + 14x + 5$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.8.

1.7 Asymptotes and End Behavior

Learning Objectives

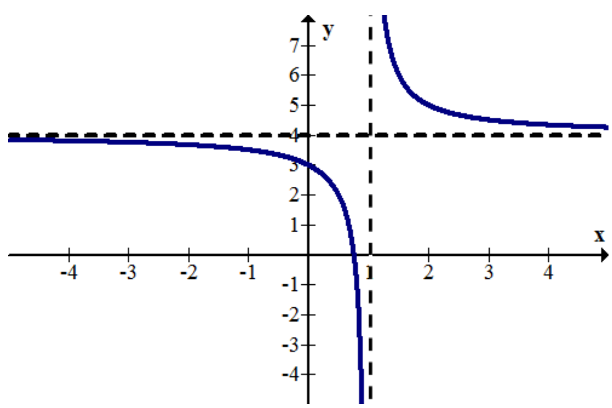
Here you will get a conceptual and graphical understanding of what is meant by asymptotes and end behavior. This will lay the groundwork for future concepts. Most functions continue beyond the viewing window in our calculator or computer. People often draw an arrow next to a dotted line to indicate the pattern specifically. How can you recognize these asymptotes?

Asymptotes and End Behavior of Functions

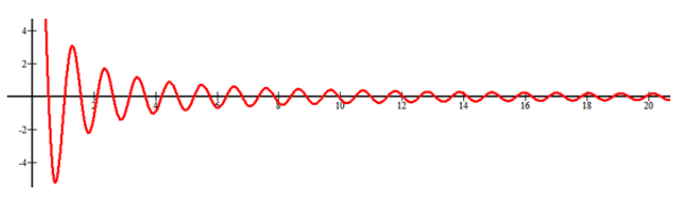
A **vertical asymptote** is a vertical line such as $x = 1$ that indicates where a function is not defined and yet gets infinitely close to.

A **horizontal asymptote** is a horizontal line such as $y = 4$ that indicates where a function flattens out as x gets very large or very small. A function may touch or pass through a horizontal asymptote.

The reciprocal function has two asymptotes, one vertical and one horizontal. Most computers and calculators do not draw the asymptotes and so they must be inserted by hand as dotted lines.

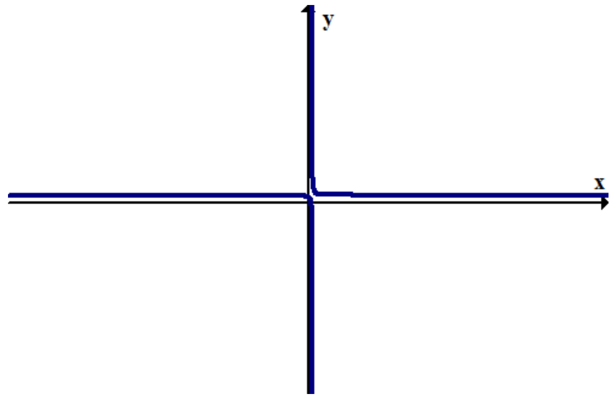


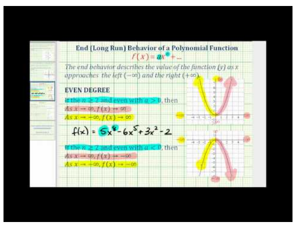
Many students have the misconception that an asymptote is a line that a function gets infinitely close to but does not touch. This is not true. Take the following function:



The graph appears to flatten as x grows larger. Thus, the horizontal asymptote is $y = 0$ even though the function clearly passes through this line an infinite number of times.

The reason why asymptotes are important is because when your perspective is zoomed way out, the asymptotes essentially become the graph.





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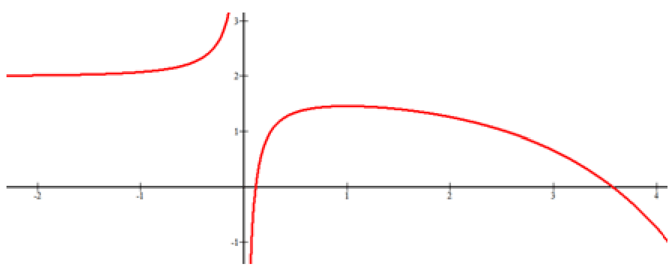
To find the asymptotes and end behavior of the function below, examine what happens to x and y as they each increase or decrease.



The function has a horizontal asymptote $y = 2$ as x approaches negative infinity. There is a vertical asymptote at $x = 0$. The right hand side seems to decrease forever and has no asymptote.

Note that slant asymptotes do exist and are called oblique asymptotes.

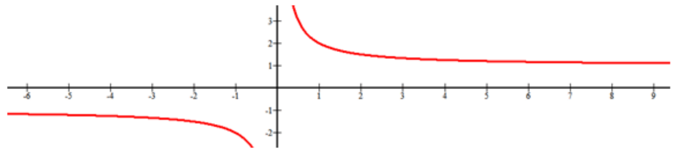
Examples

Example 1

Earlier, you were asked how to identify asymptotes on a graph. Asymptotes written by hand are usually identified with dotted lines next to the function that indicate how the function will behave outside the viewing window. The equations of these vertical and horizontal dotted lines are of the form $x =$ ____ and $y =$ ____. When problems ask you to find the asymptotes of a function, they are asking for the equations of these horizontal and vertical lines.

Example 2

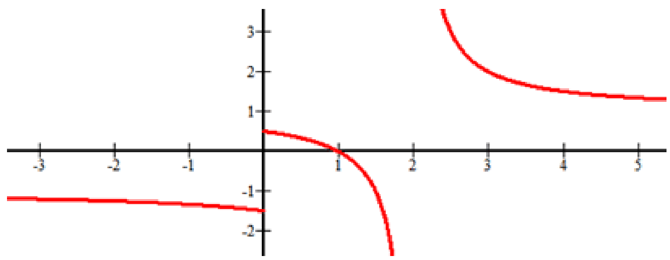
Identify the horizontal and vertical asymptotes of the following function.



There is a vertical asymptote at $x = 0$. As x gets infinitely small, there is a horizontal asymptote at $y = -1$. As x gets infinitely large, there is another horizontal asymptote at $y = 1$.

Example 3

Identify the horizontal and vertical asymptotes of the following function.

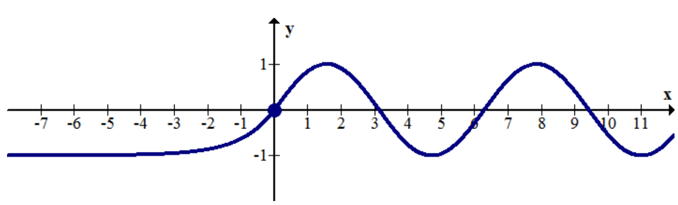


There is a vertical asymptote at $x = 2$. As x gets infinitely small there is a horizontal asymptote at $y = -1$. As x gets infinitely large, there is a horizontal asymptote at $y = 1$.

Example 4

Identify the horizontal and vertical asymptotes of the following piecewise function:

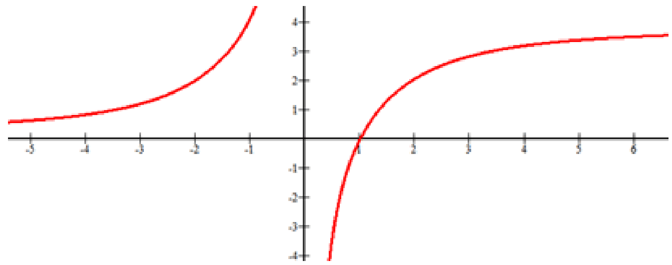
$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$



There is a horizontal asymptote at $y = -1$ as x gets infinitely small. This is because e raised to the power of a very small number becomes 0.000000... and basically becomes zero.

Example 5

Identify the asymptotes and end behavior of the following function.

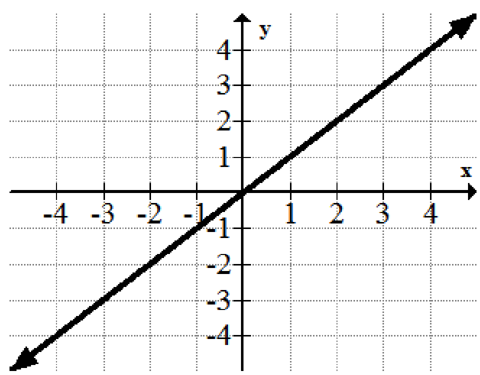


There is a vertical asymptote at $x = 0$. The end behavior of the right and left side of this function does not match. The horizontal asymptote as x approaches negative infinity is $y = 0$ and the horizontal asymptote as x approaches positive infinity is $y = 4$. At this point you can only estimate these heights because you were not given the function or the tools to find these values analytically.

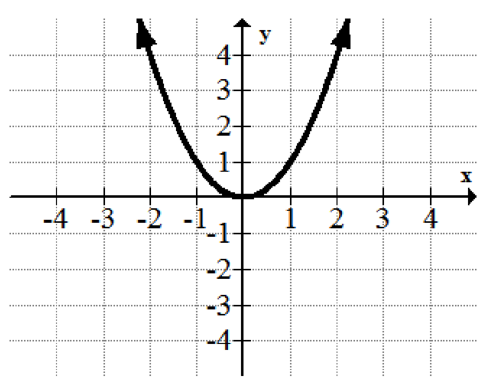
Review

Identify the asymptotes and end behavior of the following functions.

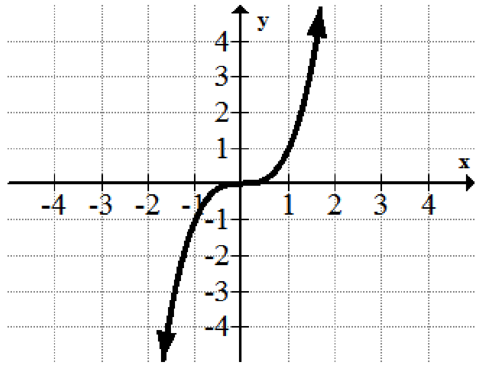
1. $y = x$



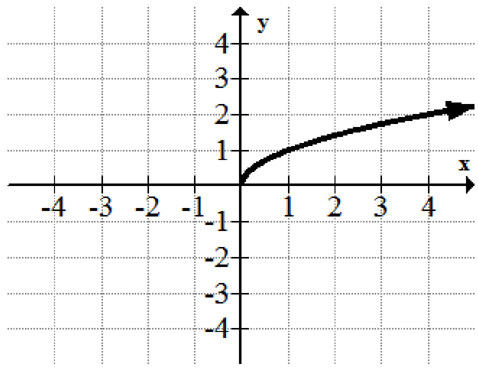
2. $y = x^2$



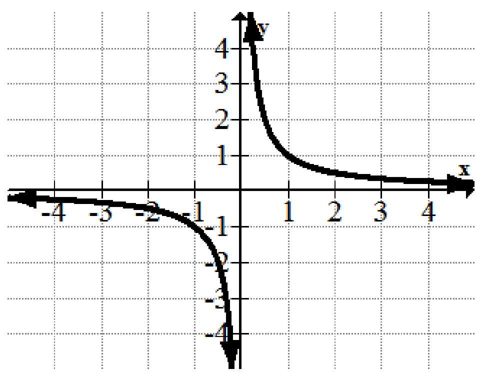
3. $y = x^3$



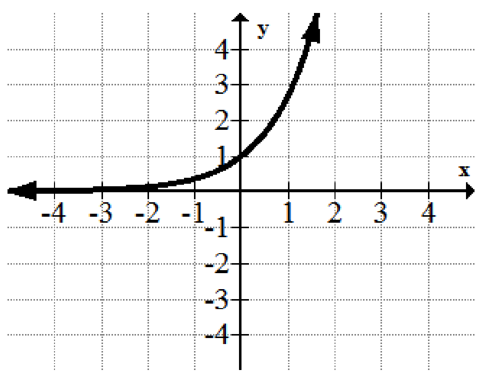
4. $y = \sqrt{x}$



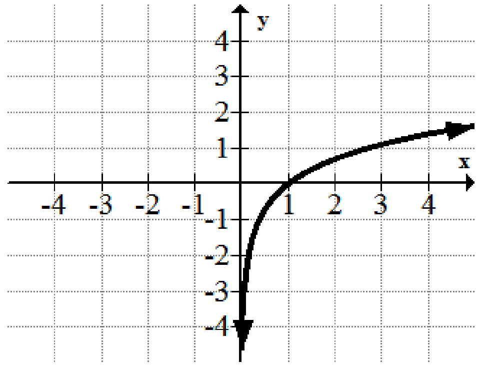
5. $y = \frac{1}{x}$



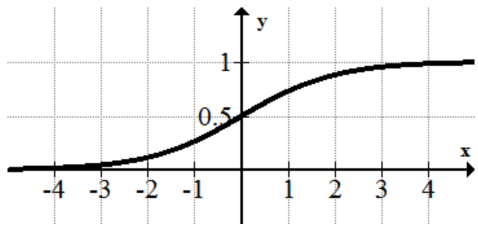
6. $y = e^x$



7. $y = \ln(x)$



8. $y = \frac{1}{1+e^{-x}}$



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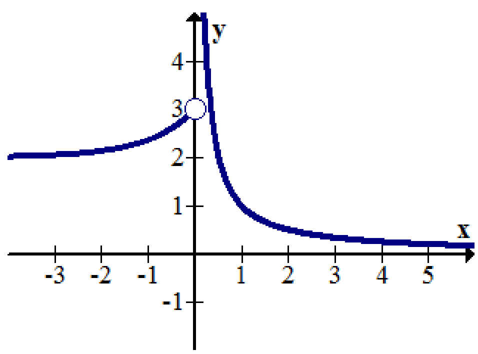

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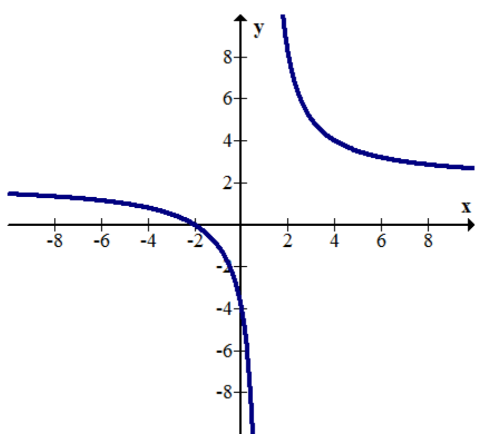

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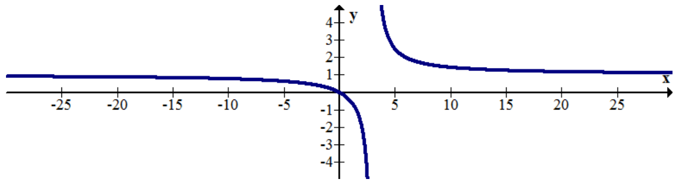


12. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x}$ has a vertical asymptote at $x = 0$.

13. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x+3}$ has a vertical asymptote at $x = -3$.

14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = \frac{1}{x-2}$.

15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = \frac{2}{x+4}$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.

1.8 Continuity and Discontinuity

Learning Objectives

Here you will learn the formal definition of continuity, the three types of discontinuities and more about piecewise functions.

Continuity is a property of functions that can be drawn without lifting your pencil. Some functions, like the reciprocal functions, have two distinct parts that are unconnected. Functions that are unconnected are discontinuous. What are the three ways functions can be discontinuous and how do they come about?

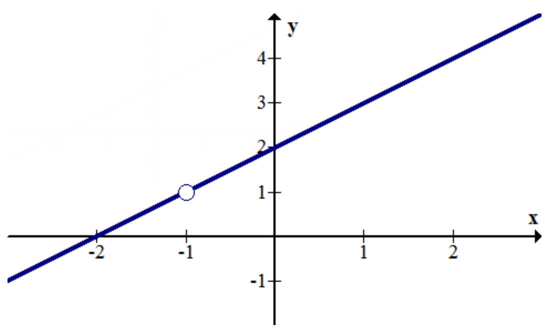
Continuity and Discontinuity of Functions

Functions that can be drawn without lifting up your pencil are called **continuous functions**. You will define continuous in a more mathematically rigorous way after you study limits.

There are three types of discontinuities: Removable, Jump and Infinite.

Removable Discontinuities

Removable discontinuities occur when a rational function has a factor with an x that exists in both the numerator and the denominator. Removable discontinuities are shown in a graph by a hollow circle that is also known as a hole. Below is the graph for $f(x) = \frac{(x+2)(x+1)}{x+1}$. Notice that it looks just like $y = x+2$ except for the hole at $x = -1$. When graphing function, you should cancel the removable factor, graph like usual and then insert a hole in the appropriate spot at the end. There is a hole at $x = -1$ because when $x = -1, f(x) = \frac{0}{0}$.

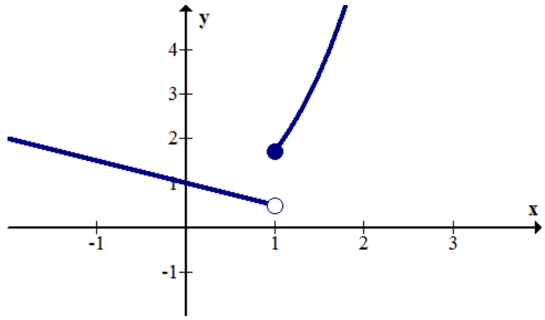


Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make $f(x)$ continuous you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

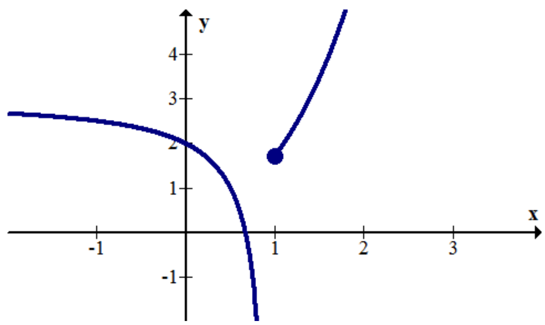
Jump Discontinuities

Jump discontinuities occur when a function has two ends that don't meet even if the hole is filled in. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity.



Infinite Discontinuities

Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below at $x = 1$.



Examples

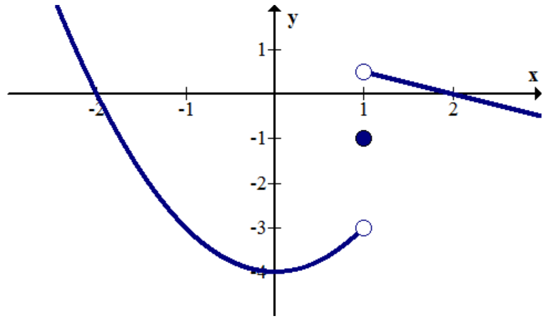
Example 1

Earlier you were asked how functions can be discontinuous. There are three ways that functions can be discontinuous. When a rational function has a vertical asymptote as a result of the denominator being equal to zero at some point, it will have an infinite discontinuity at that point. When the numerator and denominator of a rational function have one or more of the same factors, there will be removable discontinuities corresponding to each of these factors. Finally, when the different parts of a piecewise function don't "match", there will be a jump discontinuity.

Example 2

Identify the discontinuity of the piecewise function graphically.

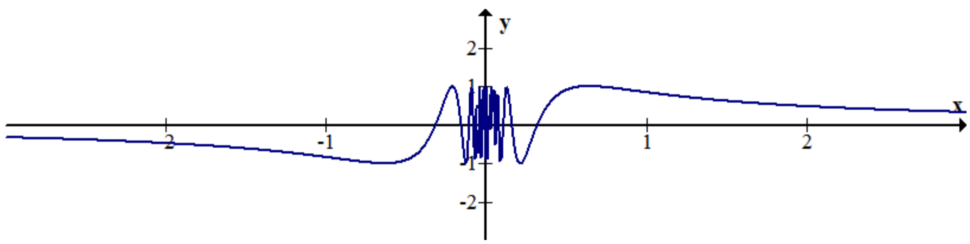
$$f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$$



There is a jump discontinuity at $x = 1$. The piecewise function describes a function in three parts; a parabola on the left, a single point in the middle and a line on the right.

Example 3

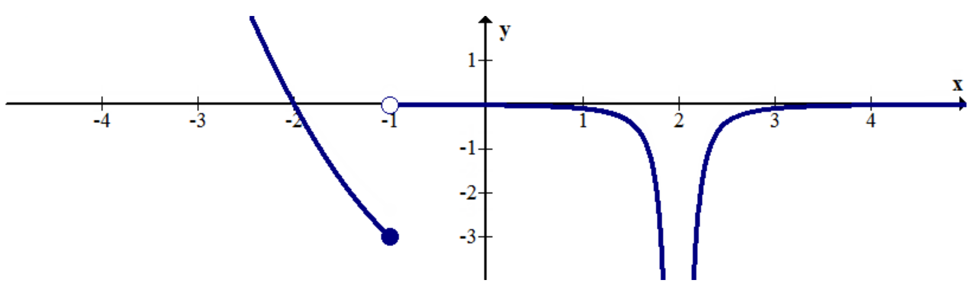
Describe the continuity or discontinuity of the function $f(x) = \sin\left(\frac{1}{x}\right)$.



The function seems to oscillate infinitely as x approaches zero. One thing that the graph fails to show is that 0 is clearly not in the domain. The graph does not shoot to infinity, nor does it have a simple hole or jump discontinuity. Calculus and Real Analysis are required to state more precisely what is going on.

Example 4

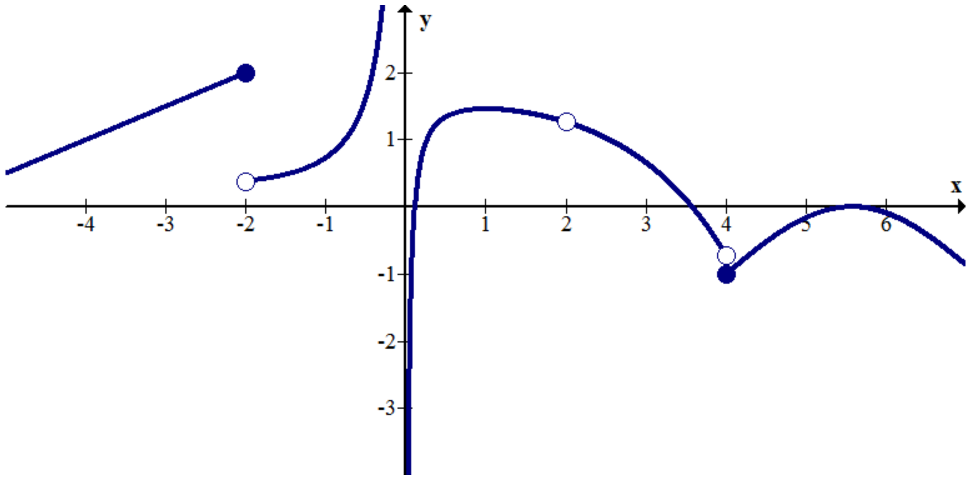
Describe the discontinuities of the function below.



There is a jump discontinuity at $x = -1$ and an infinite discontinuity at $x = 2$.

Example 5

Describe the discontinuities of the function below.

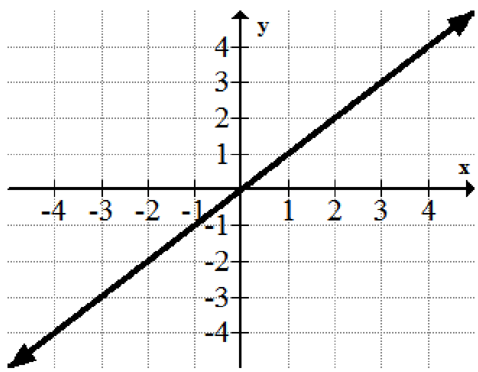


There are jump discontinuities at $x = -2$ and $x = 4$. There is a removable discontinuity at $x = 2$. There is an infinite discontinuity at $x = 0$.

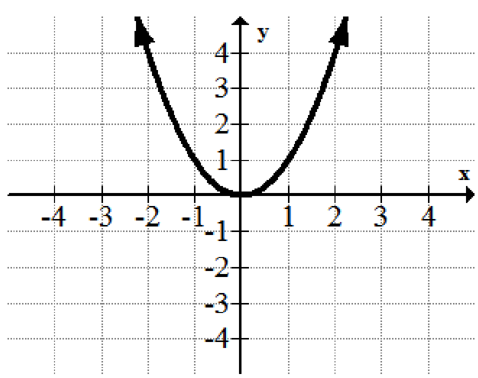
Review

Describe any discontinuities in the functions below:

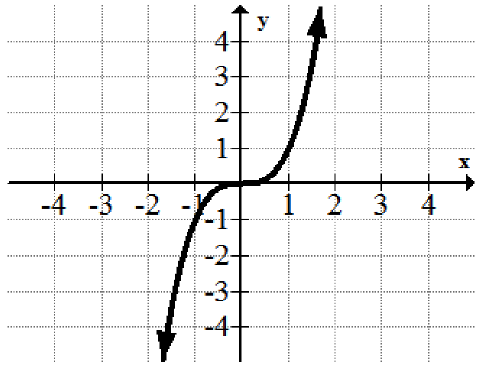
1. $y = x$



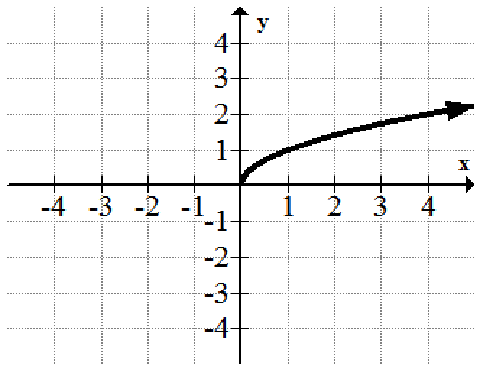
2. $y = x^2$



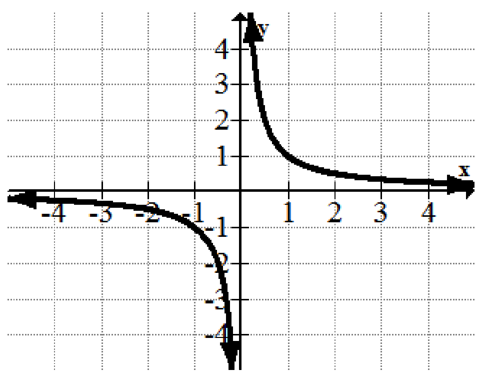
3. $y = x^3$



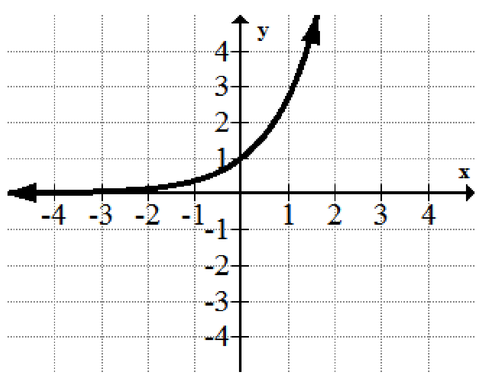
4. $y = \sqrt{x}$



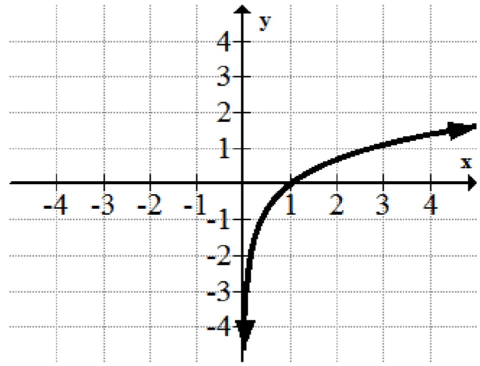
5. $y = \frac{1}{x}$



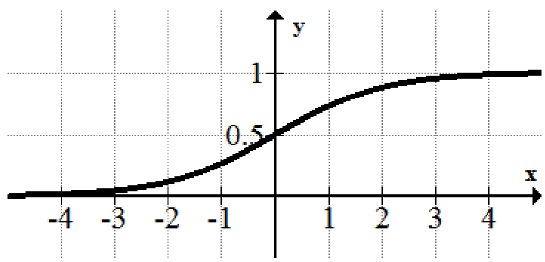
6. $y = e^x$



7. $y = \ln(x)$



8. $y = \frac{1}{1+e^{-x}}$



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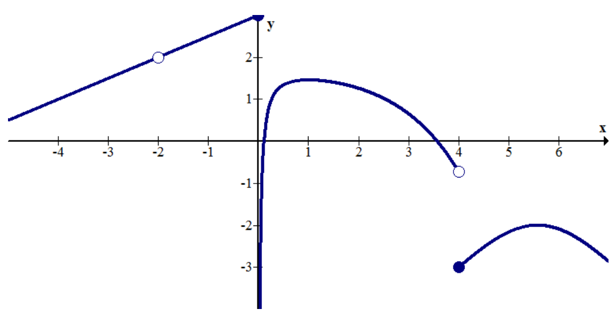

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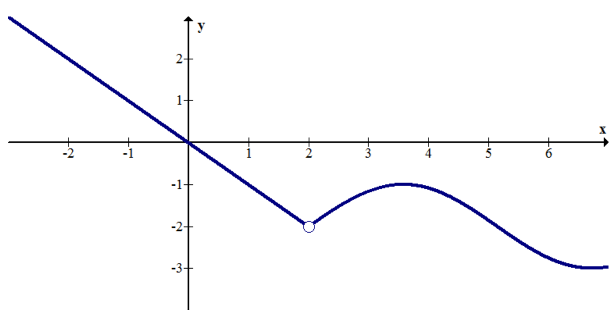

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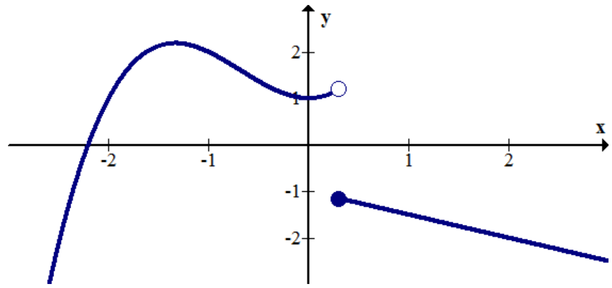


12. $f(x)$ has a jump discontinuity at $x = 3$, a removable discontinuity at $x = 5$, and another jump discontinuity at $x = 6$. Draw a picture of a graph that could be $f(x)$.

13. $g(x)$ has a jump discontinuity at $x = -2$, an infinite discontinuity at $x = 1$, and another jump discontinuity at $x = 3$. Draw a picture of a graph that could be $g(x)$.

14. $h(x)$ has a removable discontinuity at $x = -4$, a jump discontinuity at $x = 1$, and another jump discontinuity at $x = 7$. Draw a picture of a graph that could be $h(x)$.

15. $j(x)$ has an infinite discontinuity at $x = 0$, a removable discontinuity at $x = 1$, and a jump discontinuity at $x = 4$. Draw a picture of a graph that could be $j(x)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.10.

1.9 Function Families

Learning Objectives

Here you will learn to identify primary function families by their equations and graphs. This will set the stage for analyzing all types of functions.

Functions come in all different shapes. A few are very closely related and others are very different, but often confused. For example, what is the difference between x^2 and 2^x? They both have an x and a 2 and they both equal 4 when $x = 2$, but one eventually becomes much bigger than the other.

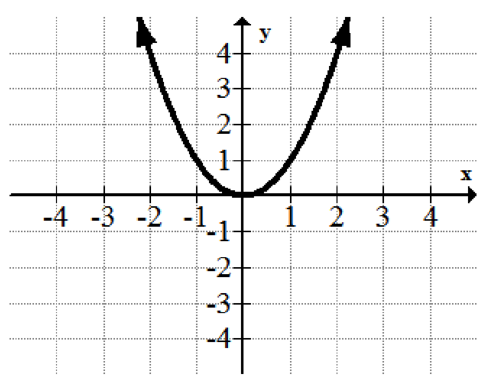
Families of Functions

If mathematicians are cooks, then families of functions are their ingredients. Each family of functions has its own flavor and personality. Before you learn to combine functions to create an infinite number of potential models, you need to get a clear idea of the name of each function family and how it acts.

The Identity Function: $f(x) = x$

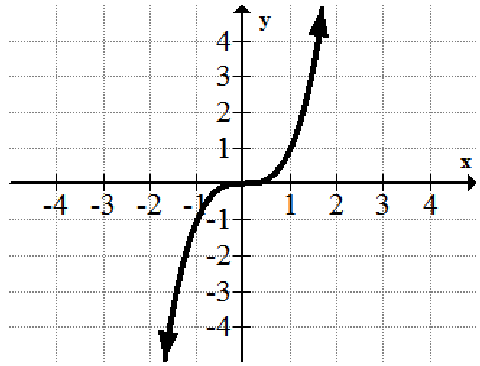
The **identity function** is the simplest function and all straight lines are transformations of the identity function family.

The Squaring Function: $f(x) = x^2$



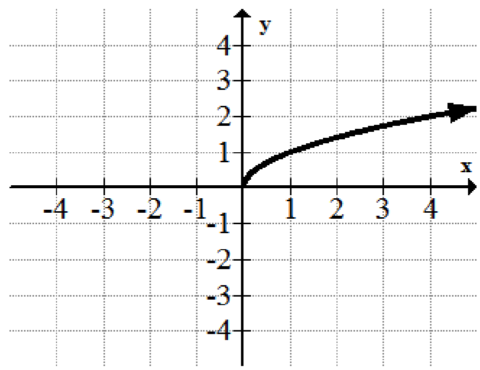
The **squaring function(quadratic function)** is commonly called a parabola and is useful for modeling the motion of falling objects. All parabolas are transformations of this squaring function.

The Cubing Function: $f(x) = x^3$



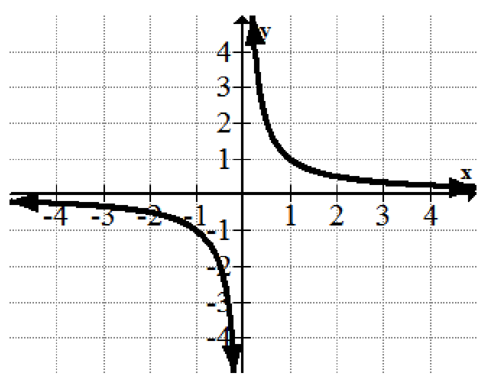
The **cubing function** has a different kind of symmetry than the squaring function. Since volume is measured in cubic units, many physics applications use the cubic function.

The Square Root Function: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$



The **square root function** is not defined over all real numbers. It introduces the possibility of complex numbers and is also closely related to the squaring function.

The Reciprocal Function: $f(x) = x^{-1} = \frac{1}{x}$



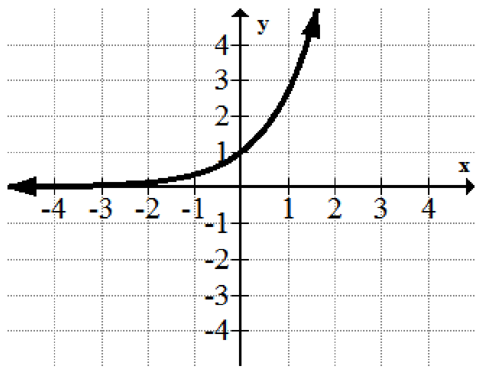
The **reciprocal function** is also known as a hyperbola and a rational function. It has two parts that are disconnected and is not defined at zero. Simple electric circuits are modeled with the reciprocal function.

So far all the functions can be grouped together into an even larger function family called the power function family.

The Power Function Family: $f(x) = cx^a$

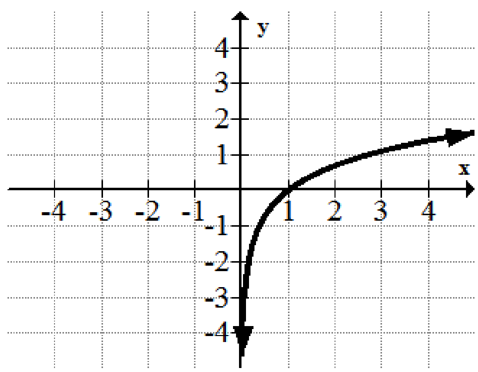
The **power function** family has two parameters. The parameter c is a vertical scale factor. The parameter a controls everything about the shape. The reason why all the functions so far are subsets of the larger power function family is because they only differ in their value of a. The power function family also shows you that there are an infinite number of other functions like quartics ($f(x) = x^4$) and quintics ($f(x) = x^5$) that don't really need a whole category of their own. The power function family can be extended to create polynomials and rational functions.

The Exponential Function Family: $f(x) = e^x$



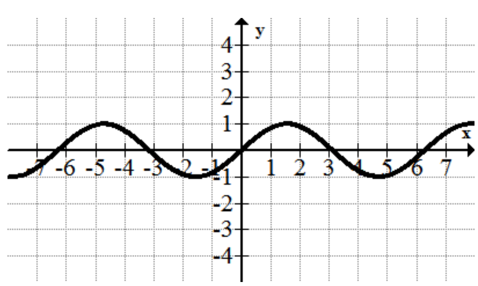
The **exponential function** family is one of the first functions you see where x is not the base of the exponent. This function eventually grows much faster than any power function. $f(x) = 2^x$ is a very common exponential function as well. Many applications like biology and finance require the use of exponential growth.

The Logarithm Function: $f(x) = \ln x$



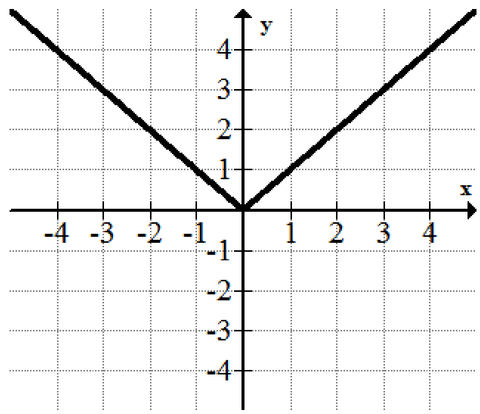
The **logarithmic function** is closely related to the exponential function family. Many people confuse the graph of the log function with the square root function. Careful analysis will show several important differences. The log function is the basis for the Richter Scale which is how earthquakes are measured.

The Periodic Function Family: $f(x) = \sin x$



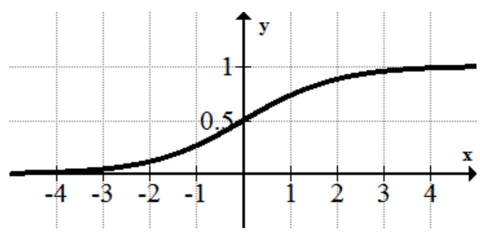
The **sine graph** is one of many periodic functions. Periodic refers to the fact that the sine wave repeats a cycle every period of time. Periodic functions are extremely important for modeling tides and other real world phenomena.

The Absolute Value Function: $f(x) = |x|$



The **absolute value function** is one of the few basic functions that is not totally smooth.

The Logistic Function: $f(x) = \frac{1}{1+e^{-x}}$



The **logistic function** is a combination of the exponential function and the reciprocal function. This curve is very powerful because it models population growths where the maximum population is limited by environmental resources.

Examples

Example 1

Earlier, you were given a problem about comparing x^2 and 2^x. While x^2 and 2^x have similar ingredients, they have very different graphical features. The squaring function is symmetric about the line $x = 0$ while the exponential function is not. When $x = 0$, the squaring function has a height of zero and the exponential function has a height of one. The squaring function has a slope that becomes steeper as x gets further from the origin while the exponential function flattens as x gets very small. All of these differences are important and not obvious at first glance.

Example 2

Which functions always have a positive slope over the entire real line?

$y = x, y = e^x, y = \frac{1}{1+e^{-x}}$. Some functions that are close but not quite: $y = x^3, y = \sqrt{x}$

Example 3

Compare and contrast the graphs of the two functions:

$f(x) = \ln x$ and $h(x) = \sqrt{x}$

Similarities: Both functions increase without bound as x gets larger. Both functions are not defined for negative numbers.

Differences: The log function approaches negative infinity as x approaches 0. The square root function, on the other hand, just ends at the point (0, 0).

Example 4

Describe the symmetry among the function families discussed in this concept. Consider both reflection symmetry and rotational symmetry.

Some function families have reflective symmetry with themselves:

$y = x, y = x^2, y = \frac{1}{x}, y = |x|$

Some function families are rotationally symmetric:

$y = x, y = x^3, y = \frac{1}{x}, y = \sin x, y = \frac{1}{1+e^{-x}}$

Some pairs of function families are full or partial reflections of other function families:

$y = x^2, y = \sqrt{x}$

$y = e^x, y = \ln x$

Review

For 1-10, sketch a graph of the function from memory.

1. $y = e^x$
2. $y = \ln(x)$
3. $y = \sin(x)$
4. $y = x^2$
5. $y = |x|$
6. $y = \frac{1}{x}$
7. $y = \frac{1}{1+e^{-x}}$
8. $y = \sqrt{x}$
9. $y = x^3$
10. $y = x$
11. Which function is not defined at 0? Why?
12. Which functions are bounded below but not above?
13. What are the differences between $y = x^2$ and $y = x^3$?
14. What is a similarity between $y = e^x$ and $y = \ln(x)$?
15. Explain why $y = \sqrt{x}$ is not defined for all values of x.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.1.

1.10 Analyzing the Graph of a Quadratic Function

Objective

Graphing, finding the vertex and $x-$intercepts, and using all the forms of a quadratic equation.

Review Queue

Solve the following equations using the method of your choice.

1. $x^2+6x-27=0$

2. $x^2-10x+29=0$

3. $x^2-8x+16=0$

4. In this lesson, we are going to be analyzing the graph of a quadratic function. Using what you know about the solutions of a quadratic equation, what do you think the shape of the function will look like? Draw it on your paper.

Finding the Parts of a Parabola

Objective

Finding the $x-$intercepts, vertex, axis of symmetry, and $y-$intercept of a parabola.

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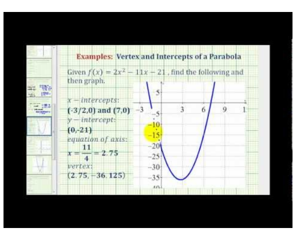

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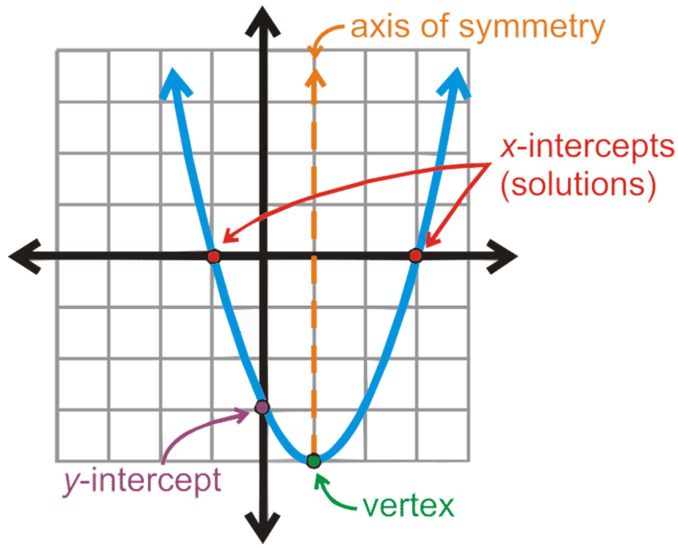
James Sousa: Ex 4: Graph a Quadratic Function in General Form by Finding Key Components

Guidance

Now that we have found the solutions of a quadratic equation we will graph the function. First, we need to introduce y or $f(x)$. A quadratic function is written $y=ax^2+bx+c$ or $f(x)=ax^2+bx+c$ (see *Finding the Domain and Range of Functions* concept). All quadratic equations are also functions.

Recall that the solutions of a quadratic equation are found when the equation is set equal to zero. This is also the same as when $y=0$. Therefore, the solutions of a quadratic equation are also the $x-$ **intercepts** of that function, when graphed.



The graph of a quadratic equation is called a **parabola** and looks like the figure to the left. A parabola always has "U" shape and depending on certain values, can be wider or narrower. The lowest part of a parabola, or minimum, is called the **vertex**. Parabolas can also be flipped upside-down, and in this case, the vertex would be the maximum value. Notice that this parabola is symmetrical about vertical line that passes through the vertex. This line is called the **axis of symmetry.** Lastly, where the parabola passes through the $y-$axis (when $x = 0$), is the $y-$**intercept**.

If you are given (or can find) the $x-$intercepts and the vertex, you can always graph a parabola.

Investigation: Finding the Vertex of a Parabola

1. The equation of the parabola above is $y = x^2 - 2x - 3$. Find a, b, and c. $a = 1, b = -2, c = -3$

2. What are the coordinates of the vertex? (1, -4)

3. Create an expression using a and b (from Step 1) that would be equal to the $x-$coordinate of the vertex. $1 = \frac{-b}{2a}$

4. Plug in $x = 1$ to the equation of the parabola. What do you get for y? $y = -4$

From this investigation, we have introduced how to find the vertex of a parabola. The $x-$coordinate of the vertex is $x = \frac{-b}{2a}$. To find y, plug in this value to the equation, also written $f\left(\frac{-b}{2a}\right)$. $x = \frac{-b}{2a}$ is also the equation of the axis of symmetry.

Example A

Find the vertex, axis of symmetry, $x-$intercepts, and $y-$intercept of $y = -\frac{1}{2}x^2 - 2x + 6$.

Solution: First, let's find the $x-$intercepts. This equation is factorable and $ac = -3$. The factors of -3 that add up to -2 are -3 and 1. Expand the $x-$term and factor.

$$\begin{aligned}
-\frac{1}{2}x^2 - 2x + 6 &= 0 \\
-\frac{1}{2}x^2 - 3x + x + 6 &= 0 \\
-x\left(\frac{1}{2}x + 3\right) + 2\left(\frac{1}{2}x + 3\right) &= 0 \\
\left(\frac{1}{2}x + 3\right)(-x + 2) &= 0
\end{aligned}$$

Solving for x, the intercepts are (-6, 0) and (2, 0).

To find the vertex, use $x = \frac{-b}{2a}$.

$x = \frac{-(-2)}{2 \cdot -\frac{1}{2}} = \frac{2}{-1} = -2$ Plug this into the equation: $y = -\frac{1}{2}(-2)^2 - 2(-2) + 6 = -2 + 4 + 6 = 8$.

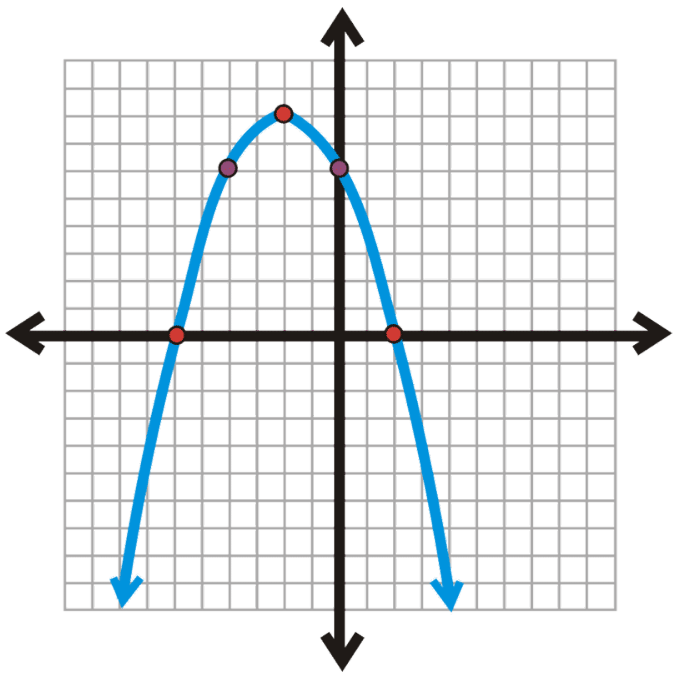
Therefore, the vertex is (-2, 8) and the axis of symmetry is $x = -2$.

To find the $y-$intercept, $x = 0$. $y = -\frac{1}{2}(0)^2 - 2(0) + 6 = 6$. Therefore, the $y-$intercept is (0, 6).

Example B

Sketch a graph of the parabola from Example A.

Solution: Plot the vertex and two $x-$intercepts (red points). Plot the $y-$intercept. Because all parabolas are symmetric, the corresponding point on the other side would be (-4, 6). Connect the five points to form the parabola.



For this parabola, the vertex is the **maximum** value. If you look at the equation, $y = -\frac{1}{2}x^2 - 2x + 6$, we see that the a value is negative. When a is negative, the sides of the parabola, will point down.

Example C

Find the vertex and $x-$intercepts of $y = 2x^2 - 5x - 25$. Then, sketch a graph.

Solution: First, this is a factorable function. $ac = -50$. The factors of -50 that add up to -5 are -10 and 5.

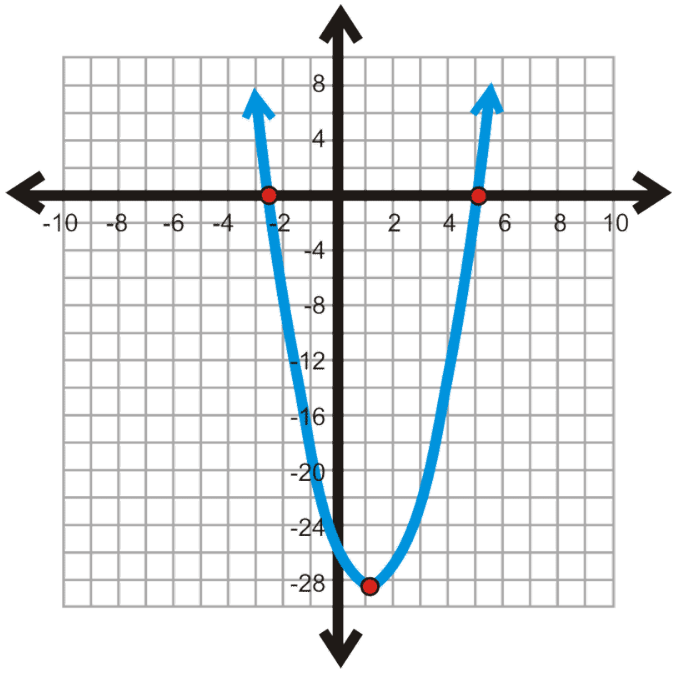
$$\begin{aligned}
2x^2 - 5x - 25 &= 0 \\
2x^2 - 10x + 5x - 25 &= 0 \\
2x(x-5) + 5(x-5) &= 0 \\
(2x+5)(x-5) &= 0
\end{aligned}$$

Setting each factor equal to zero, we get $x = 5$ and $-\frac{5}{2}$.

From this, we get that the $x-$intercepts are (5, 0) and $\left(-\frac{5}{2}, 0\right)$. To find the vertex, use $x = \frac{-b}{2a}$.

$x = \frac{5}{2 \cdot 2} = \frac{5}{4}$ Now, find y. $y = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) - 25 = \frac{25}{8} - \frac{25}{4} - 25 = -\frac{225}{8} = -28\frac{1}{8}$

The vertex is $\left(\frac{5}{4}, -28\frac{1}{8}\right)$. To graph this, we will need to estimate the vertex and draw an appropriate scale for the grid. As a decimal, the vertex is (1.25, -28.125).



Guided Practice

1. Find the $x-$intercepts, $y-$intercept, vertex, and axis of symmetry of $y = -x^2 + 7x - 12$.

2. Sketch a graph of the parabola from #1.

3. Find the vertex of $y = -4x^2 + 16x - 17$. Does the parabola open up or down?

Answers

1. This is a factorable quadratic equation.

$$\begin{aligned} -(x^2 - 7x + 12) &= 0 \\ -(x^2 - 3x - 4x + 12) &= 0 \\ -[x(x-3) - 4(x-3)] &= 0 \\ -(x-3)(x-4) &= 0 \end{aligned}$$

The $x-$intercepts are (3, 0) and (4, 0).

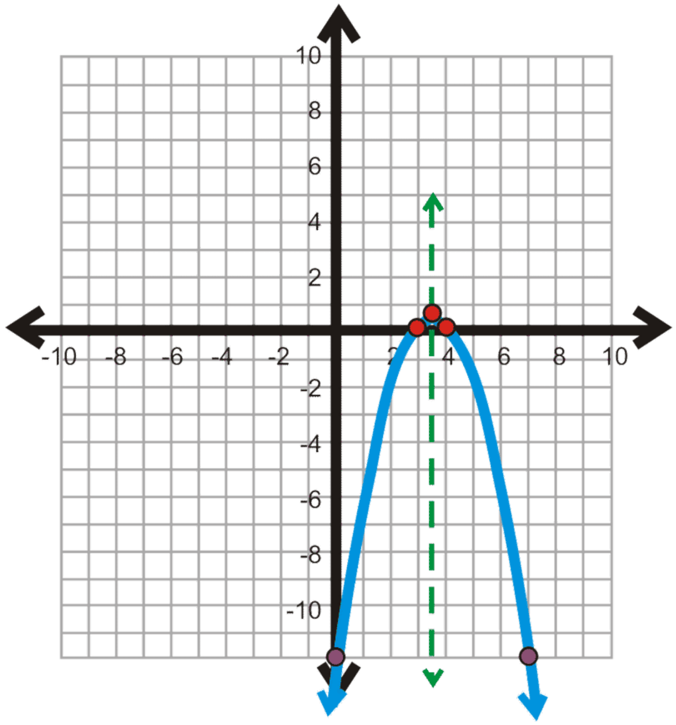
$$\begin{aligned} y &= -0^2 + 7(0) - 12 \\ y &= -12 \end{aligned}$$

The $y-$intercept is (0, -12).

The $x-$coordinate of the vertex is $x = \frac{-7}{2(-1)} = \frac{7}{2}$. The $y-$coordinate is $y = -\left(\frac{7}{2}\right)^2 + 7\left(\frac{7}{2}\right) - 12 = \frac{1}{4}$.

Therefore, the vertex is $\left(\frac{7}{2}, \frac{1}{4}\right)$ and the parabola opens down because $a < 0$. The axis of symmetry is $x = \frac{7}{2}$.

2. Plot all the points you found in #1. Then, connect the points to create the parabola.



3. First, the parabola opens <u>down</u> because a is negative. The $x-$coordinate of the vertex is $x = \frac{-16}{2(-4)} = \frac{-16}{-8} = 2$. The $y-$coordinate is $y = -4(2)^2 + 16(2) - 17 = -16 + 32 - 17 = -1$. This makes the vertex (2, -1).

Even though the problem does not ask, we can infer that this parabola does not cross the $x-$axis because it points down and the vertex is below the $x-$axis. This means that the solutions would be imaginary.

Vocabulary

Parabola
The "U" shaped graph of a quadratic equation.

Vertex
The highest or lowest point of a parabola. The $x-$coordinate of the vertex is $\frac{-b}{2a}$.

Maximum/Minimum
The highest/lowest point of a function.

$x-$intercept(s)
The point(s) where a function crosses the $x-$axis. $x-$intercepts are also called solutions, roots or zeros.

$y-$intercept
The point where a function crosses the $y-$axis. A function will only cross the $y-$axis once.

Axis of Symmetry
The line that a parabola is symmetric about. The vertex always lies on this line.

Problem Set

Find the vertex of each parabola and determine if it is a maximum or minimum.

1. $y = x^2 - 12x + 11$

2. $y = x^2 + 10x - 18$
3. $y = -3x^2 + 4x + 17$
4. $y = 2x^2 - 9x - 11$
5. $y = -x^2 + 6x - 9$
6. $y = -\frac{1}{4}x^2 + 8x - 33$

Find the vertex, $x-$intercepts, $y-$intercept, and axis of symmetry of each *factorable* quadratic equation below. Then, sketch a graph of each one.

7. $y = x^2 - 12x + 11$
8. $y = -2x^2 - 5x + 12$
9. $y = \frac{1}{3}x^2 + 4x - 15$
10. $y = 3x^2 + 26x - 9$
11. $y = -x^2 + 10x - 25$
12. $y = -\frac{1}{2}x^2 + 5x + 28$
13. If a function is not factorable, how would you find the $x-$intercepts?

Find the vertex and $x-$intercepts of the following quadratic equations. Then, sketch the graph. These equations are not factorable.

14. $y = -x^2 + 8x - 9$
15. $y = 2x^2 - x - 8$

Complete the table of values for the quadratic equations below. Then, plot the points and graph.

16. $y = x^2 - 2x + 2$

TABLE 1.3:

x	y
5	
3	
1	
-1	
-3	

17. $y = x^2 + 4x + 13$

TABLE 1.4:

x	y
4	
0	
-2	
-4	
-8	

18. **Writing** What do you notice about the two parabolas from 16 and 17? What type of solutions do these functions have? Solve #16.
19. **Writing** How many different ways can a parabola intersect the $x-$axis? Draw parabolas on an $x-y$ plane to

represent the different solution possibilities.

20. **Challenge** If the $x-$coordinate of the vertex is $\frac{-b}{2a}$ for $y = ax^2 + bx + c$, find the $y-$coordinate in terms of a, b, and c.

Vertex, Intercept, and Standard Form

Objective

To explore the different forms of the quadratic equation.

Guidance

So far, we have only used the **standard form** of a quadratic equation, $y = ax^2 + bx + c$ to graph a parabola. From standard form, we can find the vertex and either factor or use the Quadratic Formula to find the $x-$intercepts. The **intercept form** of a quadratic equation is $y = a(x-p)(x-q)$, where a is the same value as in standard form, and p and q are the $x-$intercepts. This form looks very similar to a factored quadratic equation.

Example A

Change $y = 2x^2 + 9x + 10$ to intercept form and find the vertex. Graph the parabola.

Solution: First, let's change this equation into intercept form by factoring. $ac = 20$ and the factors of 20 that add up to 9 are 4 and 5. Expand the $x-$term.

$$\begin{aligned}
y &= 2x^2 + 9x + 10 \\
y &= 2x^2 + 4x + 5x + 10 \\
y &= 2x(x+2) + 5(x+2) \\
y &= (2x+5)(x+2)
\end{aligned}$$

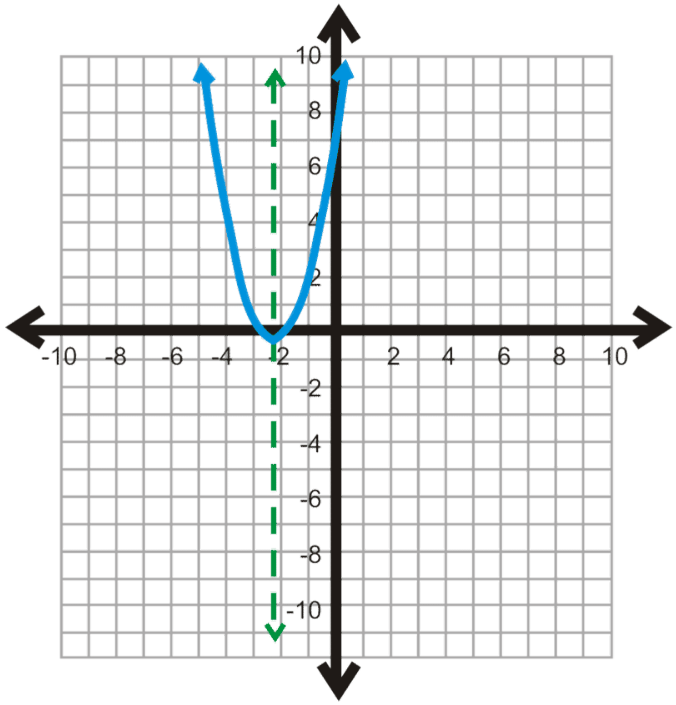
Notice, this does not exactly look like the definition. The factors cannot have a number in front of x. Pull out the 2 from the first factor to get $y = 2\left(x + \frac{5}{2}\right)(x+2)$. Now, find the vertex. Recall that all parabolas are symmetrical. This means that the axis of symmetry is *halfway* between the $x-$intercepts or their average.

$$\text{axis of symmetry} = \frac{p+q}{2} = \frac{-\frac{5}{2} - 2}{2} = -\frac{9}{2} \div 2 = -\frac{9}{2} \cdot \frac{1}{2} = -\frac{9}{4}$$

This is also the $x-$coordinate of the vertex. To find the $y-$coordinate, plug the $x-$value into either form of the quadratic equation. We will use Intercept form.

$$\begin{aligned}
y &= 2\left(-\frac{9}{4} + \frac{5}{2}\right)\left(-\frac{9}{4} + 2\right) \\
y &= 2 \cdot \frac{1}{4} \cdot -\frac{1}{4} \\
y &= -\frac{1}{8}
\end{aligned}$$

The vertex is $\left(-2\frac{1}{4}, -\frac{1}{8}\right)$. Plot the $x-$intercepts and the vertex to graph.

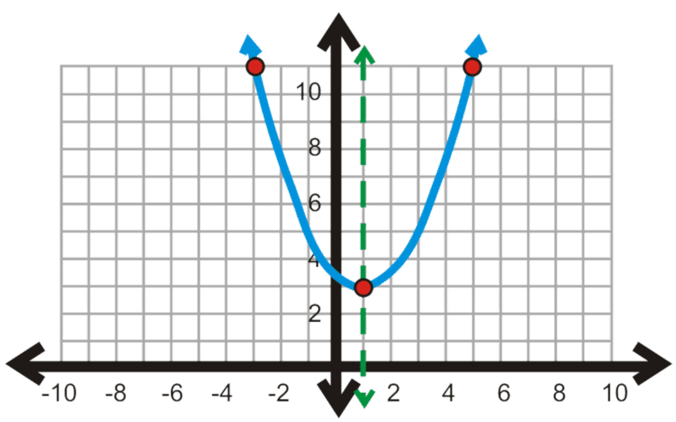


The last form is vertex form. **Vertex form** is written $y = a(x-h)^2 + k$, where (h,k) is the vertex and a is the same is in the other two forms. Notice that h is negative in the equation, but positive when written in coordinates of the vertex.

Example B

Find the vertex of $y = \frac{1}{2}(x-1)^2 + 3$ and graph the parabola.

Solution: The vertex is going to be (1, 3). To graph this parabola, use the symmetric properties of the function. Pick a value on the left side of the vertex. If $x = -3$, then $y = \frac{1}{2}(-3-1)^2 + 3 = 11$. -3 is 4 units away from 1 (the $x-$coordinate of the vertex). 4 units on the *other* side of 1 is 5. Therefore, the $y-$coordinate will be 11. Plot (1, 3), (-3, 11), and (5, 11) to graph the parabola.



Example C

Change $y = x^2 - 10x + 16$ into vertex form.

Solution: To change an equation from standard form into vertex form, you must complete the square. Review the *Completing the Square* Lesson if needed. The major difference is that you will not need to solve this equation.

$$\begin{aligned}
y &= x^2 - 10x + 16 \\
y - 16 + {\color{red}25} &= x^2 - 10x + {\color{red}25} && \text{Move 16 to the other side and add } \left(\frac{b}{2}\right)^2 \text{ to both sides.} \\
y + 9 &= (x-5)^2 && \text{Simplify left side and factor the right side} \\
y &= (x-5)^2 - 9 && \text{Subtract 9 from both sides to get } y \text{ by itself.}
\end{aligned}$$

To solve an equation in vertex form, set $y = 0$ and solve for x.

$$\begin{aligned}
(x-5)^2 - 9 &= 0 \\
(x-5)^2 &= 9 \\
x - 5 &= \pm 3 \\
x &= 5 \pm 3 \ \mathit{or}\ 8 \ \mathit{and}\ 2
\end{aligned}$$

Guided Practice

1. Find the intercepts of $y = 2(x-7)(x+2)$ and change it to standard form.

2. Find the vertex of $y = -\frac{1}{2}(x+4)^2 - 5$ and change it to standard form.

3. Change $y = x^2 + 18x + 45$ to intercept form and graph.

4. Change $y = x^2 - 6x - 7$ to vertex form and graph.

Answers

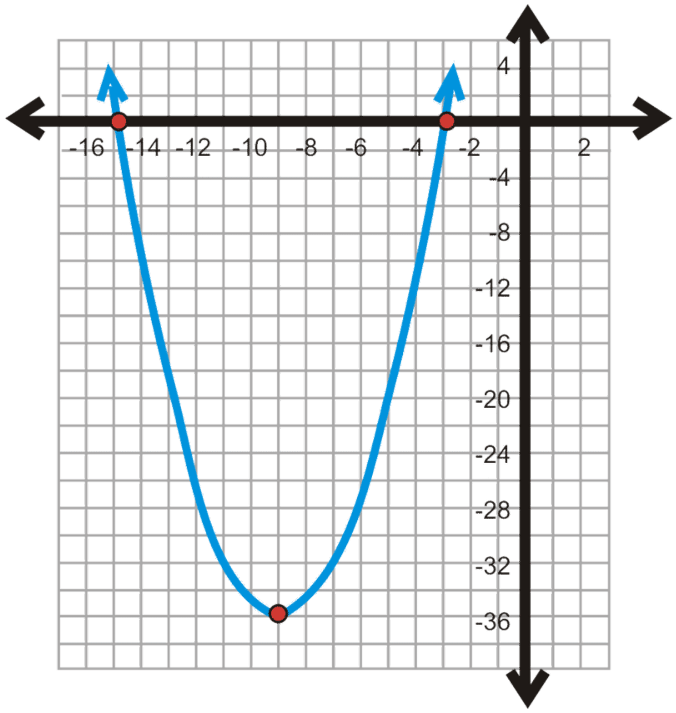
1. The intercepts are the opposite sign from the factors; (7, 0) and (-2, 0). To change the equation into standard form, FOIL the factors and distribute a.

$$\begin{aligned}
y &= 2(x-7)(x+2) \\
y &= 2(x^2 - 5x - 14) \\
y &= 2x^2 - 10x - 28
\end{aligned}$$

2. The vertex is (-4, -5). To change the equation into standard form, FOIL $(x+4)^2$, distribute a, and then subtract 5.

$$\begin{aligned}
y &= -\frac{1}{2}(x+4)(x+4) - 5 \\
y &= -\frac{1}{2}(x^2 + 8x + 16) - 5 \\
y &= -\frac{1}{2}x^2 - 4x - 21
\end{aligned}$$

3. To change $y = x^2 + 18x + 45$ into intercept form, factor the equation. The factors of 45 that add up to 18 are 15 and 3. Intercept form would be $y = (x+15)(x+3)$. The intercepts are (-15, 0) and (-3, 0). The $x-$coordinate of the vertex is halfway between -15 and -3, or -9. The $y-$coordinate of the vertex is $y = (-9)^2 + 18(-9) + 45 = -36$. Here is the graph:

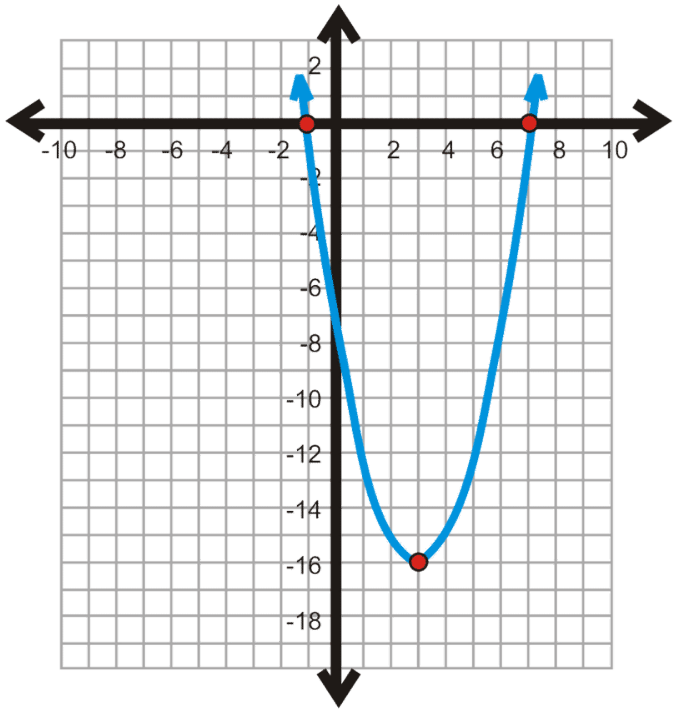


4. To change $y = x^2 - 6x - 7$ into vertex form, complete the square.

$$\begin{aligned} y + 7 + 9 &= x^2 - 6x + 9 \\ y + 16 &= (x - 3)^2 \\ y &= (x - 3)^2 - 16 \end{aligned}$$

The vertex is (3, -16).

For vertex form, we could solve the equation by using square roots or we could factor the standard form. Either way, we will get that the $x-$intercepts are (7, 0) and (-1, 0).



Vocabulary

Standard form
: $y = ax^2 + bx + c$

Intercept form
: $y = a(x - p)(x - q)$, where p and q are the $x-$intercepts.

Vertex form
: $y = a(x - h)^2 + k$, where (h, k) is the vertex.

Problem Set

1. Fill in the table below.

TABLE 1.5:

	Equation	**Vertex**	**Intercepts** (or how to find the intercepts)
Standard form			
Intercept form			
Vertex form			

Find the vertex and $x-$intercepts of each function below. Then, graph the function. If a function does not have any $x-$intercepts, use the symmetry property of parabolas to find points on the graph.

2. $y = (x - 4)^2 - 9$
3. $y = (x + 6)(x - 8)$
4. $y = x^2 + 2x - 8$

5. $y=-(x-5)(x+7)$
6. $y=2(x+1)^2-3$
7. $y=3(x-2)^2+4$
8. $y=\frac{1}{3}(x-9)(x+3)$
9. $y=-(x+2)^2+7$
10. $y=4x^2-13x-12$

Change the following equations to intercept form.

11. $y=x^2-3x+2$
12. $y=-x^2-10x+24$
13. $y=4x^2+18x+8$

Change the following equations to vertex form.

14. $y=x^2+12x-28$
15. $y=-x^2-10x+24$
16. $y=2x^2-8x+15$

Change the following equations to standard form.

17. $y=(x-3)^2+8$
18. $y=2\left(x-\frac{3}{2}\right)(x-4)$
19. $y=-\frac{1}{2}(x+6)^2-11$

Using the Graphing Calculator to Graph Quadratic Equations

Objective

To use the graphing calculator to graph parabolas, find their intercepts, and the vertex.

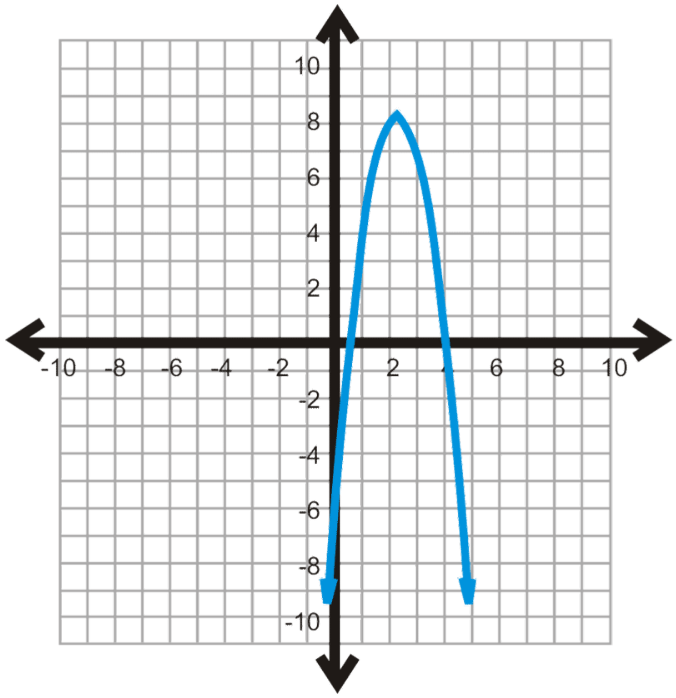
Guidance

A graphing calculator can be a very helpful tool when graphing parabolas. This concept outlines how to use the TI-83/84 to graph and find certain points on a parabola.

Example A

Graph $y=-3x^2+14x-8$ using a graphing calculator.

Solution: Using a TI-83/84, press the $Y=$ button. Enter in the equation. Be careful not to confuse the negative sign and the subtraction sign. The equation should look like $y=-3x^2+14x-8$ or $y=-3x^2+14x-8$. Press GRAPH.



If your graph does not look like this one, there may be an issue with your window. Press ZOOM and then **6:ZStandard,** ENTER. This should give you the standard window.

Example B

Using your graphing calculator, find the vertex of the parabola from Example A.

Solution: To find the vertex, press 2^{nd} TRACE (CALC). The Calculate menu will appear. In this case, the vertex is a maximum, so select **4:maximum,** ENTER. The screen will return to your graph. Now, you need to tell the calculator the Left Bound. Using the arrows, arrow over to the left side of the vertex, press ENTER. Repeat this for the Right Bound. The calculator then takes a guess, press ENTER again. It should give you that the maximum is $X = 2.3333333$ and $Y = 8.3333333$. As fractions, the coordinates of the vertex are $\left(2\frac{1}{3}, 8\frac{1}{3}\right)$. Make sure to write the coordinates of the vertex as a point.

Example C

Using your graphing calculator, find the $x-$intercepts of the parabola from Example A.

Solution: To find the $x-$intercepts, press 2^{nd} TRACE (CALC). The Calculate menu will appear. Select **2:Zero,** ENTER. The screen will return to your graph. Let's focus on the left-most intercept. Now, you need to tell the calculator the Left Bound. Using the arrows, arrow over to the left side of the vertex, press ENTER. Repeat this for the Right Bound (keep the bounds close to the intercept). The calculator then takes a guess, press ENTER again. This intercept is $X = .666667$, or $\left(\frac{2}{3}, 0\right)$. Repeat this process for the second intercept. You should get (4, 0).

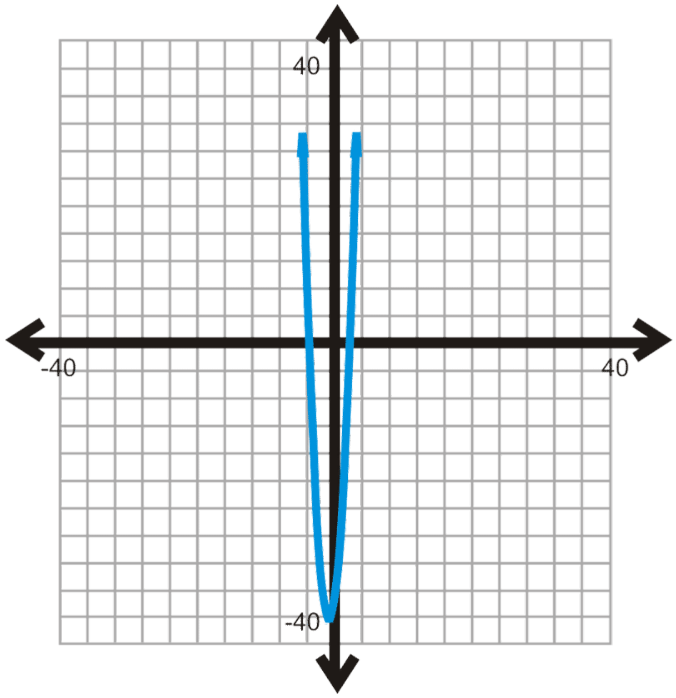
NOTE: When graphing parabolas and the vertex does not show up on the screen, you will need to zoom out. The calculator will not find the value(s) of any $x-$intercepts or the vertex that do not appear on screen. To zoom out, press ZOOM, **3:Zoom Out**, ENTER, ENTER.

Guided Practice

1. Graph $y = 6x^2 + 11x - 35$ using a graphing calculator. Find the vertex and $x-$intercepts. Round your answers to the nearest hundredth.

Answers

1. Using the steps above, the vertex is (-0.917, -40.04) and is *a minimum*. The $x-$intercepts are (1.67, 0) and (-3.5, 0).



Problem Set

Graph the quadratic equations using a graphing calculator. Find the vertex and $x-$intercepts, if there are any. If there are no $x-$intercepts, use algebra to find the imaginary solutions. Round all real answers to the nearest hundredth.

1. $y = x^2 - x - 6$
2. $y = -x^2 + 3x + 28$
3. $y = 2x^2 + 11x - 40$
4. $y = x^2 - 6x + 7$
5. $y = x^2 + 8x + 13$
6. $y = x^2 + 6x + 34$
7. $y = 10x^2 - 13x - 3$
8. $y = -4x^2 + 12x - 3$
9. $y = \frac{1}{3}(x-4)^2 + 12$
10. **Calculator Investigation** The *parent graph* of a quadratic equation is $y = x^2$.

    a. Graph $y = x^2, y = 3x^2$, and $y = \frac{1}{2}x^2$ on the same set of axes in the calculator. Describe how a effects the shape of the parabola.
    b. Graph $y = x^2, y = -x^2$, and $y = -2x^2$ on the same set of axes in the calculator. Describe how a effects the shape of the parabola.
    c. Graph $y = x^2, y = (x-1)^2$, and $y = (x+4)^2$ on the same set of axes in the calculator. Describe how h effects the location of the parabola.
    d. Graph $y = x^2, y = x^2 + 2$, and $y = x^2 - 5$ on the same set of axes in the calculator. Describe how k effects the location of the parabola.

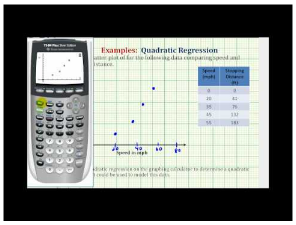
11. The path of a baseball hit by a bat follows a parabola. A batter hits a home run into the stands that can be modeled by the equation $y = -0.003x^2 + 1.3x + 4$, where x is the horizontal distance and y is the height (in feet) of the ball. Find the maximum height of the ball and its total distance travelled.

Modeling with Quadratic Functions

Objective

To find the quadratic equation that fits to a data set.

Watch This



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/60113

James Sousa: Ex: Quadratic Regression on the TI84 - Stopping Distance

Guidance

When finding the equation of a parabola, you can use any of the three forms. If you are given the vertex and any other point, you only need two points to find the equation. However, if you are not given the vertex you must have at least three points to find the equation of a parabola.

Example A

Find the equation of the parabola with vertex (-1, -4) and passes through (2, 8).

Solution: Use vertex form and substitute -1 for h and -4 for k.

$$
\begin{aligned}
y &= a(x-(-1))^2 - 4 \\
y &= a(x+1)^2 - 4
\end{aligned}
$$

Now, take the second point and plug it for x and y and solve for a.

$$
\begin{aligned}
8 &= a(2+4)^2 - 4 \\
12 &= 36a \\
\frac{1}{3} &= a
\end{aligned}
$$

The equation is $y = \frac{1}{3}(x+1)^2 - 4$.

Like in the *Analyzing Scatterplots* lesson, we can also fit a set of data to a quadratic equation. In this concept, we will be using **quadratic regression** and a TI-83/84.

Example B

Determine the quadratic equation of best fit for the data set below.

TABLE 1.6:

x	0	4	7	12	17
y	7	9	10	8	3

Solution: We need to enter the $x-$coordinates as a list of data and the $y-$coordinates as another list.

1. Press STAT.

2. In EDIT, select **1:Edit. . . .** Press ENTER.

3. The List table appears. If there are any current lists, you will need to clear them. To do this, arrow up to L1 so that it is highlighted (black). Press CLEAR, then ENTER. Repeat with L2, if necessary.

4. Now, enter the data into the lists. Enter all the entries into L1 (x) first and press enter between each entry. Then, repeat with L2 and y.

5. Press 2^{nd} MODE (QUIT).

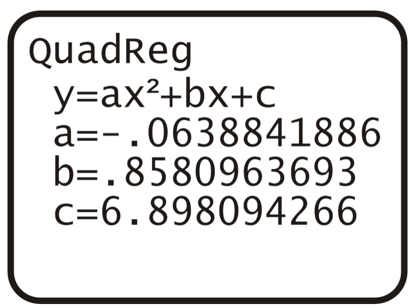
Now that we have everything in the lists, we can use *quadratic* regression to determine the equation of best fit.

6. press STAT and then arrow over to the CALC menu.

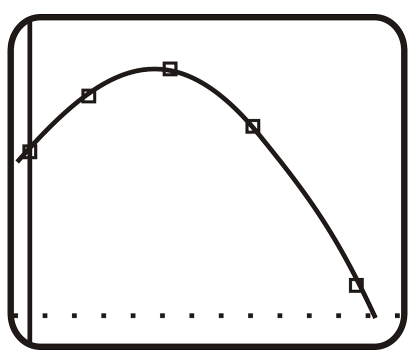
7. Select **5:QuadReg.** Press ENTER.

8. You will be taken back to the main screen. Type **(L1,L2)** and press ENTER. L1 is 2^{nd} 1, L2 is 2^{nd} 2.

9. The following screen appears. The equation of best fit is $y = -0.64x^2 + 0.86x + 6.90$.



If you would like to plot the equation on the scatterplot follow the steps from the *Finding the Equation of Best Fit using a Graphing Calculator* concept. The scatterplot and parabola are to the right.



This technique can be applied to real-life problems. You can also use technique to find the equation of any parabola, given three points.

Example C

Find the equation of the parabola that passes through (1, 11), (2, 20), (-3, 75).

Solution: You can use the same steps from Example B to find the equation of the parabola. Doing this, you should get the equation is $y = 5x^2 - 6x + 12$.

This problem can also be done by solving three equations, with three unknowns. If we plug in (x, y) to $y = ax^2 + bx + c$, we would get:

$$\begin{aligned}11 &= a+b+c\\20 &= 4a+2b+c\\75 &= 9a-3b+c\end{aligned}$$

Use linear combinations to solve this system of equations (see *Solving a System in Three Variables Using Linear Combinations* concept). This problem will be finished in the Problem Set.

Guided Practice

1. Find the equation of the parabola with $x-$intercepts (4, 0) and (-5, 0) that passes through (-3, 8).

2. A study compared the speed, x (in miles per hour), and the average fuel economy, y (in miles per gallon) of a sports car. Here are the results.

TABLE 1.7:

speed	30	40	50	55	60	65	70	80
fuel economy	11.9	16.1	21.1	22.2	25.0	26.1	25.5	23.2

Plot the scatterplot and use your calculator to find the equation of best fit.

Answers

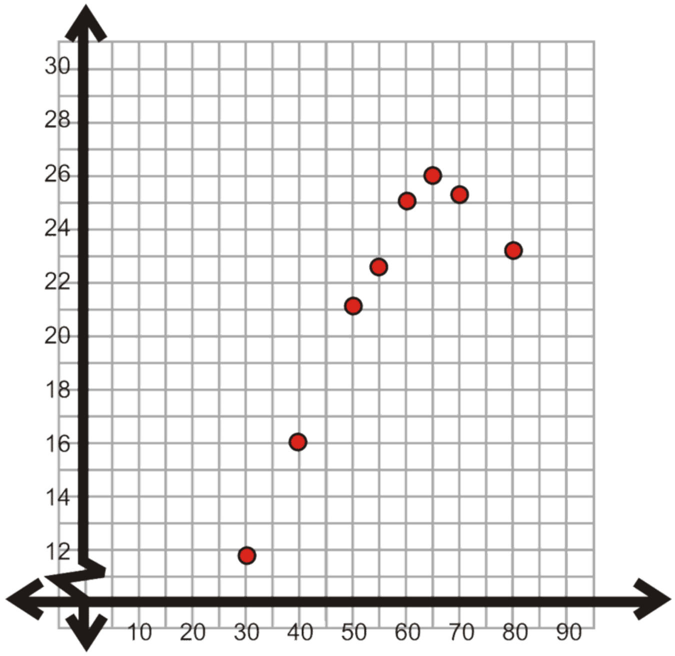
1. Because we are given the intercepts, use intercept form to find the equation.

$y = a(x-4)(x+5)$ Plug in (-3,8) and solve for a

$$\begin{aligned}8 &= a(-3-4)(-3+5)\\8 &= -14a\\-\frac{4}{7} &= a\end{aligned}$$

The equation of the parabola is $y = -\frac{4}{7}(x-4)(x+5)$.

2. Plotting the points, we have:



Using the steps from Example B, the quadratic regression equation is $y = -0.009x^2 + 1.24x - 18.23$.

Vocabulary

Quadratic Regression
The process through which the equation of best fit is a quadratic equation.

Problem Set

Find the equation of the parabola given the following points. No decimal answers.

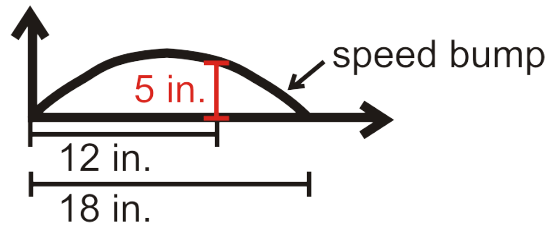
1. vertex: (-1, 1) point: (1, -7)
2. $x-$intercepts: -2, 2 point: (4, 3)
3. vertex: (9, -4) point: (5, 12)
4. $x-$intercepts: 8, -5 point: (3, 20)
5. $x-$intercepts: -9, -7 point: (-3, 36)
6. vertex: (6, 10) point: (2, -38)
7. vertex: (-4, -15) point: (-10, 1)
8. vertex: (0, 2) point: (-4, -12)
9. $x-$intercepts: 3, 16 point: (7, 24)

Use a graphing calculator to find the quadratic equation (in standard form) that passes through the given three points. No decimal answers.

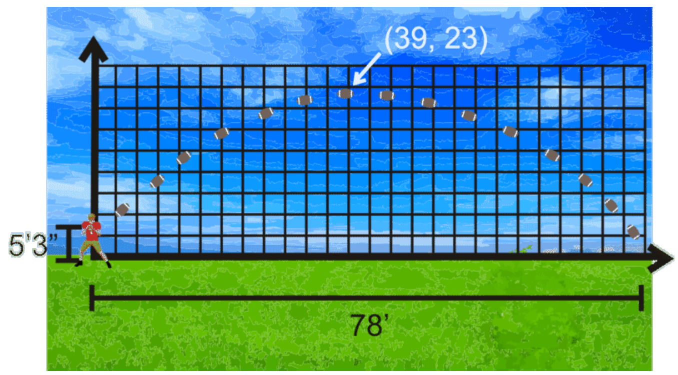
10. (-4, -51), (-1, -18), (4, -43)
11. (-5, 131), (-1, -5), (3, 51)
12. (-2, 9), (2, 13), (6, 41)
13. **Challenge** Finish computing Example C using linear combinations.

For the quadratic modeling questions below, use a graphing calculator. Round any decimal answers to the nearest hundredth.

14. The surface of a speed bump is shaped like a parabola. Write a quadratic model for the surface of the speed bump shown.



15. **Physics and Photography Connection** Your physics teacher gives you a project to analyze parabolic motion. You know that when a person throws a football, the path is a parabola. Using your camera, you take an long exposure picture of a friend throwing a football. A sketch of the picture is below.



You put the path of the football over a grid, with the $x-$axis as the horizontal distance and the $y-$axis as the height, both in 3 feet increments. The release point, or shoulder height, of your friend is 5 ft, 3 in and you estimate that the maximum height is 23 feet. Find the equation of the parabola.

16. An independent study was done linking advertising to the purchase of an object. 400 households were used in the survey and the commercial exposure was over a one week period. See the data set below.

TABLE 1.8:

# of times commercial was shown, x	1	7	14	21	28	35	42	49
# of households bought item, y	2	25	96	138	88	37	8	6

a) Find the quadratic equation of best fit.

b) Why do you think the amount of homes that purchased the item went down after more exposure to the commercial?

1.11 Graphing Absolute Value Functions

Objective

To understand and graph a basic absolute value function.

Review Queue

Solve the following equations.

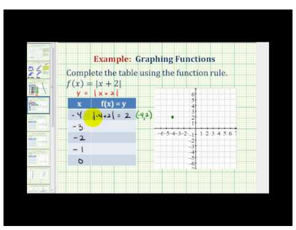
1. $|x-1| = 6$
2. $|3x+5| = 16$
3. $2\left|\frac{1}{4}x-3\right|+9 = 23$

Graphing Basic Absolute Value Functions

Objective

To learn about the basic properties of absolute value functions.

Watch This



MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/60091

James Sousa: Ex: Graph an Absolute Value Function Using a Table of Values

Guidance

In the *Solving Absolute Value Equations* concept, we learned how to solve and define absolute value equations. We will now take this idea one step further and graph absolute value equations.

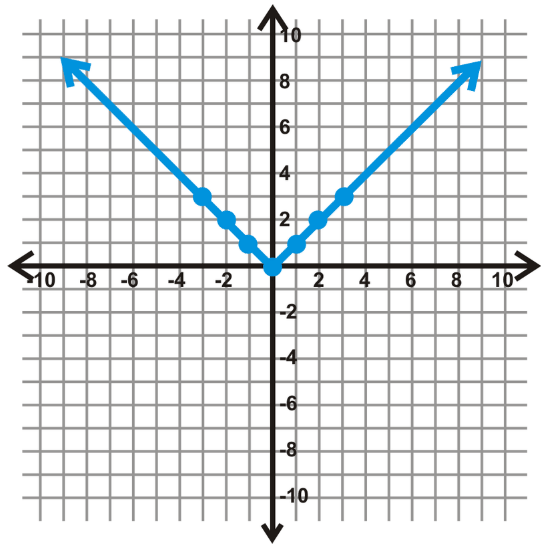
Investigation: Graphing the Parent Graph of an Absolute Value Function

1. We are going to graph $y = |x|$. Draw a table for x and y, with the $x-$values ranging from -3 to 3.

TABLE 1.9:

x	$\|x\|$	y
-3	$\|-3\|$	3
-2	$\|-2\|$	2
-1	$\|-1\|$	1
0	$\|0\|$	0
1	$\|1\|$	1
2	$\|2\|$	2
3	$\|3\|$	3

2. Recall that the absolute value of a number is always positive. Now that you have 7 sets of points, plot each one and graph the function.



3. Notice that this function is very similar to the linear function, $y = x$. Draw this line on the graph in a different color or with a dashed line.

4. Now, fold your graph on the $x-$axis. What do you notice?

In the investigation, you should discover that when you fold your graph on the $x-$axis, the line $y = x$ becomes the absolute value equation, $y = |x|$. That is because the absolute value of a number can never be zero; therefore, the range will always be positive. We call $y = |x|$ the **parent graph** because it is the most basic of all the absolute value functions. We will also compare other absolute value functions to this graph. All linear absolute value functions have this "V" shape.

In general, we can define the graph of $y = |x|$ as $y = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$. From this, we see that each side, is the mirror image of the other over a vertical line, through the vertex.

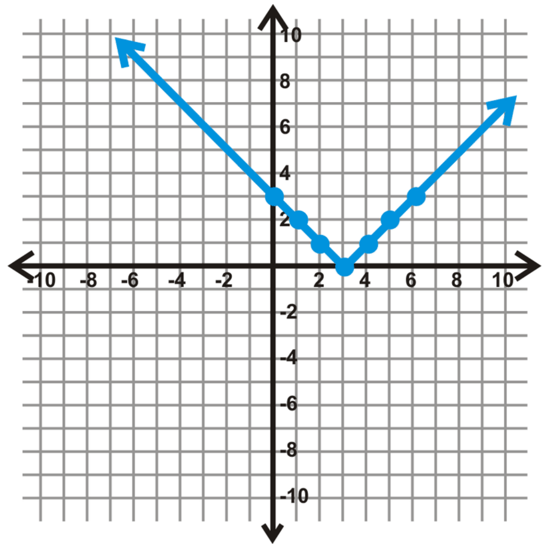
Example A

Use a table to graph $y = |x-3|$. Determine the domain and range.

Solution: In general, when you use a table to graph a function, pick some positive and negative numbers, as well as zero. Use the equation to help you determine which $x-$values to pick. Setting what is inside the absolute value equal to zero, we get that $x = 3$. Pick three values on either side of $x = 3$ and then graph.

TABLE 1.10:

x	$\|x-3\|$	y
0	$\|-3\|$	3
1	$\|-2\|$	2
2	$\|-1\|$	1
3	$\|0\|$	0
4	$\|1\|$	1
5	$\|2\|$	2
6	$\|3\|$	3



Notice that this graph shifts to the right 3 when compared to the parent graph. The domain will be all real numbers, $x \in \mathbb{R}$, and the range will be all positive real numbers, including zero, $y \in [0, \infty)$.

Example B

Use a table to graph $y = |x| - 5$. Determine the domain and range.

Solution: Be careful! Here, the minus 5 is not inside the absolute value. So, first take the absolute value of the $x-$value and then subtract 5. In cases like these, the range can include negative numbers.

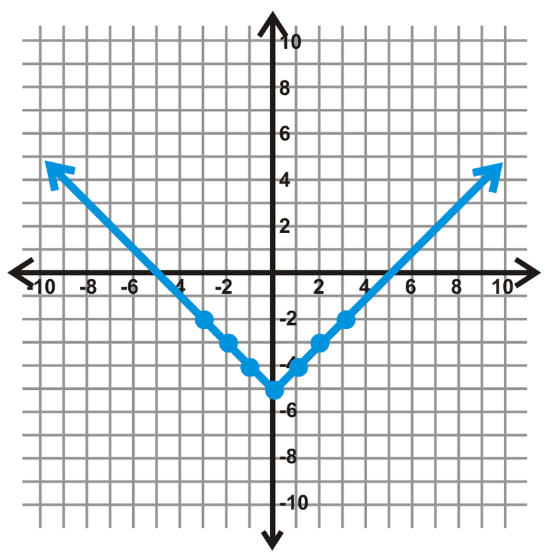


TABLE 1.11:

x	$\|x\| - 5$	y
-3	$\|-3\| - 5$	-2
-2	$\|-2\| - 5$	-3
-1	$\|-1\| - 5$	-4
0	$\|0\| - 5$	-5
1	$\|1\| - 5$	-4
2	$\|2\| - 5$	-3
3	$\|3\| - 5$	-2

Here, the graph shifts down 5 when compared to the parent graph. The domain will be all real numbers, $x \in \mathbb{R}$, and the range will be all real numbers greater than or equal to -5, $y \in [-5, \infty)$.

In these three absolute value graphs, you may have noticed that there is a **minimum** point. This point is called the **vertex.** For example, in Example B, the vertex is (0, -5). The vertex can also be a **maximum.** See the next example.

Example C

Use a table to graph $y = -|x-1|+2$. Determine the vertex, domain, and range.

Solution: Determine what makes the inside of the absolute value equation zero, $x = 1$. Then, to make your table of values, pick a couple values on either side of $x = 1$.

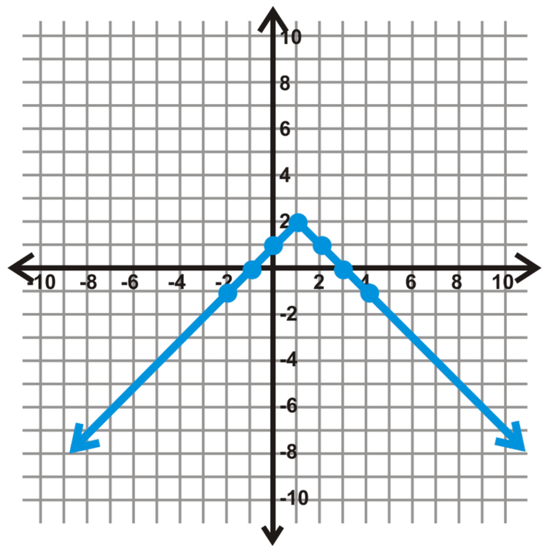


TABLE 1.12:

x	$-\|x-1\|+2$	y
-2	$-\|-2-1\|+2$	-1
-1	$-\|-1-1\|+2$	0
0	$-\|0-1\|+2$	1
1	$-\|1-1\|+2$	2
2	$-\|2-1\|+2$	1
3	$-\|3-1\|+2$	0
4	$-\|4-1\|+2$	-1

The vertex is (1, 2) and in this case, it is the maximum value. The domain is $x \in \mathbb{R}$, and the range is $y \in (-\infty, 2]$.

Vocabulary

Absolute Value
The distance away from zero a number is. The absolute value is always positive.

Parent Graph
The simplest form of a particular type of function. All other functions of this type are usually compared to the parent graph.

Vertex
The highest or lowest point of a graph.

Minimum
The lowest point of a graph. The minimum will yield the smallest value of the range.

Maximum
The highest point of a graph. The maximum will yield the largest value of the range.

Guided Practice

Graph the following functions using a table. Determine the vertex, domain, and range of each function.

1. $y = -|x-5|$

2. $y = |x+4|-2$

Answers

1. Determine what makes the inside of the absolute value equation zero, $x = 5$. Then, to make your table of values, pick a couple values on either side of $x = 5$.

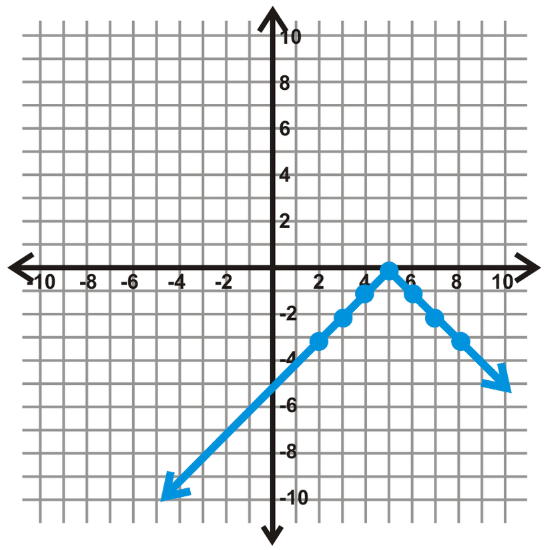


TABLE 1.13:

x	$-\|x-5\|$	y
2	$-\|2-5\|$	-3
3	$-\|3-5\|$	-2
-4	$-\|4-5\|$	-1
5	$-\|5-5\|$	0
6	$-\|6-5\|$	-1
7	$-\|7-5\|$	-2
8	$-\|8-5\|$	-3

The vertex is (5, 0) and in this case, it is the maximum value. The domain is $x \in \mathbb{R}$, and the range is $y \in (-\infty, 0]$.

2. Determine what makes the inside of the absolute value equation zero, $x = -4$. Then, to make your table of values, pick a couple values on either side of $x = -4$.

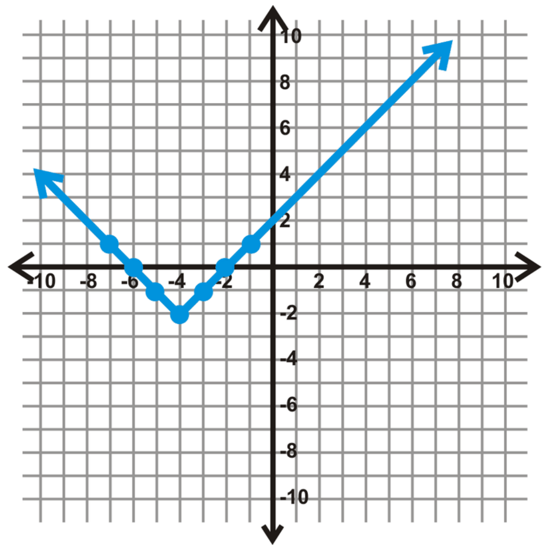


TABLE 1.14:

x	$\lvert x+4\rvert-2$	y
-1	$\lvert -1+4\rvert-2$	1
-2	$\lvert -2+4\rvert-2$	0
-3	$\lvert -3+4\rvert-2$	-1
-4	$\lvert -4+4\rvert-2$	-2
-5	$\lvert -5+4\rvert-2$	-1
-6	$\lvert -6+4\rvert-2$	0
-7	$\lvert -7+4\rvert-2$	1

The vertex is (-4, -2) and in this case, it is the minimum value. The domain is $x \in \mathbb{R}$, and the range is $y \in [-2, \infty)$.

Problem Set

Graph the following functions using a table. Determine the vertex, domain, and range of each function.

1. $y = |x+6|$
2. $y = |x-4|$
3. $y = -|x|+3$
4. $y = |x|-2$
5. $y = -|x+3|+7$
6. $y = |x-1|-6$
7. $y = 2|x|$
8. $y = -3|x|$
9. $y = \frac{1}{3}|x|$

Use problems 1-9 to answer fill in the blanks.

10. If there is a negative sign in front of the absolute value, the graph is ________________ (when compared to the parent graph).
11. If the equation is $y = |x-h|+k$, the vertex will be ___________________.
12. The domain of an absolute value function is always _____________________________.
13. For $y = a|x|$, if $a > 1$, then the graph will be ___________________ than the parent graph.
14. For $y = a|x|$, if $0 < a < 1$, then the graph will be ___________________ than the parent graph.
15. Without making a table, what is the vertex of $y = |x-9|+7$?

Using the General Form and the Graphing Calculator

Objective

To graph more complicated absolute value functions and use the graphing calculator.

Guidance

In the problem set of the previous concept, we were introduced to the general equation of an absolute value function. Let's formally define it here.

General Form of an Absolute Value Function: For any absolute value function, the general form is $y = a|x-h|+k$, where a controls the width of the "V" and (h,k) is the vertex.

You probably made these connections during the problem set from the previous concept. Now, we will put it all to use together.

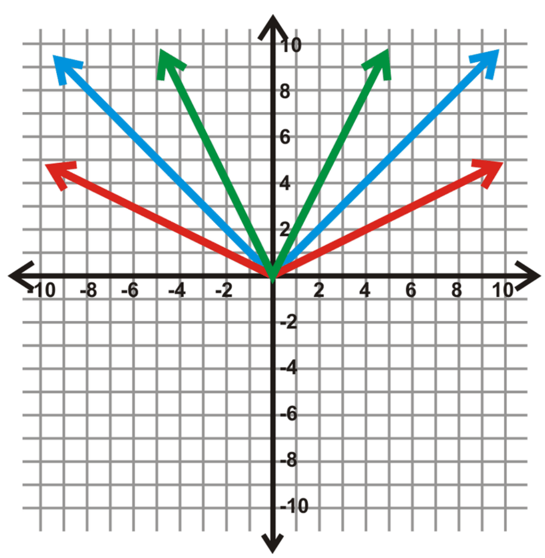
Example A

Graph $y = |x|$, $y = \frac{1}{2}|x|$, and $y = 2|x|$ on the same set of axes. Compare the three functions.

Solution: You can make a table for all three of these functions. However, now that we have a better understanding of absolute value functions, let's use some patterns. First, look at the vertex. Nothing is being added or subtracted, so the vertex for all three will be (0, 0). Second, look at "a." For an absolute value function, we can think of a like the slope. Referring back to the definition of the parent graph, each function above can be rewritten as:

$$y = \begin{cases} x; x \geq 0 \\ -x; x < 0 \end{cases} \quad \text{(blue)}, \quad y = \begin{cases} \frac{1}{2}x; x \geq 0 \\ -\frac{1}{2}x; x < 0 \end{cases} \quad \text{(red), and } y = \begin{cases} 2x; x \geq 0 \\ -2x; x < 0 \end{cases} \quad \text{(green)}$$

Comparing the three, we see that if the slope is between 1 and 0, the opening is wider than the parent graph. If the slope, or a, is greater than 1, then that opening is narrower. The amount of the opening between the two sides of an absolute value function (and other functions) is called the **breadth.**

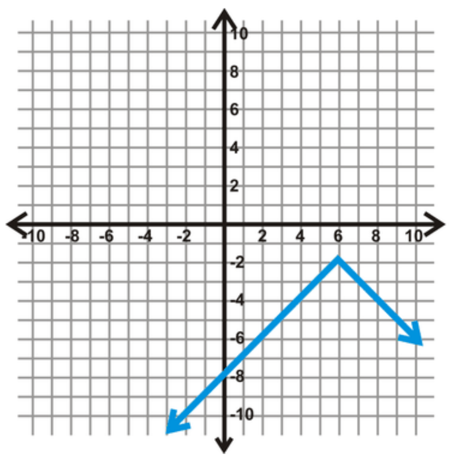


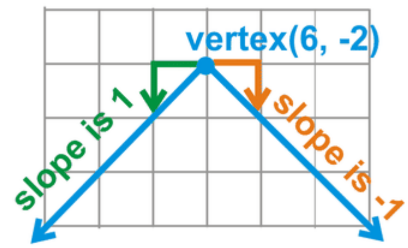
Now, in addition to drawing a table, we can use the general form of an absolute value equation and the value of a to find the shape of the V.

Example B

Without making a table, sketch the graph of $y=-|x-6|-2$.

Solution: First, determine the vertex. From the general form, we know that it will be (6, -2). Notice that the $x-$ *variable is the opposite sign* of what is in the equation; the $y-$variable is the same. That is our starting point. Then, we have a negative sign in front of the absolute value. This means our V will open down. Finally, there is no a term, so we can assume it is 1, meaning that the slope of each side of the V will be 1 and -1.





Lastly, we can use a graphing calculator to help us graph absolute value equations. The directions given here pertain to the TI-83/84 series; however every graphing calculator should be able to graph absolute value functions.

Example C

Use a graphing calculator to graph $y=|4x+1|-2$. Find the vertex, domain, and range.

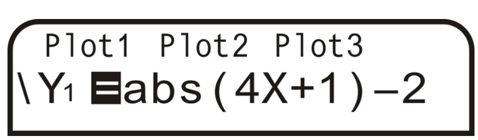
Solution: *For the TI-83/84*

1. Press the $Y=$ button.

2. Clear any previous functions (press CLEAR) and turn off any previous plots (arrow up to Plot 1 and press ENTER).

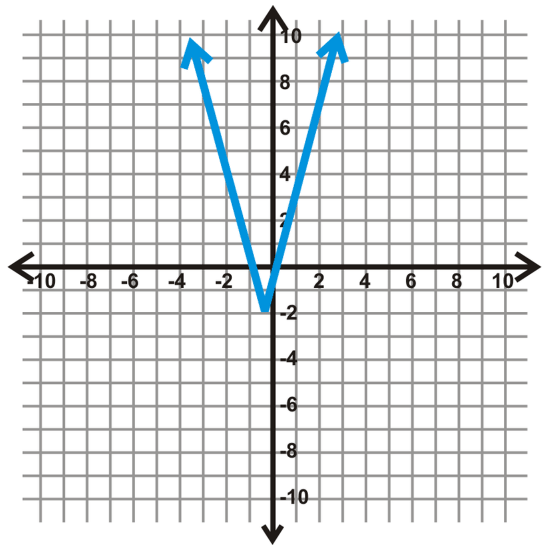
3. Press the MATH button, arrow over to NUM and highlight **1:abs(**. Press ENTER.

4. Type in the remaining portion of the function. The screen:



5. Press GRAPH. If your screen is off, press ZOOM, scroll down to **6:ZStandard**, and press ENTER.

The graph looks like:



As you can see from the graph, the vertex is not (-1, -2). The $y-$coordinate is -2, but the 4 inside the absolute value affects the $x-$ coordinate. Set what is inside the absolute value equal to zero to solve for the $x-$coordinate of the vertex.

$$\begin{aligned} 4x+1&=0\\ 4x&=-1\\ x&=-\frac{1}{4} \end{aligned}$$

The vertex is $\left(-\frac{1}{4},-2\right)$. From the previous concept, we know that the domain is all real numbers. The range will be any number greater than and including -2. In this function, the "a" term was inside the absolute value. When this happens, it will always affect the $x-$coordinate of the vertex.

Vocabulary

General Form of an Absolute Value Function
For any absolute value function, the general form is $y=a|x-h|+k$, where a controls the width of the "V" and (h,k) is the vertex.

Breadth
The wideness or narrowness of a function with two symmetric sides.

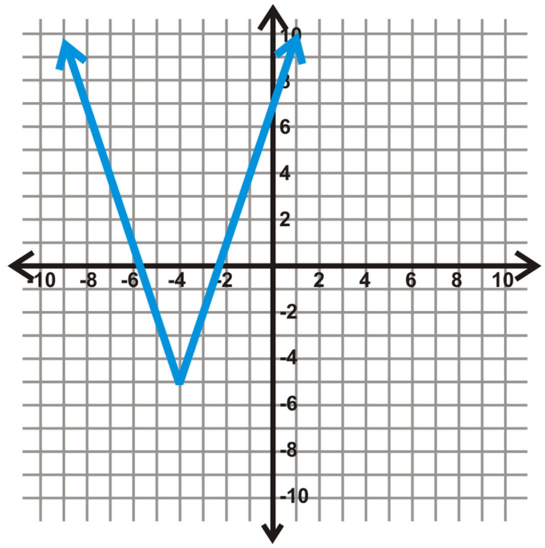
Guided Practice

1. Graph $y=3|x+4|-5$ without a graphing calculator or making a table. Find the vertex, domain, and range of the function.

2. Graph $y=-2|x-5|+1$ using a graphing calculator.

Answers

1. First, use the general form to find the vertex, (-4, -5). Then, use a to determine the breadth of the function. $a=3$, so we will move up 3 and over 1 *in both directions* to find the points on either side of the vertex.

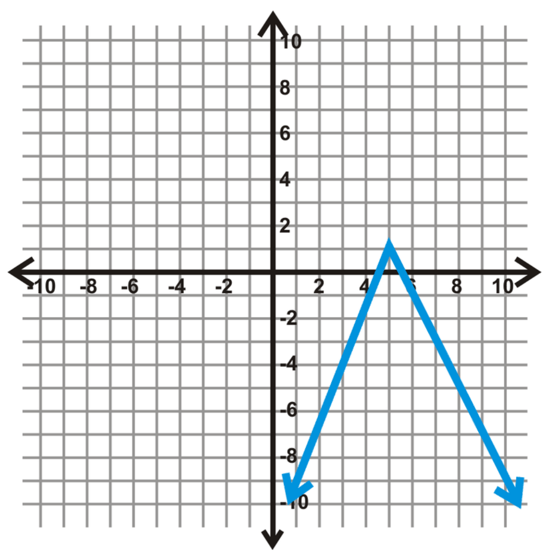


The domain is all real numbers and the range is all reals greater than and including -5.

Domain: $x \in \mathbb{R}$

Range: $y \in [-5, \infty)$

2. Using the steps from Example C, the function looks like:



Problem Set

1. Graph $y = 3|x|$, $y = -3|x|$, and $y = |-3x|$ on the same set of axes. Compare the graphs.
2. Graph $y = \frac{1}{4}|x+1|$, and $y = \frac{1}{4}|x|+1$ on the same set of axes. Compare the graphs.
3. Without graphing, do you think that $y = 2|x|, y = |2x|$, and $y = |-2x|$ will all produce the same graph? Why or why not?
4. We know that the domain of all absolute value functions is all real numbers. What would be a general rule for the range?

Use the general form and pattern recognition to graph the following functions. Determine the vertex, domain, and range. No graphing calculators!

5. $y = |x-2|+5$

6. $y = -2|x+3|$
7. $y = \frac{1}{3}|x| + 4$
8. $y = 2|x+1| - 2$
9. $y = -\frac{1}{2}|x-7|$
10. $y = -|x-8| + 6$

Use a graphing calculator to graph the following functions. Sketch a copy of the graph on your paper. Identify the vertex, domain, and range.

11. $y = -4|2x+1|$
12. $y = \frac{2}{3}|x-4| + \frac{1}{2}$
13. $y = \frac{4}{3}|2x-3| - \frac{7}{2}$

Graphing Calculator Extension Use the graphing calculator to answer questions 14-16.

14. Graph $y = x^2 - 4$ on your calculator. Sketch the graph and determine the domain and range.
15. Graph $y = |x^2 - 4|$ on your calculator. Sketch graph and determine the domain and range.
16. How do the two graphs compare? How are they different? What could you do to the first graph to get the second?

1.12 Solving Linear Systems by Graphing

Objective

Review the concept of the solution to a linear system in the context of a graph.

Review Queue

1. Graph the equation $y = \frac{1}{2}x + 3$.
2. Write the equation $4x - 3y = 6$ in slope intercept form.
3. Solve for x in $7x - 3y = 26$, given that $y = 3$.

Checking a Solution for a Linear System

Objective

Determine whether an ordered pair is a solution to a given system of linear equations.

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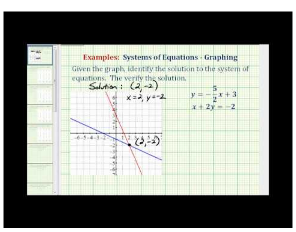

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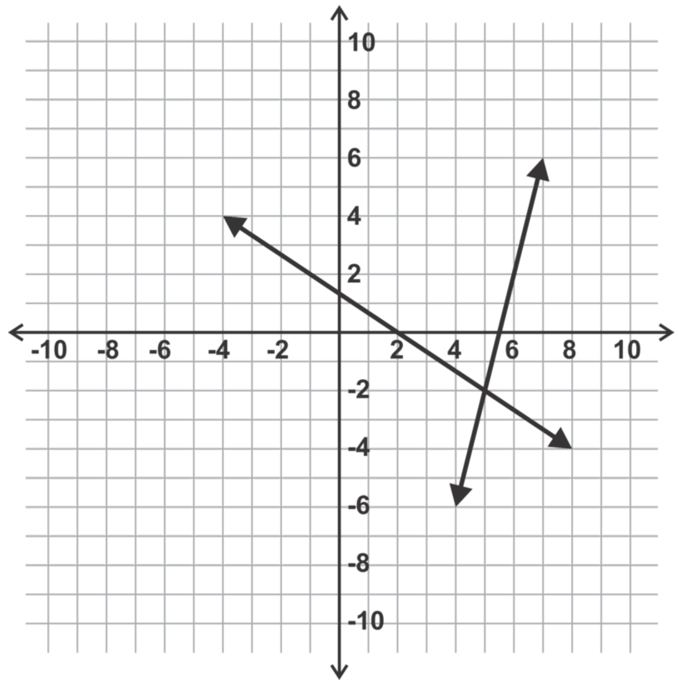
James Sousa: Ex: Identify the Solution to a System of Equations Given a Graph, Then Verify

Guidance

A system of linear equations consists of the equations of two lines.The solution to a system of linear equations is the point which lies on both lines. In other words, the solution is the point where the two lines intersect. To verify whether a point is a solution to a system or not, we will either determine whether it is the point of intersection of two lines on a graph (Example A) or we will determine whether or not the point lies on both lines algebraically (Example B.)

Example A

Is the point (5, -2) the solution of the system of linear equations shown in the graph below?



Solution: Yes, the lines intersect at the point (5, -2) so it is the solution to the system.

Example B

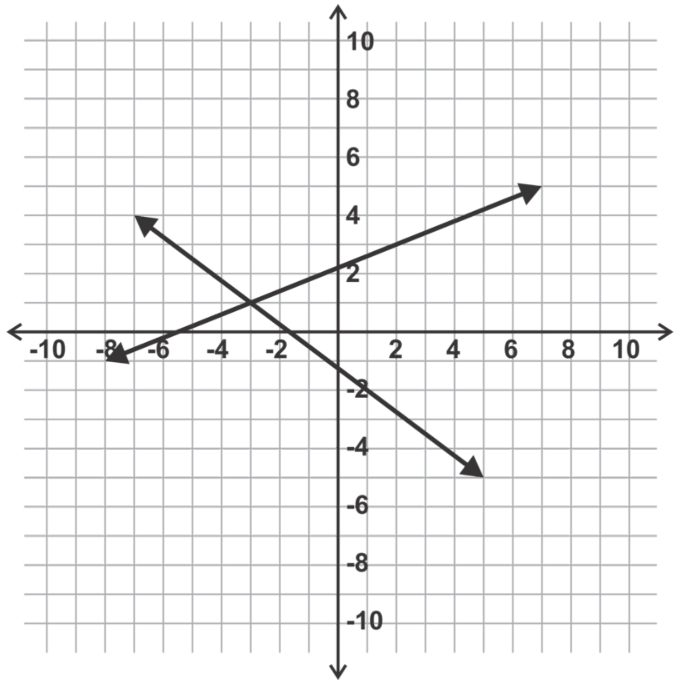
Is the point (-3, 4) the solution to the system given below?

$$\begin{aligned} 2x - 3y &= -18 \\ x + 2y &= 6 \end{aligned}$$

Solution: No, (-3, 4) is not the solution. If we replace the x and y in each equation with -3 and 4 respectively, only the first equation is true. The point is not on the second line; therefore it is not the solution to the system.

Guided Practice

1. Is the point (-2, 1) the solution to the system shown below?



2. Verify algebraically that (6, -1) is the solution to the system shown below.

$$3x - 4y = 22$$
$$2x + 5y = 7$$

3. Explain why the point (3, 7) is the solution to the system:

$$y = 7$$
$$x = 3$$

Answers

1. No, (-2, 1) is not the solution. The solution is where the two lines intersect which is the point (-3, 1).

2. By replacing x and y in both equations with 6 and -1 respectively (shown below), we can verify that the point (6, -1) satisfies both equations and thus lies on both lines.

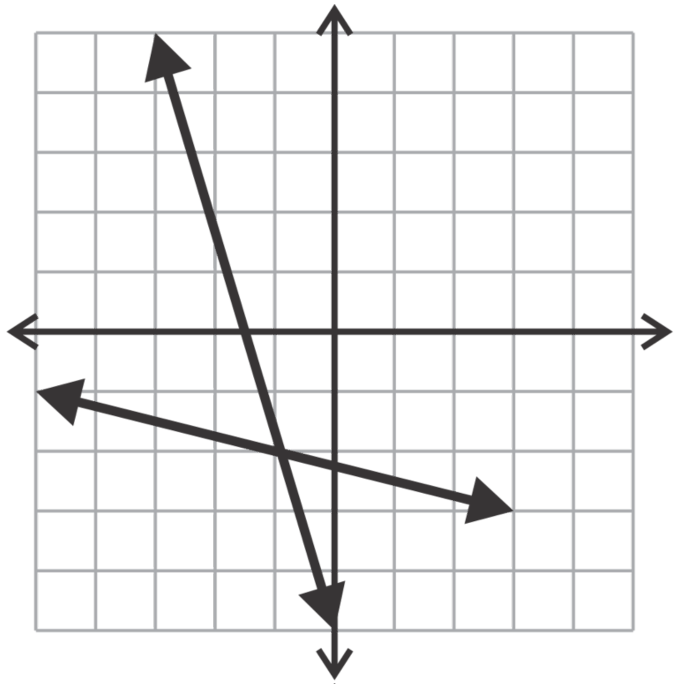
$$3(6) - 4(-1) = 18 + 4 = 22$$
$$2(6) + 5(-1) = 12 - 5 = 7$$

3. The horizontal line is the line containing all points where the y-coordinate is 7. The vertical line is the line containing all points where the $x-$coordinate is 3. Thus, the point (3, 7) lies on both lines and is the solution to the system.

Problem Set

Match the solutions with their systems.

1. (1, 2)



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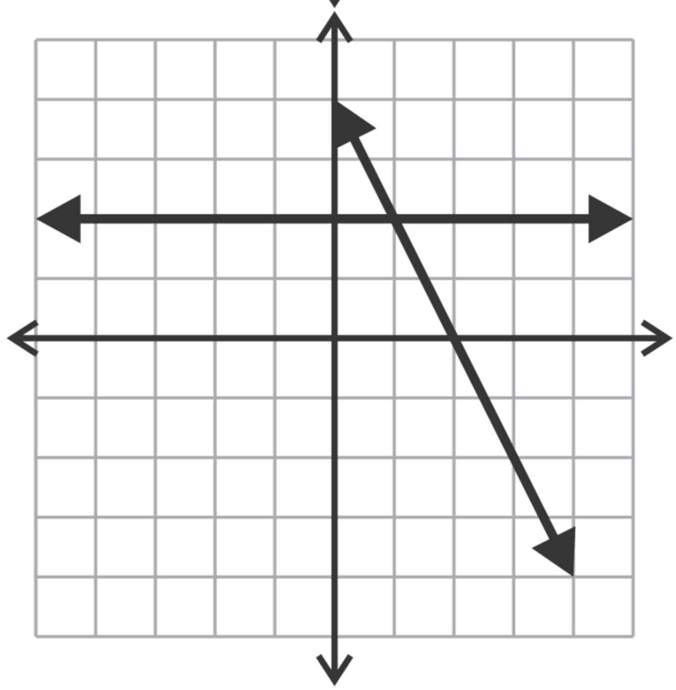

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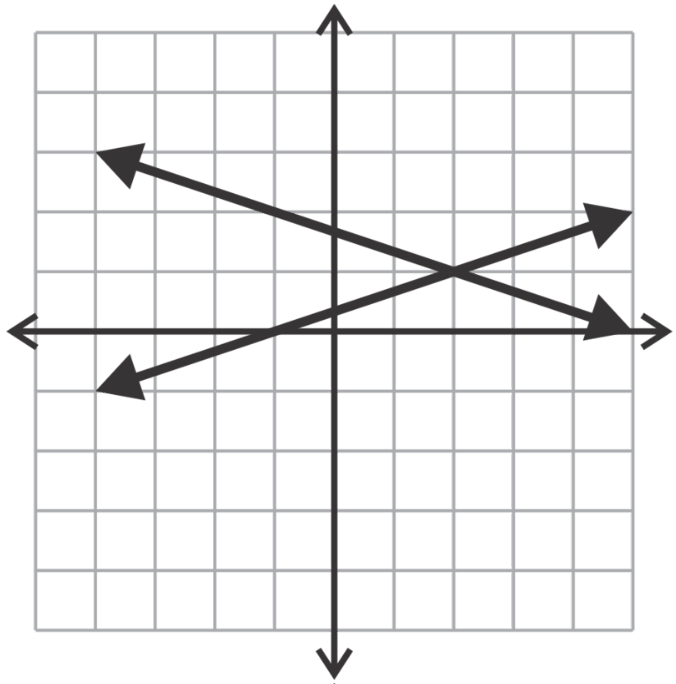

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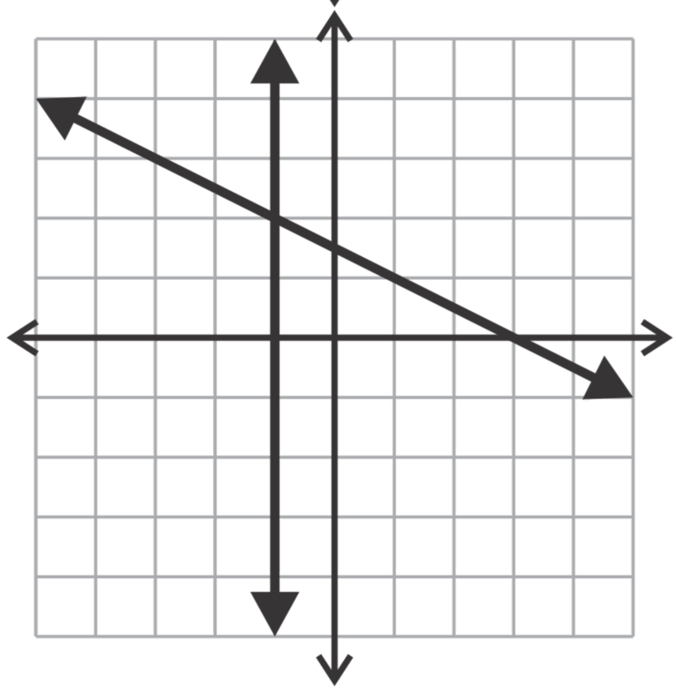
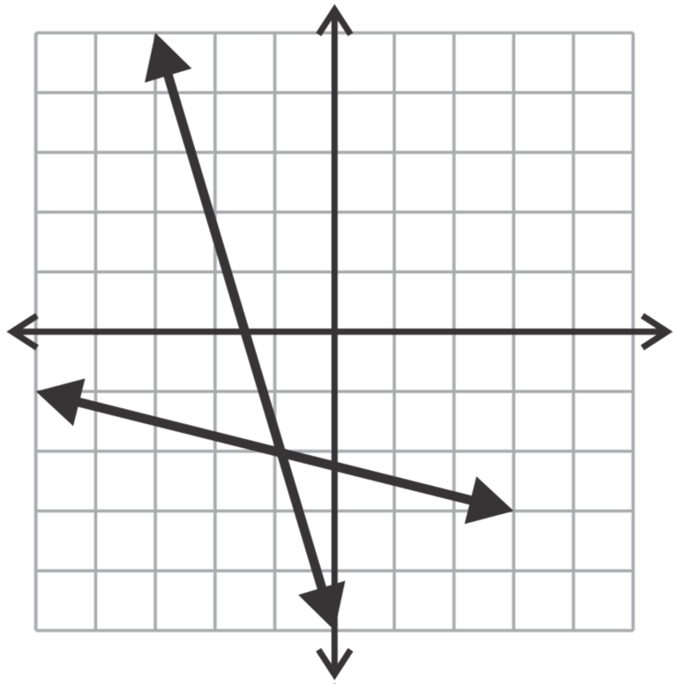


d.

2. (2, 1)



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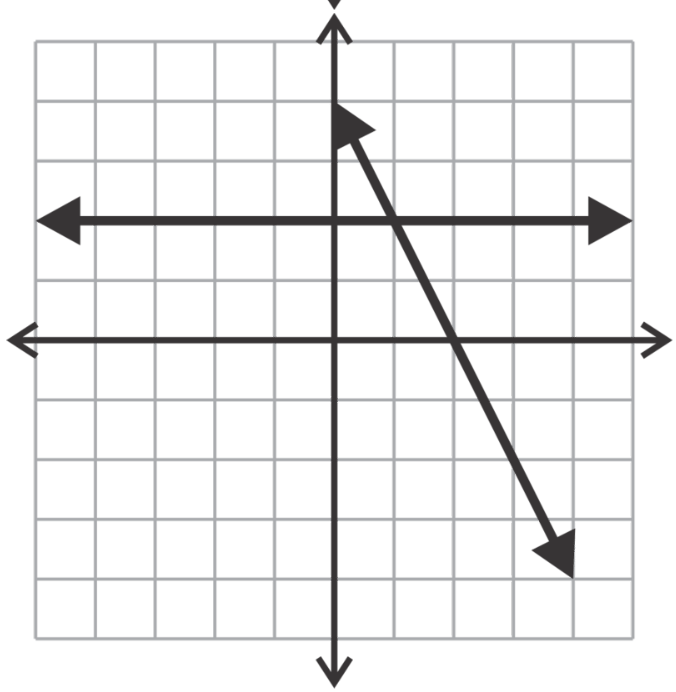

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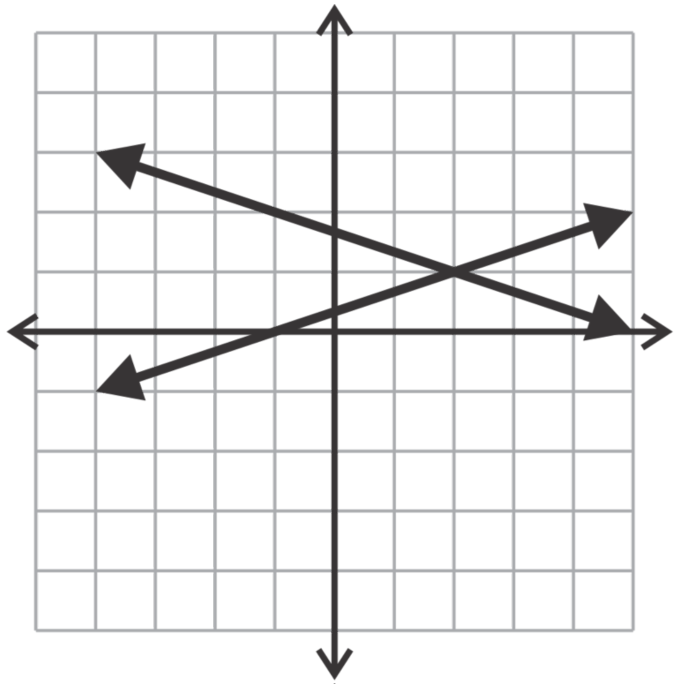

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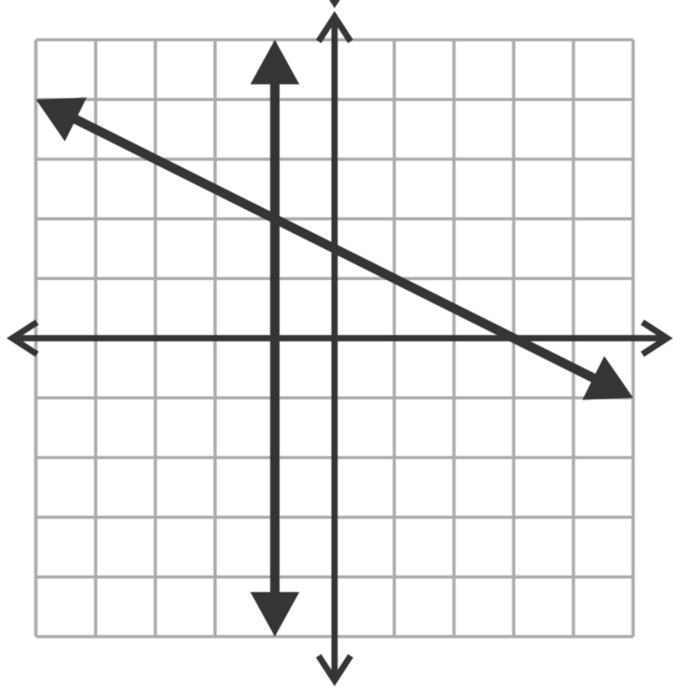
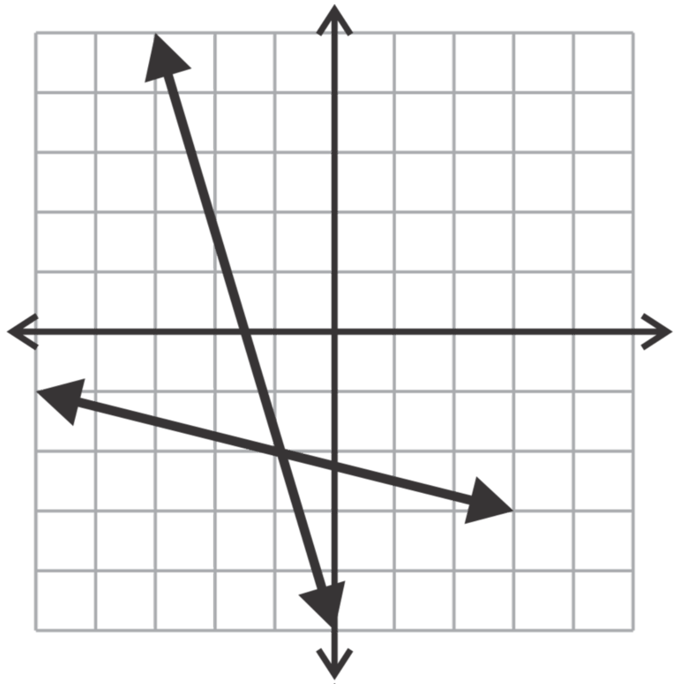


d.

3. (-1, 2)



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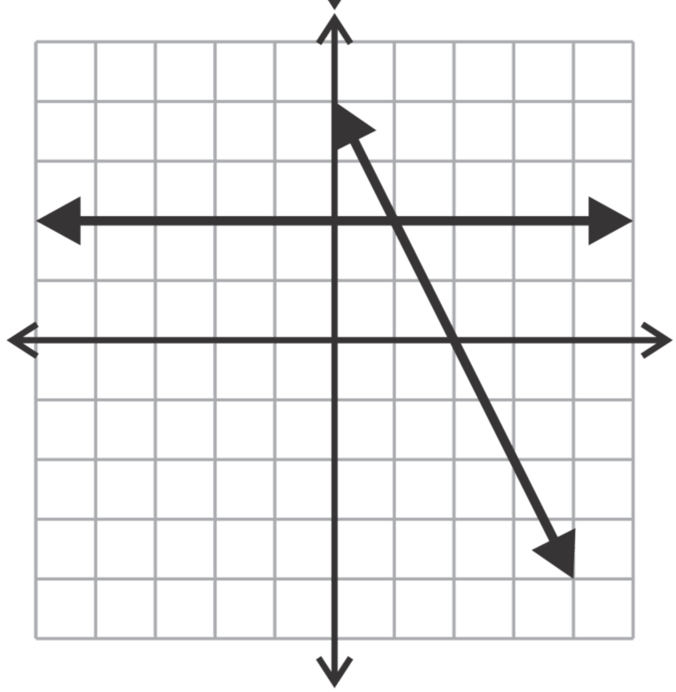

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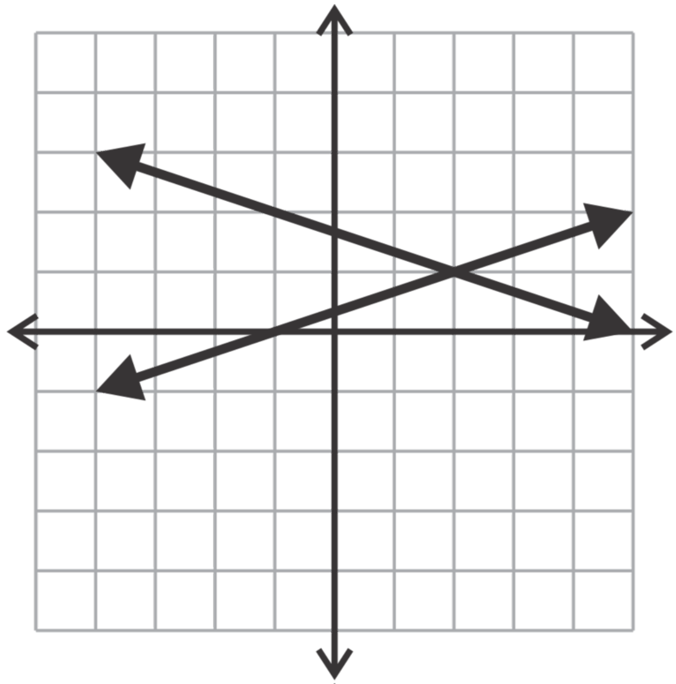

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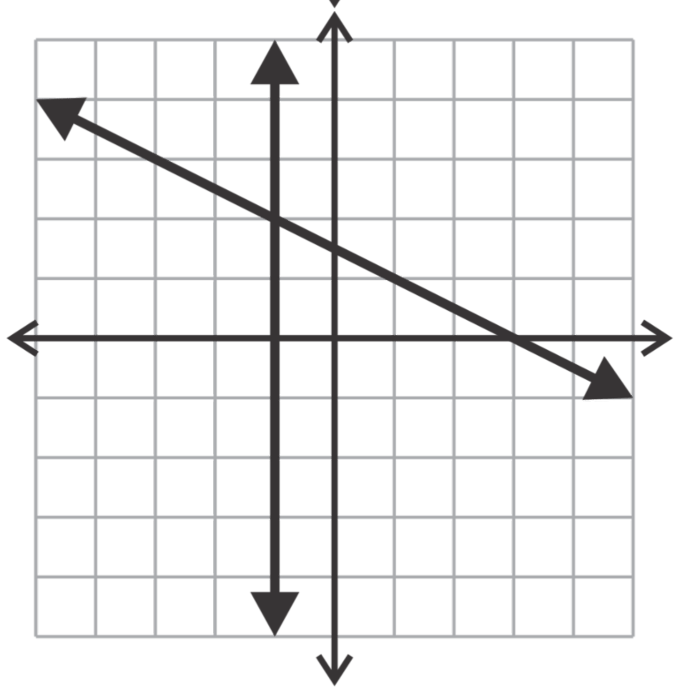
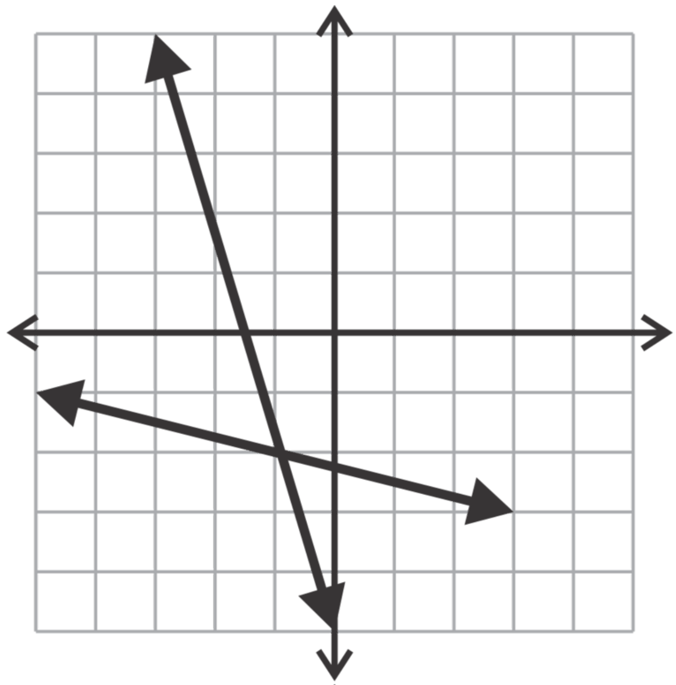


d.

4. (-1, -2)



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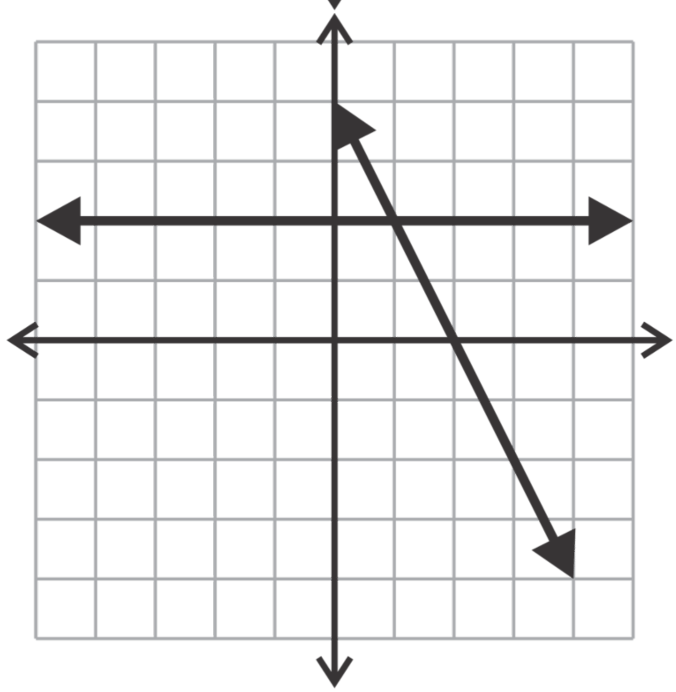

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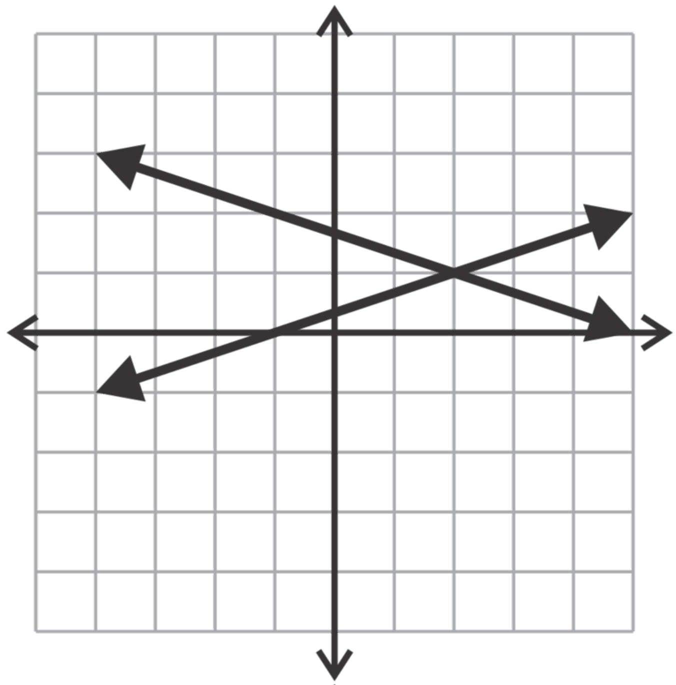

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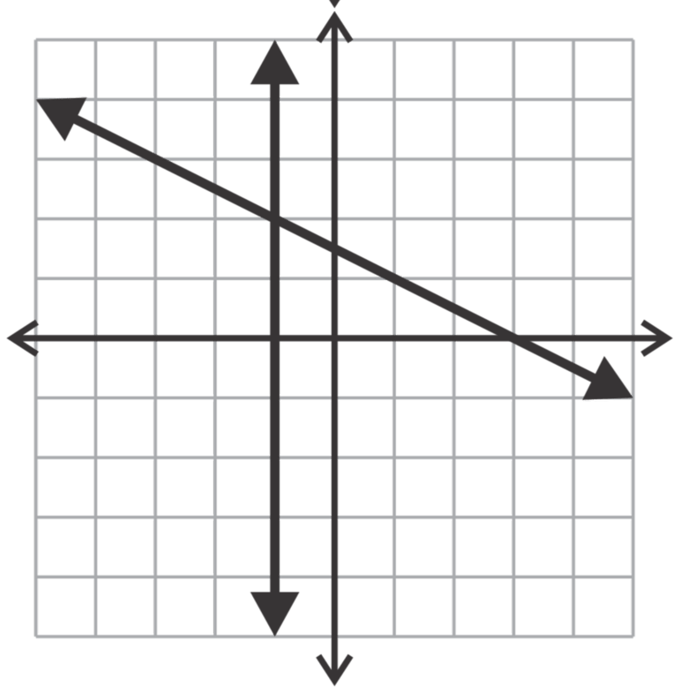


d.

Determine whether each ordered pair represents the solution to the given system.

5.

$$\begin{aligned} 4x+3y&=12\\ 5x+2y&=1;\ (-3,8) \end{aligned}$$

6.

$$\begin{aligned} 3x-y&=17\\ 2x+3y&=5;\ (5,-2) \end{aligned}$$

7.

$$\begin{aligned} 7x - 9y &= 7 \\ x + y &= 1; \ (1,0) \end{aligned}$$

8.

$$\begin{aligned} x + y &= -4 \\ x - y &= 4; \ (5,-9) \end{aligned}$$

9.

$$\begin{aligned} x &= 11 \\ y &= 10; \ (11,10) \end{aligned}$$

10.

$$\begin{aligned} x + 3y &= 0 \\ y &= -5; \ (15,-5) \end{aligned}$$

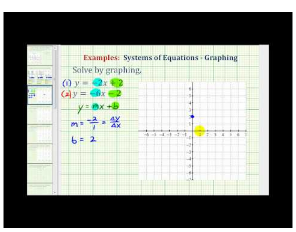
11. Describe the solution to a system of linear equations.
12. Can you think of why a linear system of two equations would not have a unique solution?

Solving Systems with One Solution Using Graphing

Objective

Graph lines to identify the unique solution to a system of linear equations.

Watch This



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James Sousa: Ex 1: Solve a System of Equations by Graphing

Guidance

In this lesson we will be using various techniques to graph the pairs of lines in systems of linear equations to identify the point of intersection or the solution to the system. It is important to use graph paper and a straightedge to graph the lines accurately. Also, you are encouraged to check your answer algebraically as described in the previous lesson.

Example A

Graph and solve the system:

$$y = -x + 1$$
$$y = \frac{1}{2}x - 2$$

Solution:

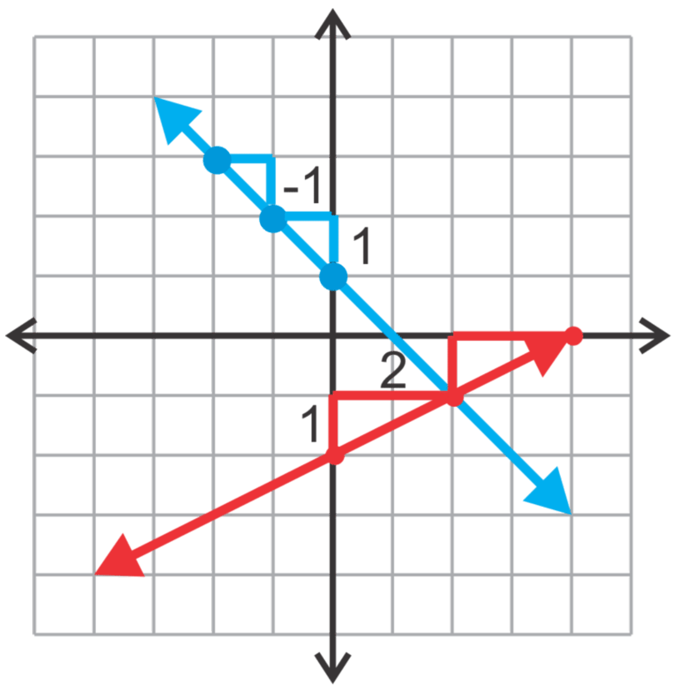
Since both of these equations are written in slope intercept form, we can graph them easily by plotting the $y-$intercept point and using the slope to locate additional points on each line.

The equation, $y = -x+1$, graphed in **blue**, has $y-$intercept 1 and slope $-\frac{1}{1}$.

The equation, $y = \frac{1}{2}x - 2$, graphed in **red**, has $y-$intercept -2 and slope $\frac{1}{2}$.

Now that both lines have been graphed, the intersection is observed to be the point (2, -1).

Check this solution algebraically by substituting the point into both equations.



Equation 1: $y = -x+1$, making the substitution gives: $(-1) = (-2)+1$. $\boxed{\checkmark}$

Equation 2: $y = \frac{1}{2}x - 2$, making the substitution gives: $-1 = \frac{1}{2}(2) - 2$. $\boxed{\checkmark}$

(2, -1) is the solution to the system.

Example B

Graph and solve the system:

$$3x + 2y = 6$$
$$y = -\frac{1}{2}x - 1$$

Solution: This example is very similar to the first example. The only difference is that equation 1 is not in slope intercept form. We can either solve for y to put it in slope intercept form or we can use the intercepts to graph the equation. To review using intercepts to graph lines, we will use the latter method.

Recall that the $x-$intercept can be found by replacing y with zero and solving for x:

$$\begin{aligned}3x+2(0)&=6\\3x&=6\\x&=2\end{aligned}$$

Similarly, the $y-$intercept is found by replacing x with zero and solving for y:

$$\begin{aligned}3(0)+2y&=6\\2y&=6\\y&=3\end{aligned}$$

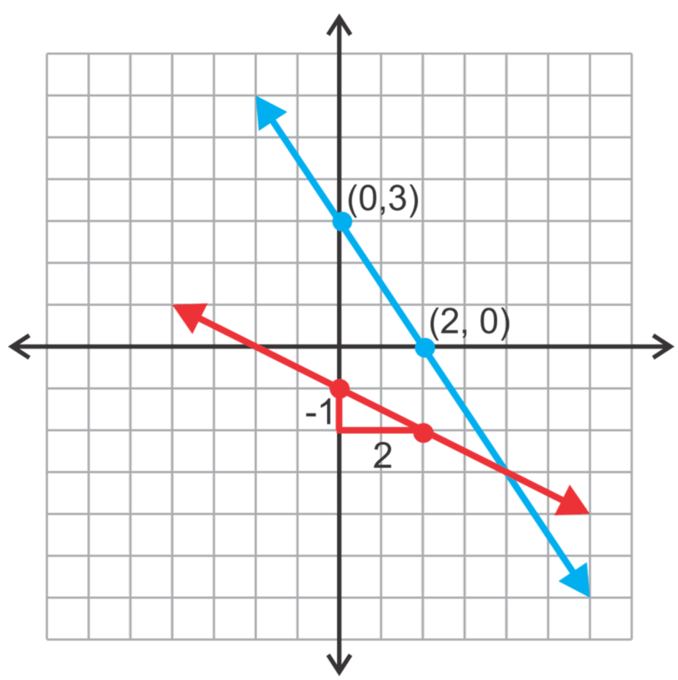
We have two points, (2, 0) and (0, 3) to plot and graph this line. Equation 2 can be graphed using the $y-$intercept and slope as shown in Example A.

Now that both lines are graphed we observe that their intersection is the point (4, -3).

Finally, check this solution by substituting it into each of the two equations.

Equation 1: $3x+2y=6; 3(4)+2(-3)=12-6=6$ $\boxed{\checkmark}$

Equation 2: $y=-\frac{1}{2}x-1; -3=-\frac{1}{2}(4)-1$ $\boxed{\checkmark}$



Example C

In this example we will use technology to solve the system:

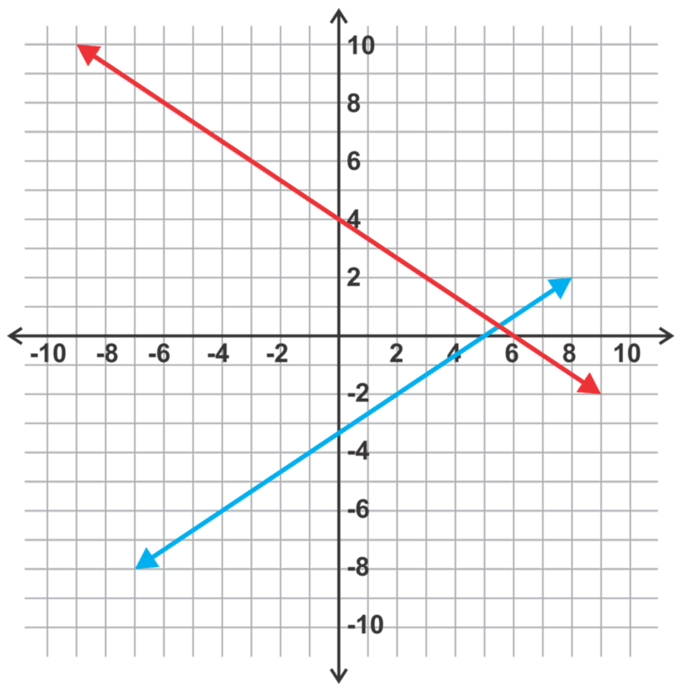
$$\begin{aligned}2x-3y&=10\\y&=-\frac{2}{3}x+4\end{aligned}$$

This process may vary somewhat based on the technology you use. All directions here can be applied to the TI-83 or 84 (plus, silver, etc) calculators.

Solution: The first step is to graph these equations on the calculator. The first equation must be rearranged into slope intercept form to put in the calculator.

$$\begin{aligned}2x-3y&=10\\-3y&=-2x+10\\y&=\frac{-2x+10}{-3}\\y&=\frac{2}{3}x-\frac{10}{3}\end{aligned}$$

The graph obtained using the calculator should look like this:



The first equation, $y=\frac{2}{3}x-\frac{10}{3}$, is graphed in **blue**. The second equation, $y=-\frac{2}{3}x+4$, is graphed in **red**.

The solution does not lie on the "grid" and is therefore difficult to observe visually. With technology we can calculate the intersection. If you have a TI-83 or 84, use the CALC menu, select INTERSECT. Then select each line by pressing ENTER on each one. The calculator will give you a "guess." Press ENTER one more time and the calculator will then calculate the intersection of (5.5, .333...). We can also write this point as $\left(\frac{11}{2},\frac{1}{3}\right)$. Check the solution algebraically.

Equation 1: $2x-3y=10; 2\left(\frac{11}{2}\right)-3\left(\frac{1}{3}\right)=11-1=10\ \boxed{\checkmark}$

Equation 2: $y=-\frac{2}{3}x+4; -\frac{2}{3}\left(\frac{11}{2}\right)+4=-\frac{11}{3}+\frac{12}{3}=\frac{1}{3}\ \boxed{\checkmark}$

If you do not have a TI-83 or 84, the commands might be different. Check with your teacher.

Guided Practice

Solve the following systems by graphing. Use technology for problem 3.

1.

$$\begin{aligned}y &= 3x-4\\ y &= 2\end{aligned}$$

2.

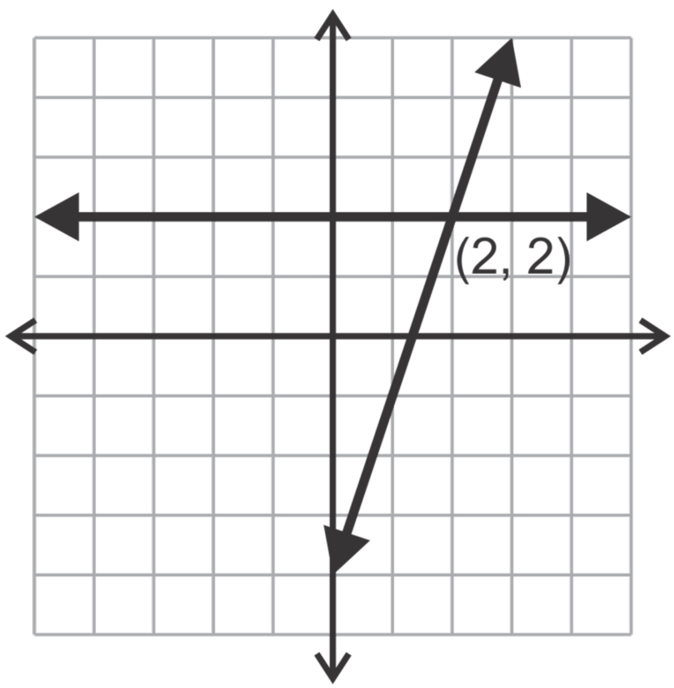
$$\begin{aligned}2x-y &= -4\\ 2x+3y &= -12\end{aligned}$$

3.

$$\begin{aligned}5x+y &= 10\\ y &= \frac{2}{3}x-7\end{aligned}$$

Answers

1.



The first line is in slope intercept form and can be graphed accordingly.

The second line is a horizontal line through (0, 2).

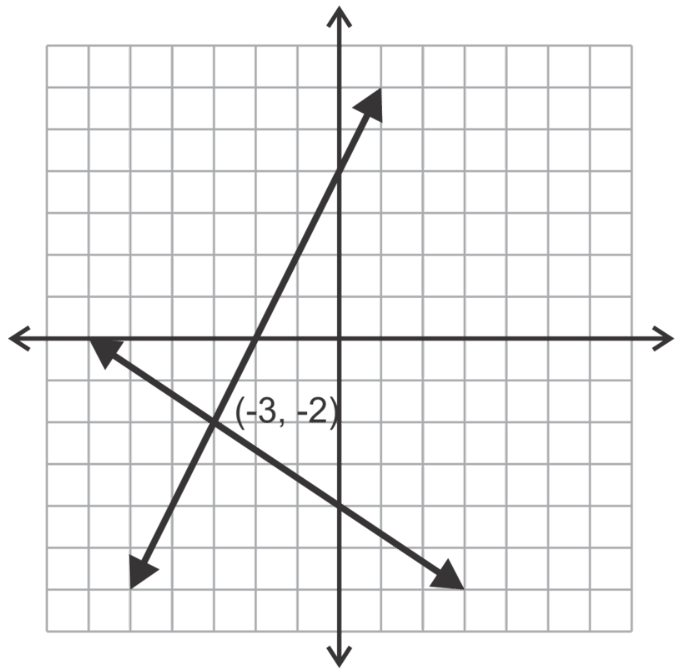
The graph of the two equations is shown below. From this graph the solution appears to be (2, 2).

Checking this solution in each equation verifies that it is indeed correct.

Equation 1: $2 = 3(2) - 4 \boxed{\checkmark}$

Equation 2: $2 = 2 \boxed{\checkmark}$

2.



Neither of these equations is in slope intercept form. The easiest way to graph them is to find their intercepts as shown in Example B.

Equation 1: Let $y = 0$ to find the $x-$intercept.

$$\begin{aligned}2x - y &= -4\\2x - 0 &= -4\\x &= -2\end{aligned}$$

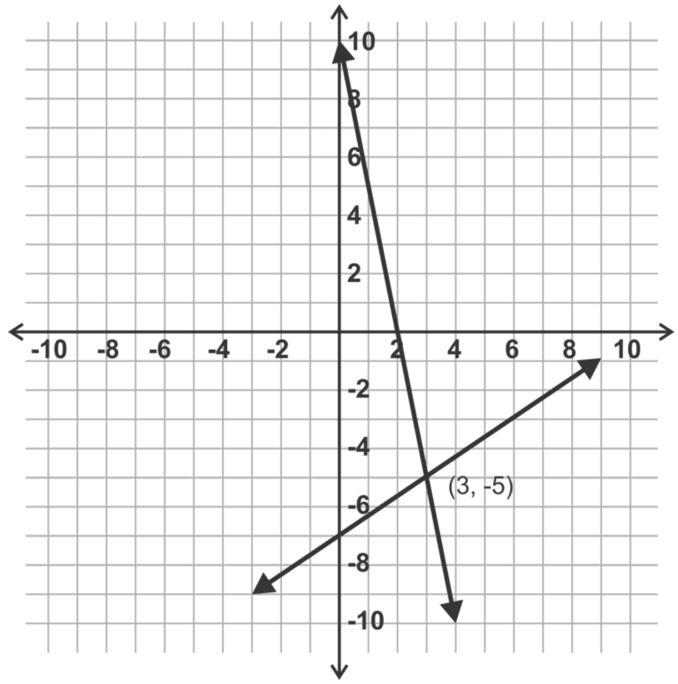
Now let $x = 0$, to find the $y-$intercept.

$$\begin{aligned}2x - y &= -4\\2(0) - y &= -4\\y &= 4\end{aligned}$$

Now we can use (-2, 0) and (0, 4) to graph the line as shown in the diagram. Using the same process, the intercepts for the second line can be found to be (-6, 0) and (0, -4).

Now the solution to the system can be observed to be (-3, -2). This solution can be verified algebraically as shown in the first problem.

3.



The first equation here must be rearranged to be $y = -5x + 10$ before it can be entered into the calculator. The second equation can be entered as is.

Using the calculate menu on the calculator the solution is (3, -5).

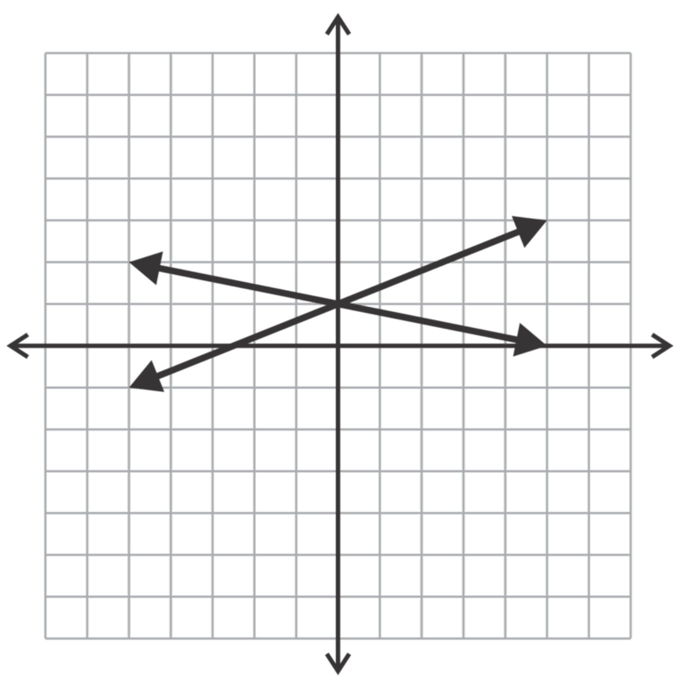
Remember to verify this solution algebraically as a way to check your work.

Problem Set

Match the system of linear equations to its graph and state the solution.

1.

$$\begin{aligned} 3x + 2y &= -2 \\ x - y &= -4 \end{aligned}$$



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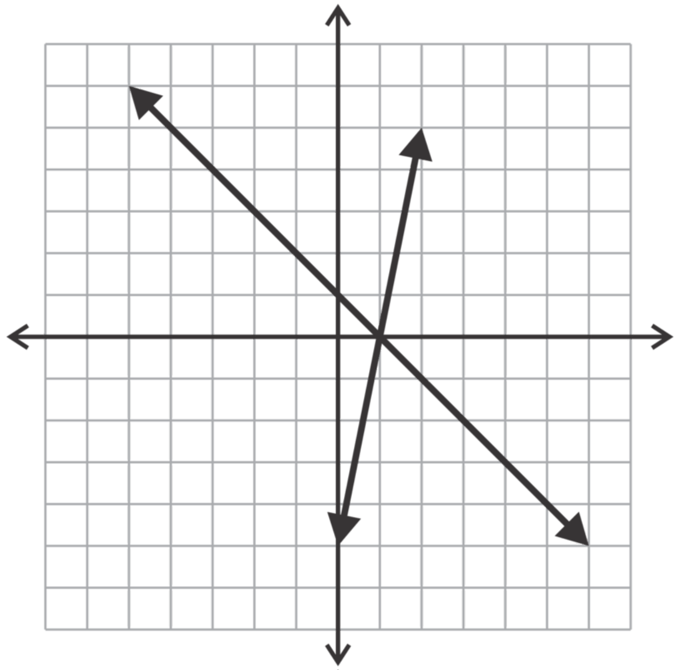

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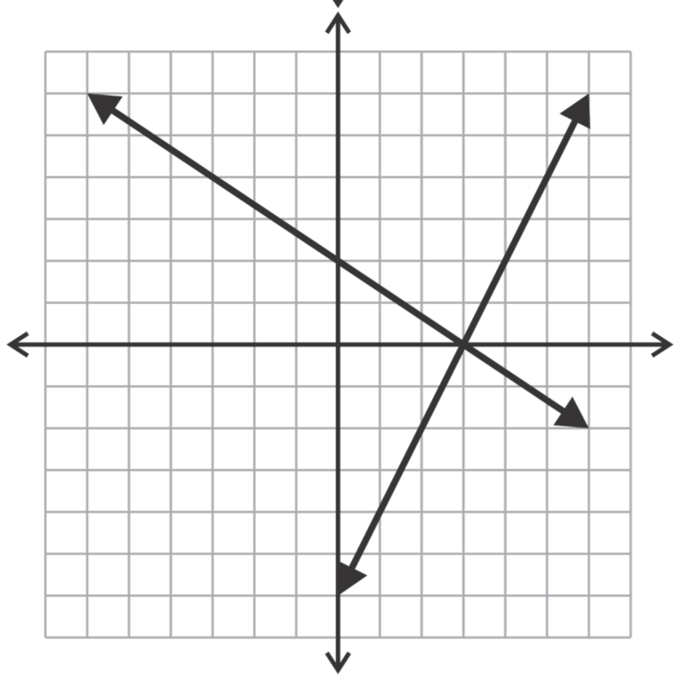

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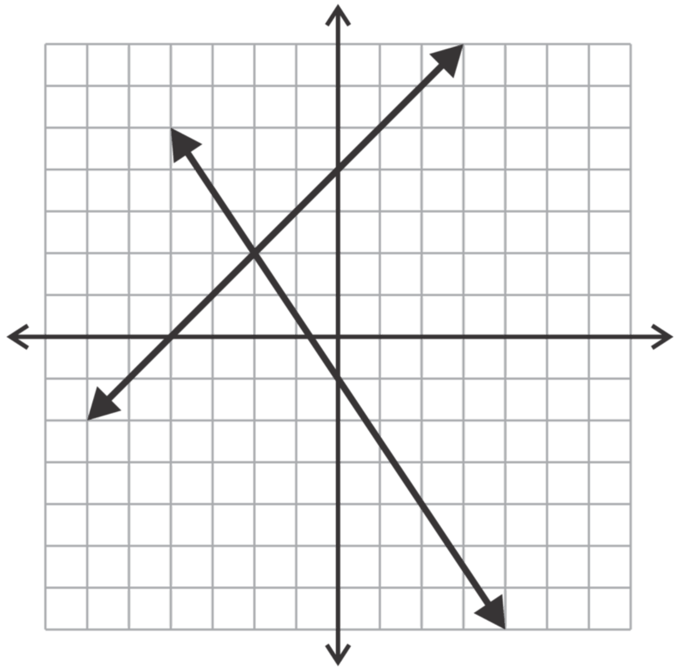
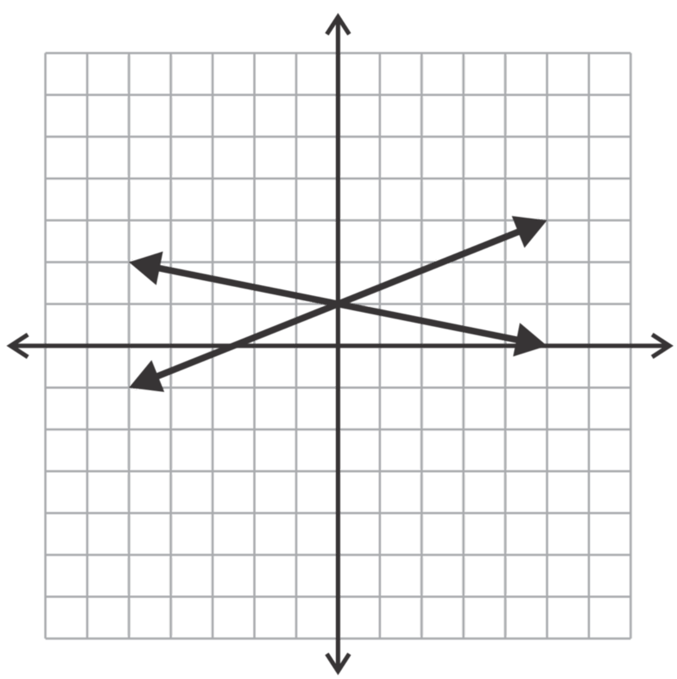


d.

2.

$$2x - y = 6$$
$$2x + 3y = 6$$



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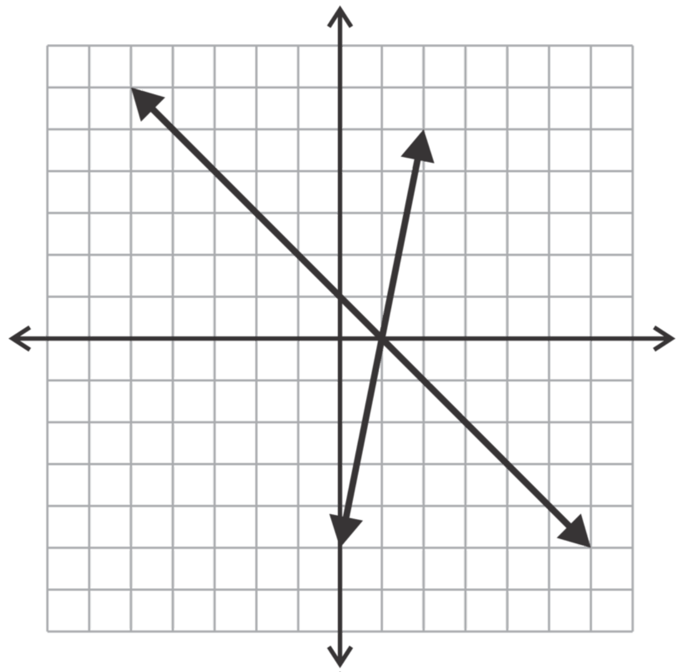

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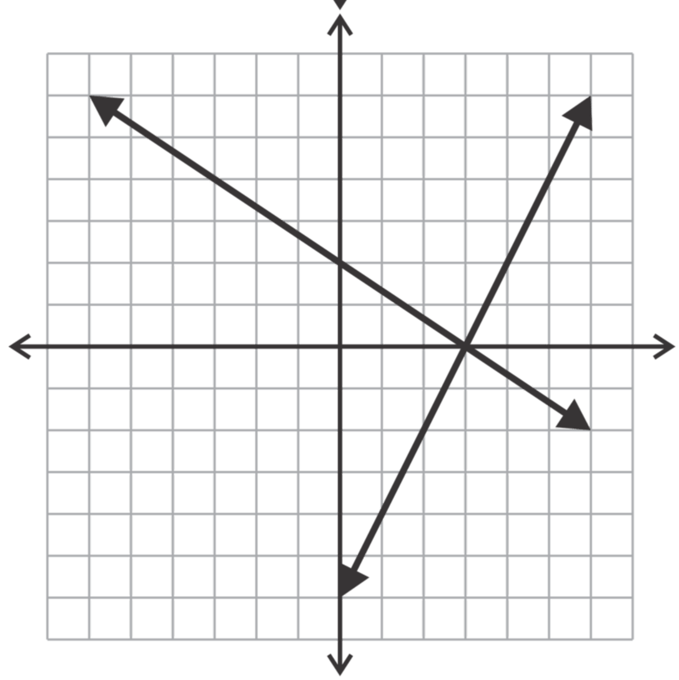

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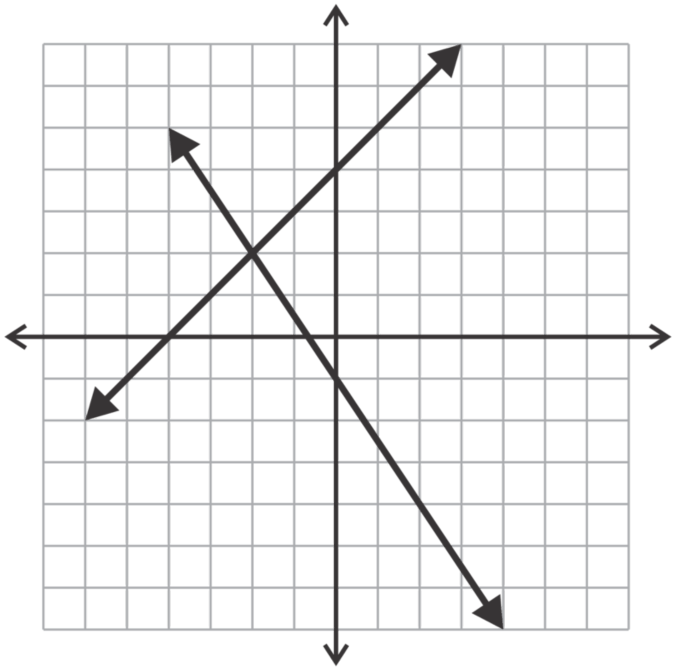
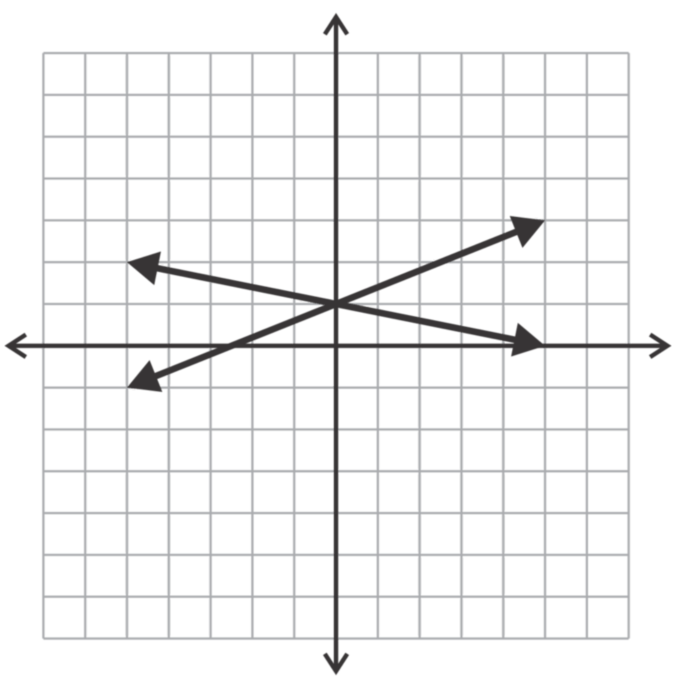


d.

3.

$$2x-5y=-5$$
$$x+5y=5$$



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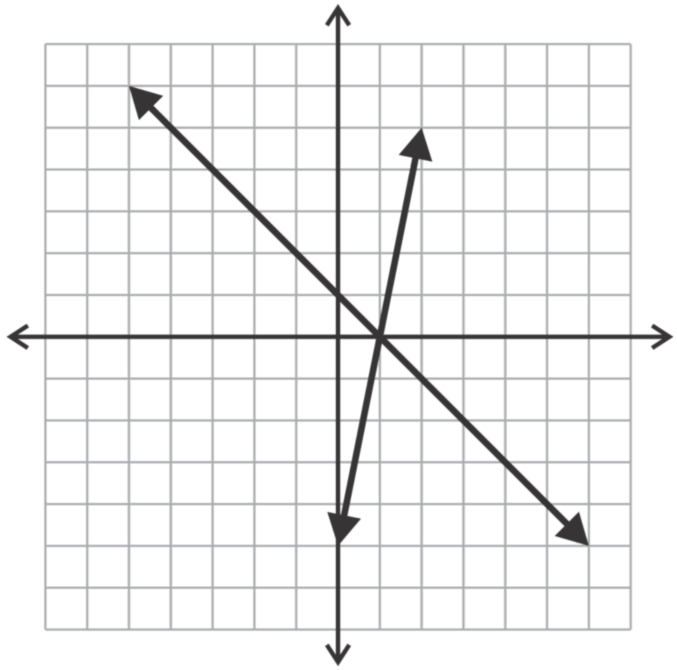

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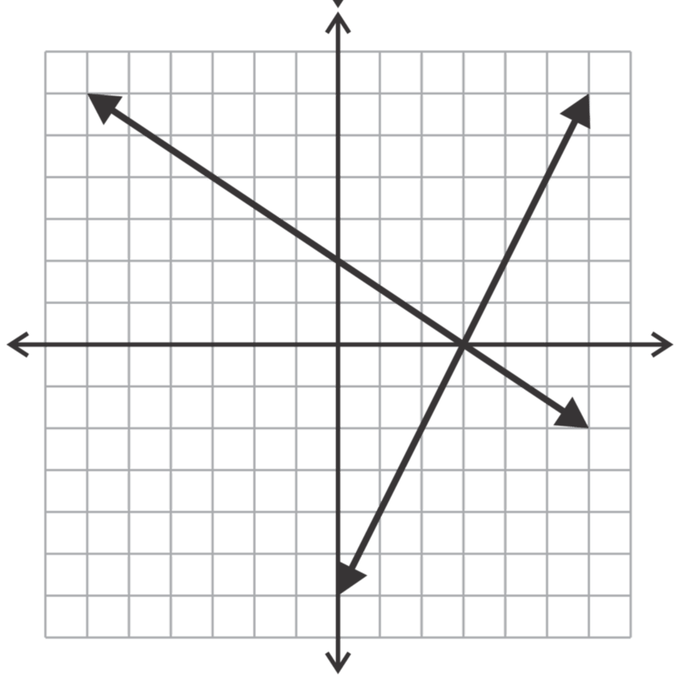

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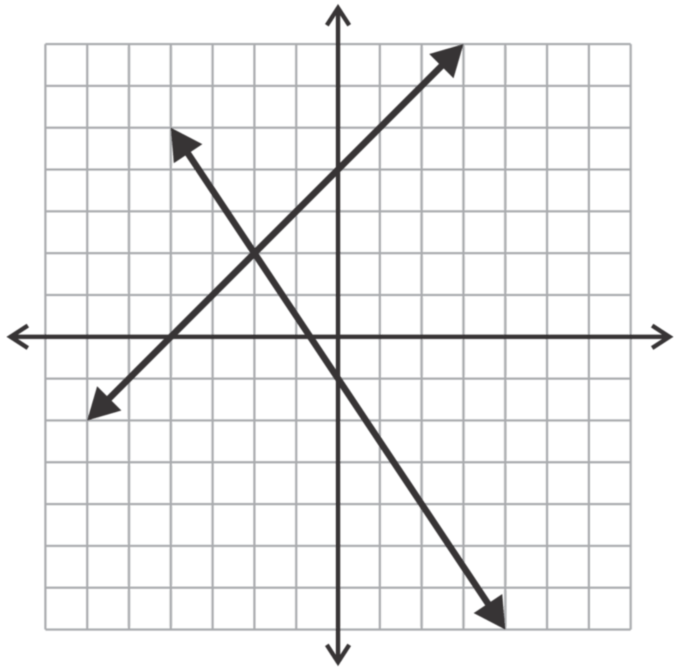
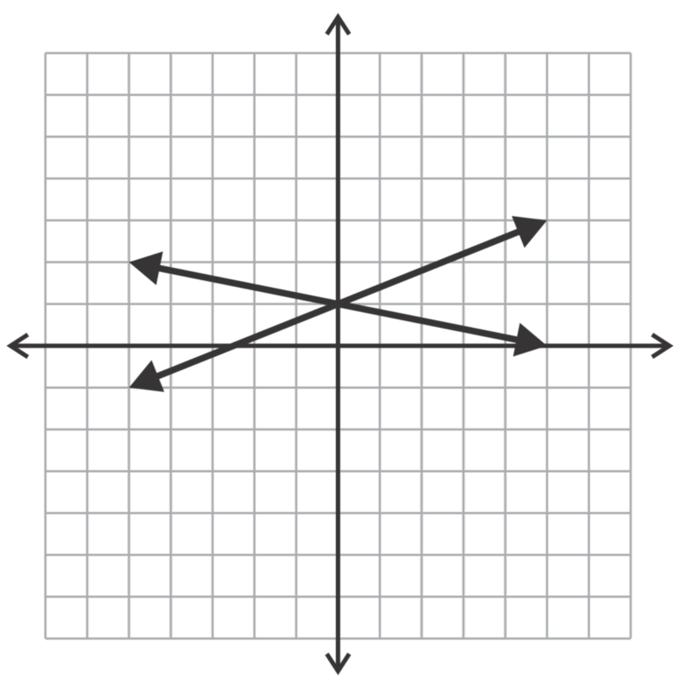


d.

4.

$$y = 5x - 5$$
$$y = -x + 1$$



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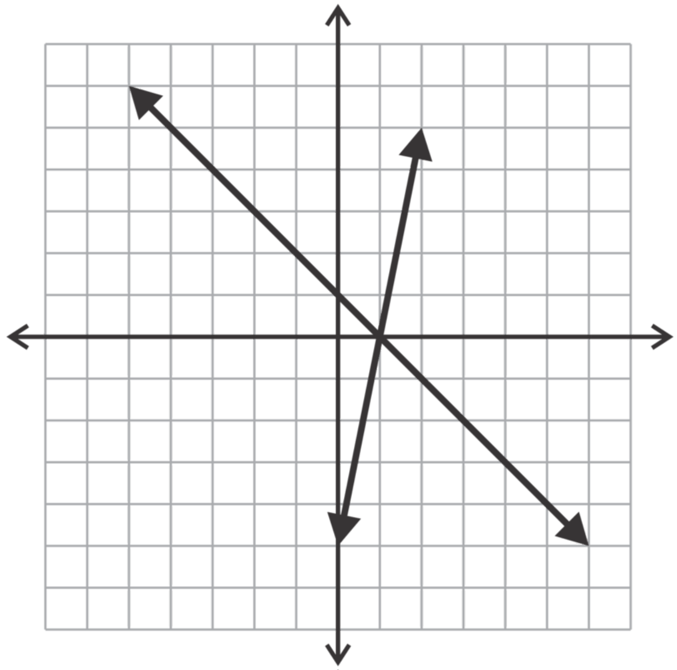

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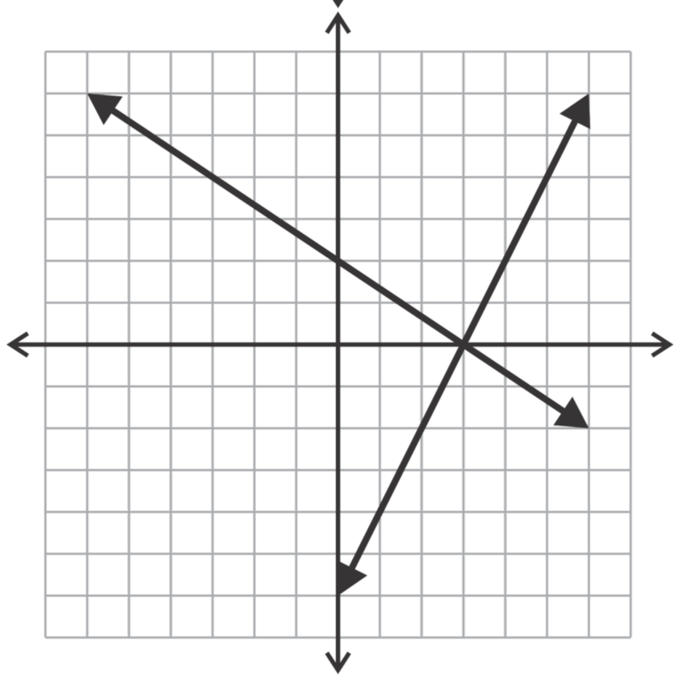

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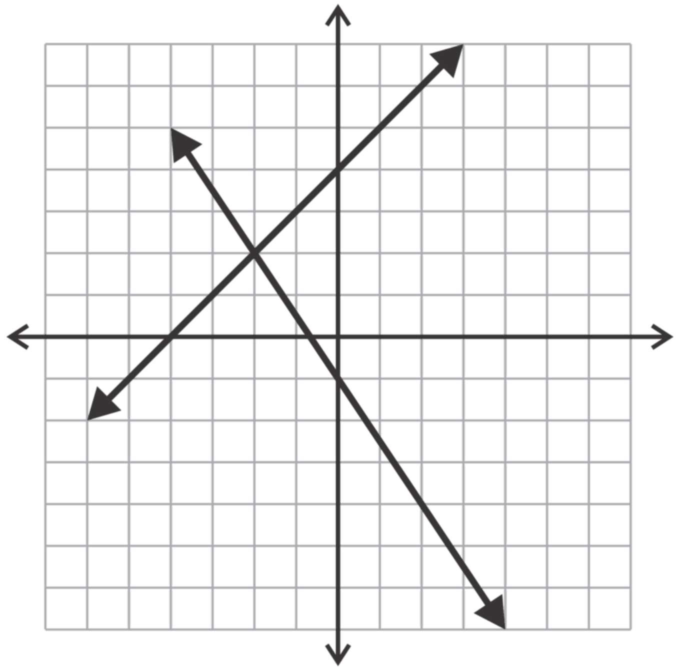


d.

Solve the following linear systems by graphing. Use graph paper and a straightedge to insure accuracy. You are encouraged to verify your answer algebraically.

5.

$$y = -\frac{2}{5}x + 1$$
$$y = \frac{3}{5}x - 4$$

6.

$$y = -\frac{2}{3}x + 4$$
$$y = 3x - 7$$

7.

$$y = -2x + 1$$
$$x - y = -4$$

8.

$$3x + 4y = 12$$
$$x - 4y = 4$$

9.

$$7x - 2y = -4$$
$$y = -5$$

10.

$$\begin{aligned} x-2y &= -8 \\ x &= -3 \end{aligned}$$

Solve the following linear systems by graphing using technology. Solutions should be rounded to the nearest hundredth as necessary.

11.

$$\begin{aligned} y &= \frac{3}{7}x+11 \\ y &= -\frac{13}{2}x-5 \end{aligned}$$

12.

$$\begin{aligned} y &= 0.95x-8.3 \\ 2x+9y &= 23 \end{aligned}$$

13.

$$\begin{aligned} 15x-y &= 22 \\ 3x+8y &= 15 \end{aligned}$$

Use the following information to complete exercises 14-17.

Clara and her brother, Carl, are at the beach for vacation. They want to rent bikes to ride up and down the boardwalk. One rental shop, Bargain Bikes, advertises rates of $5 plus $1.50 per hour. A second shop, Frugal Wheels, advertises a rate of $6 plus $1.25 an hour.

14. How much does it cost to rent a bike for one hour from each shop? How about 10 hours?
15. Write equations to represent the cost of renting a bike from each shop. Let x represent the number of hours and y represent the total cost.
16. Solve your system to figure out when the cost is the same.
17. Clara and Carl want to rent the bikes for about 3 hours. Which shop should they use?

Solving Systems with No or Infinitely Many Solutions Using Graphing

Objective

Determine whether a system has a unique solution or not based on its graph. If no unique solution exists, determine whether there is no solution or infinitely many solutions.

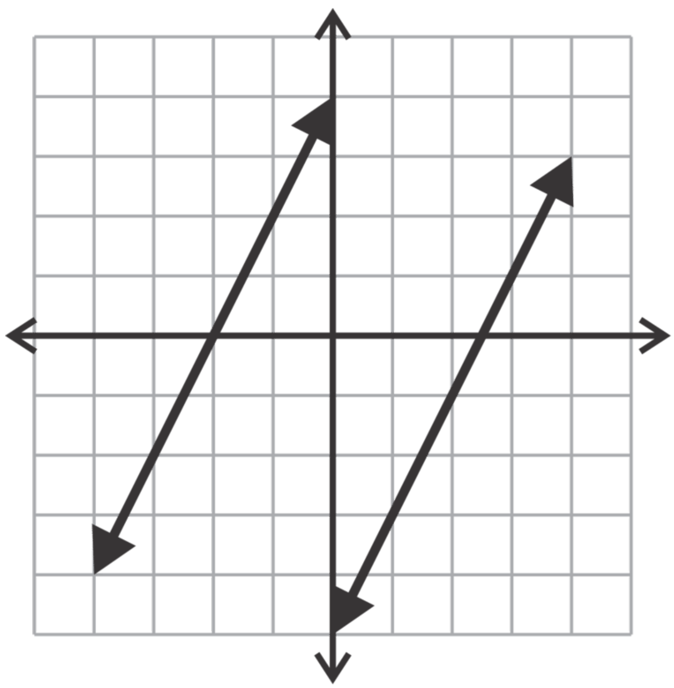
Guidance

So far we have looked at linear systems of equations in which the lines always intersected in one, unique point. What happens if this is not the case? What could the graph of the two lines look like? In Examples A and B below we will explore the two possibilities.

Example A

Graph the system:

$$y = 2x - 5$$
$$y = 2x + 4$$



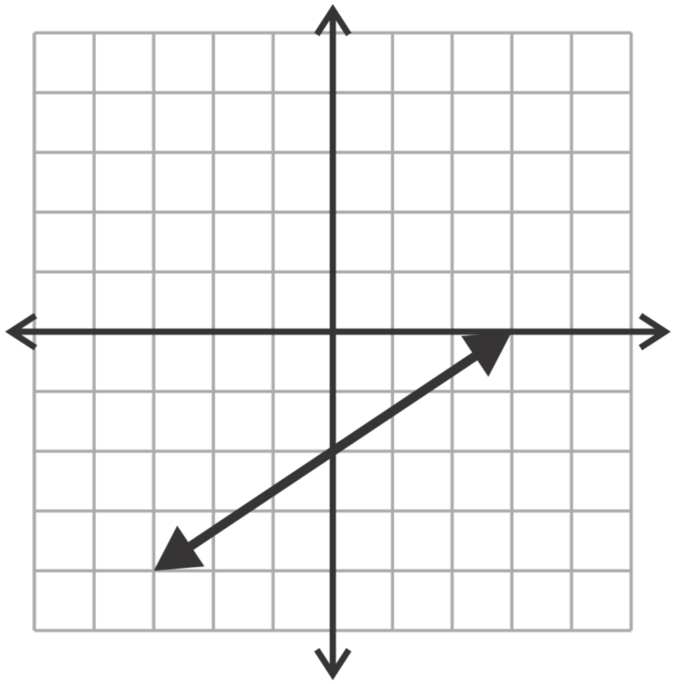
Solution:

In this example both lines have the same slope but different $y-$intercepts. When graphed, they are **parallel** lines and never intersect. This system has no solution. Another way to say this is to say that it is **inconsistent**.

Example B

Graph the system:

$$2x - 3y = 6$$
$$-4x + 6y = -12$$



Solution:

In this example both lines have the same slope and $y-$intercept. This is more apparent when the equations are written in slope intercept form:

$$y = \frac{2}{3}x - 2 \text{ and } y = \frac{2}{3}x - 2$$

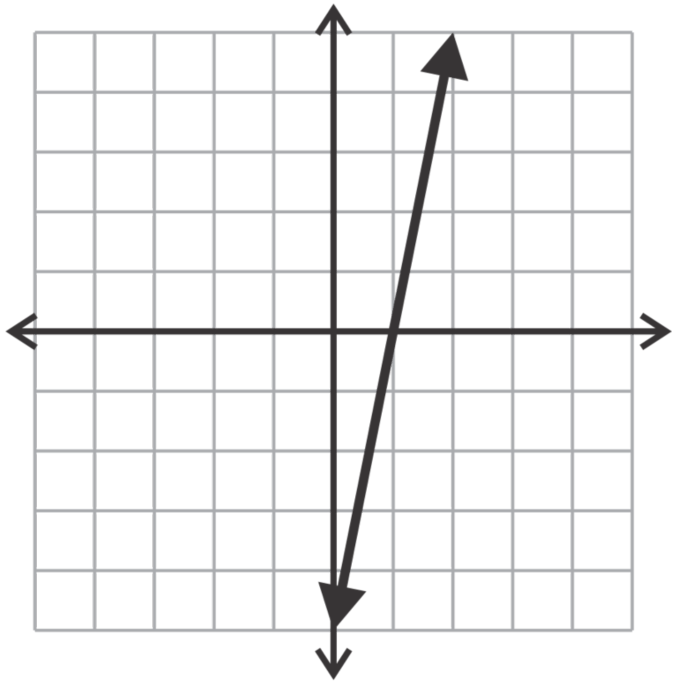
When we graph them, they are one line, **coincident**, meaning they have all points in common. This means that there are an infinite number of solutions to the system. Because this system has at least one solution it is considered to be **consistent**.

Consistent systems are systems which have at least one solution. If the system has exactly one, unique solution then it is **independent**. All of the systems we solved in the last section were independent. If the system has infinite solutions, like the system in Example B, then it is called **dependent**.

Example C

Classify the following system:

$$\begin{aligned} 10x - 2y &= 10 \\ y &= 5x - 5 \end{aligned}$$



Solution:

Rearranging the first equation into slope intercept form we get $y = 5x - 5$, which is exactly the same as the second equation. This means that they are the same line. Therefore the system is consistent and dependent.

Guided Practice

Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.

1.

$$\begin{aligned} 5x - y &= 15 \\ x + 5y &= 15 \end{aligned}$$

2.

$$\begin{aligned} 9x - 12y &= -24 \\ -3x + 4y &= 8 \end{aligned}$$

3.

$$\begin{aligned} 6x + 8y &= 12 \\ -3x - 4y &= 10 \end{aligned}$$

Answers

1. The first step is to rearrange both equations into slope intercept form so that we can compare these attributes.

$$\begin{aligned} 5x - y = 15 &\rightarrow y = 5x - 15 \\ x + 5y = 15 &\rightarrow y = -\frac{1}{5}x + 3 \end{aligned}$$

The slopes are not the same so the lines are neither parallel nor coincident. Therefore, the lines must intersect in one point. The system is consistent and independent.

2. Again, rearrange the equations into slope intercept form:

$$9x - 12y = -24 \rightarrow y = \frac{3}{4}x + 2$$
$$-3x + 4y = 8 \rightarrow y = \frac{3}{4}x + 2$$

Now, we can see that both the slope and the $y-$intercept are the same and therefore the lines are coincident. The system is consistent and dependent.

3. The equations can be rewritten as follows:

$$6x + 8y = 12 \rightarrow y = -\frac{3}{4}x + \frac{3}{2}$$
$$-3x - 4y = 10 \rightarrow y = -\frac{3}{4}x - \frac{5}{2}$$

In this system the lines have the same slope but different $y-$intercepts so they are parallel lines. Therefore the system is inconsistent. There is no solution.

Vocabulary

Parallel
Two or more lines in the same plane that never intersect. They have the same slope and different $y-$intercepts.

Coincident
Lines which have all points in common. They are line which “coincide” with one another or are the same line.

Consistent
Describes a system with at least one solution.

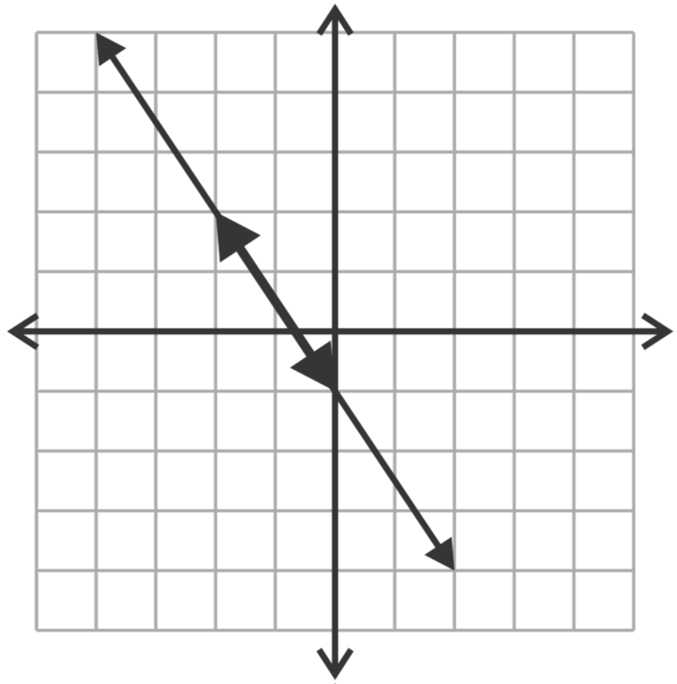
Inconsistent
Describes a system with no solution.

Dependent
Describes a consistent system with infinite solutions.

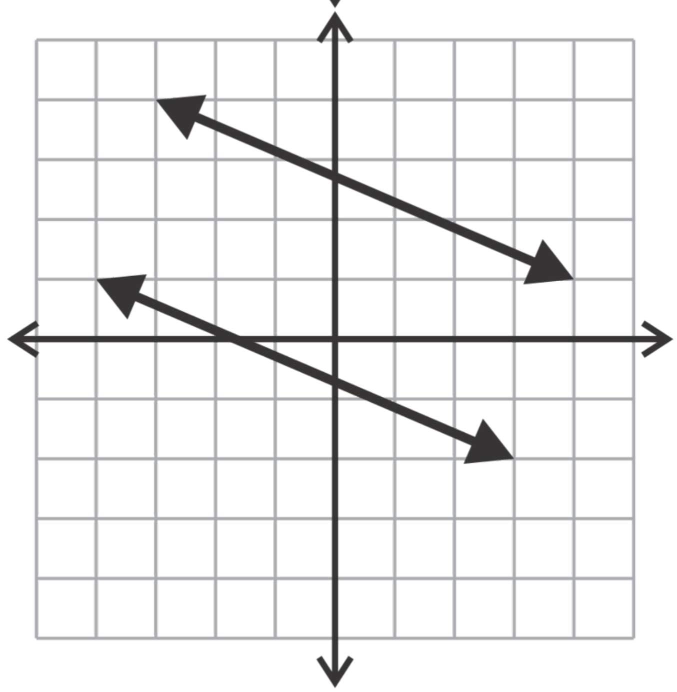
Independent
Describes a consistent system with exactly one solution.

Problem Set

Describe the systems graphed below both algebraically (consistent, inconsistent, dependent, independent) and geometrically (intersecting lines, parallel lines, coincident lines).



1.



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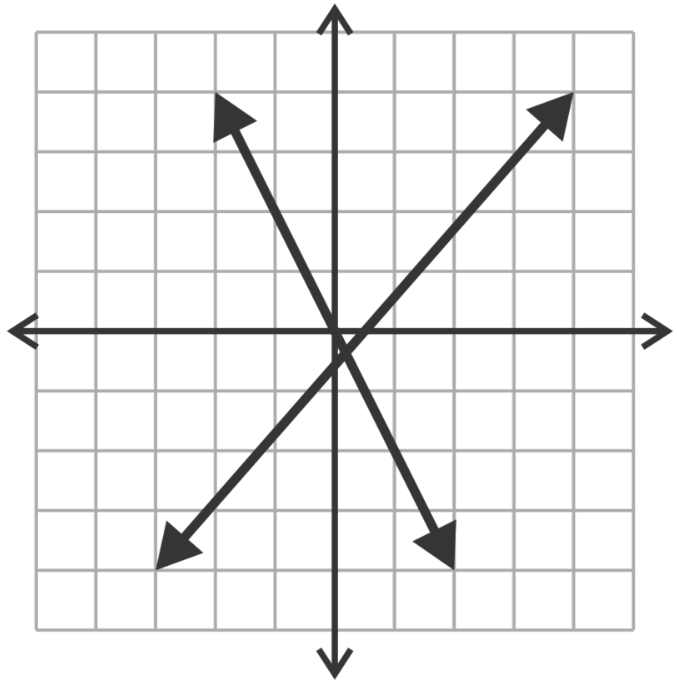


3.

Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.

4.

$$\begin{aligned} 4x - y &= 8 \\ y &= 4x + 3 \end{aligned}$$

5.

$$\begin{aligned} 5x + y &= 10 \\ y &= 5x + 10 \end{aligned}$$

6.

$$\begin{aligned} 2x - 2y &= 11 \\ y &= x + 13 \end{aligned}$$

7.

$$\begin{aligned} -7x + 3y &= -21 \\ 14x - 6y &= 42 \end{aligned}$$

8.

$$\begin{aligned} y &= -\frac{3}{5}x + 1 \\ 3x + 5y &= 5 \end{aligned}$$

9.

$$\begin{aligned} 6x - y &= 18 \\ y &= \frac{1}{6}x + 3 \end{aligned}$$

In problems 10-12 you will be writing your own systems. Your equations should be in standard form, $Ax+By=C$. Try to make them *look* different even if they are the same equation.

10. Write a system which is consistent and independent.
11. Write a system which is consistent and dependent.
12. Write a system which is inconsistent.

1.13 Analyzing the Graph of Polynomial Functions

Objective

To learn about the parts of a polynomial function and how to graph them. The graphing calculator will be used to aid in graphing.

Review Queue

1. Graph $4x-5y=25$. Find the slope, $x-$ and $y-$intercepts.

2. Graph $y=x^2-2x-8$. Find the $x-$intercepts, $y-$intercept, and vertex.

3. Find the vertex of $y=-4x^2+24x+5$. Is the vertex a maximum or minimum?

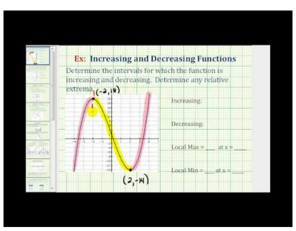
Finding and Defining Parts of a Polynomial Function Graph

Objective

Learning about the different parts of graphs for higher-degree polynomials.

Watch This

First watch this video.



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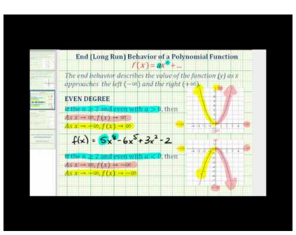
James Sousa: Ex: Increasing/ Decreasing/ Relative Extrema from Analyzing a Graph

Then watch this video.



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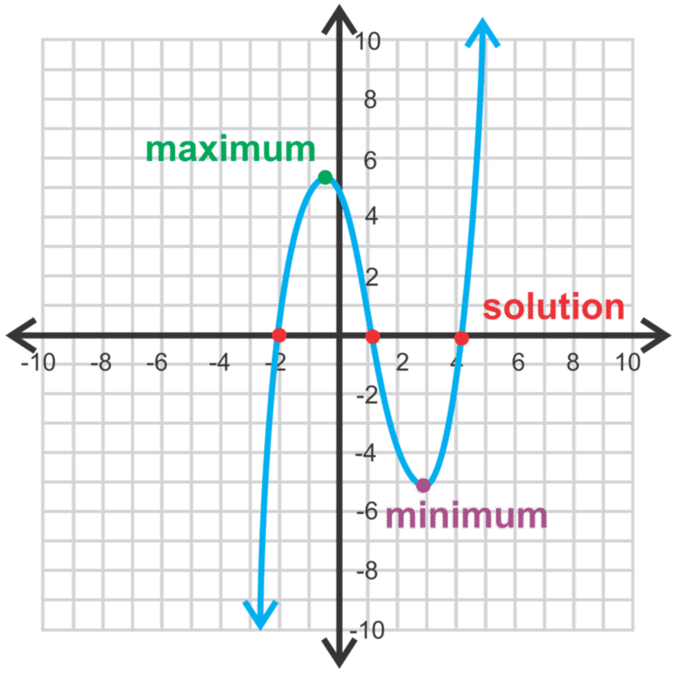
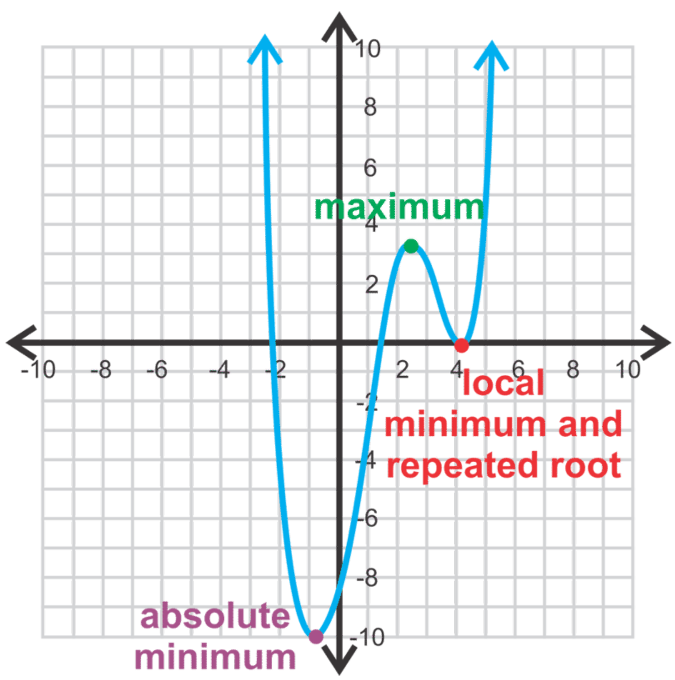
James Sousa: Summary of End Behavior or Long Run Behavior of Polynomial Functions

Guidance

By now, you should be familiar with the general idea of what a polynomial function graph does. It should cross the $x-$axis as many times as the degree, unless there are imaginary solutions. It will curve up and down and can have a maximum and a minimum. Let's define the parts of a polynomial function graph here.





Notice that in both the cubic (third degree, on the left) and the quartic (fourth degree, on the right) functions, there is no vertex. We now have minimums and maximums. If there are more than one minimum or maximum, there will be an **absolute maximum/minimum**, which is the lowest/highest point of the graph. A **local maximum/minimum** is a maximum/minimum relative to the points around it. The places where the function crosses the $x-$axis are still the **solutions** (also called $x-$intercepts, roots or zeros). In the quartic function, there is a repeated root at $x = 4$. A repeated root will touch the $x-$axis without passing through or it can also have a "jump" in the curve at that point (see Example A). All of these points together (maximums, minimums, $x-$intercepts, and $y-$intercept) are called **critical values.**

Another important thing to note is **end behavior.** It is exactly what it sounds like; how the "ends" of the graph behaves or points. The cubic function above has ends that point in the opposite direction. We say that from left to right, this function is *mostly increasing.* The quartic function's ends point in the same direction, both positive, just like a quadratic function. When considering end behavior, look at the leading coefficient and the degree of the polynomial.

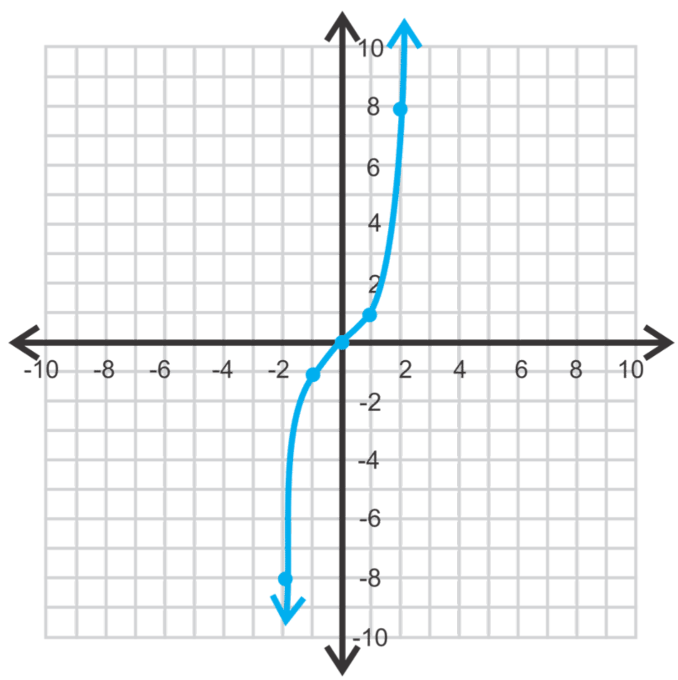
Example A

Use a table to graph $y = x^3$.

Solution: Draw a table and pick at least 5 values for x.

TABLE 1.15:

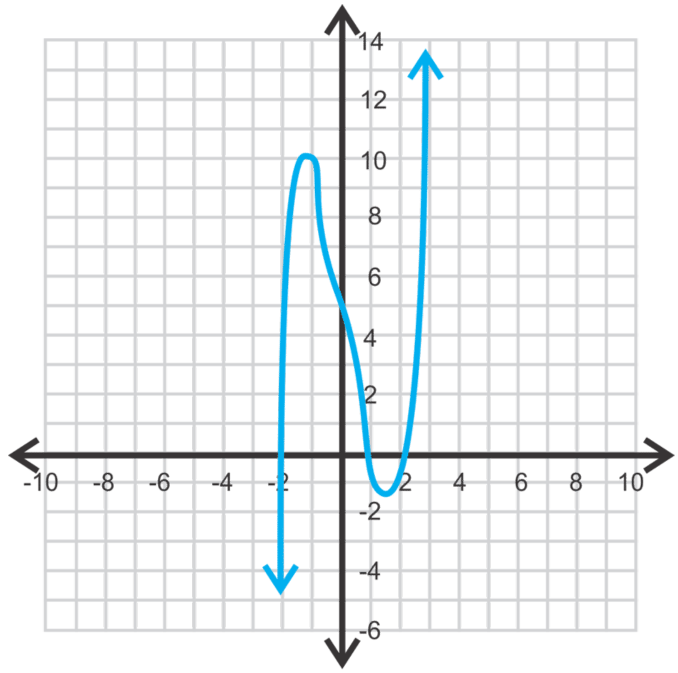
x	x^3	y
-2	$(-2)^3$	-8
-1	$(-1)^3$	-1
0	0^3	0
1	1^3	1
2	2^3	8



Plot the points and connect. This particular function is the **parent graph** for cubic functions. Recall from quadratic functions, that the parent graph has a leading coefficient of 1, no other $x-$terms, and no $y-$intercept. $y = x^4$ and $y = x^5$ are also parent graphs.

Example B

Analyze the graph below. Find the critical values, end behavior, and find the domain and range.



Solution: First, find the solutions. They appear to be (-2, 0), (1, 0), and (2, 0). Therefore, this function has a minimum degree of 3. However, look at the $y-$intercept. The graph slightly bends between the maximum and minimum. This movement in the graph tells us that there are two imaginary solutions (recall that imaginary solutions always come in pairs). Therefore, the function has a degree of 5. Approximate the other critical values:

maximum: (-1.1, 10)

minimum: (1.5, -1.3)

$y-$intercept: (0, 5)

In general, this function is mostly increasing and the ends go in opposite directions. The domain and range are both all real numbers.

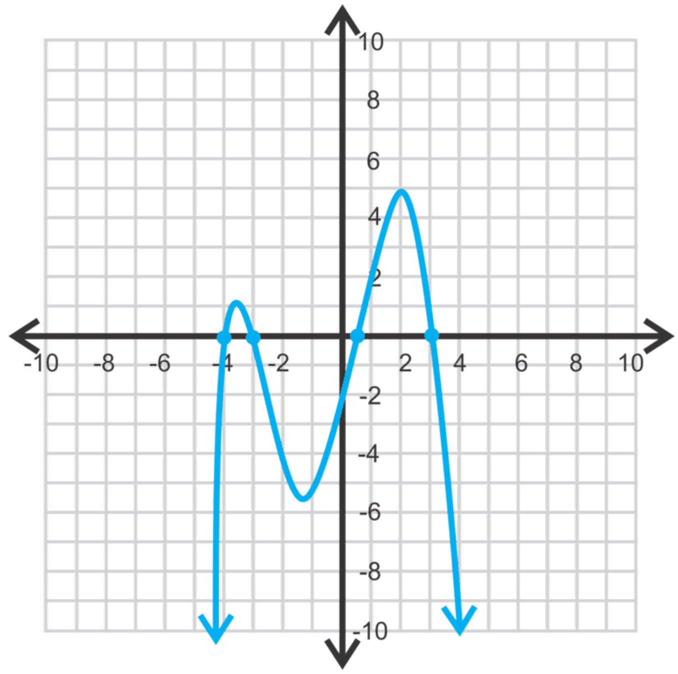
When describing critical values, you may approximate their location. In the next concept, we will use the graphing calculator to find these values exactly.

Sometime it can be tricky to see if a function has imaginary solutions from the graph. Compare the graph in Example B to the cubic function above. Notice that it is smooth between the maximum and minimum. As was pointed out earlier, the graph from Example B bends. Any function with imaginary solutions will have a slightly irregular shape or bend like this one does.

Example C

Sketch a graph of a function with roots $-4, -3, \frac{1}{2}$, and 3, has an absolute maximum at (2, 5), and has negative end behavior. This function does not have any imaginary roots.

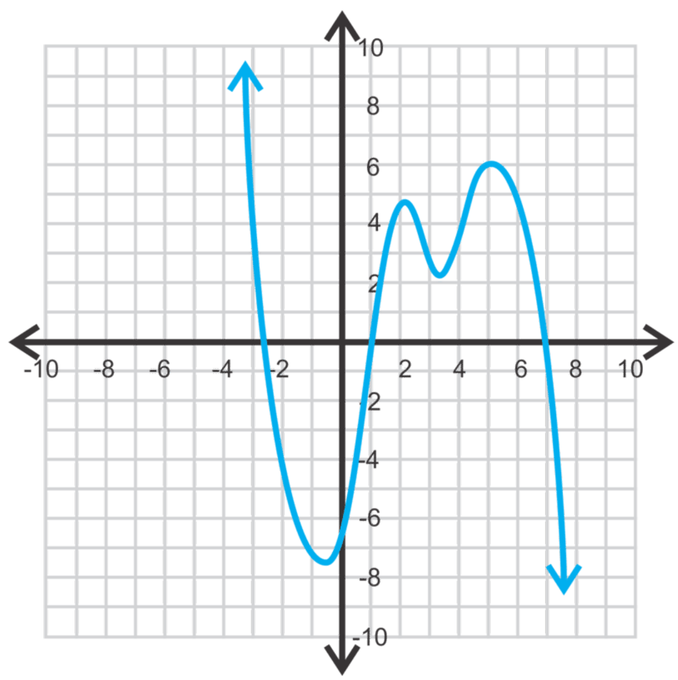
Solution: There are several possible answers for this graph because we are only asking for a sketch. You would need more information to get an exact answer. Because this function has negative end behavior and four roots, we know that it will pass through the $x-$axis four times and face down. The absolute maximum is located between the roots $\frac{1}{2}$ and 3. Plot these five points and connect to form a graph.



Guided Practice

1. Use a table to graph $f(x) = -(x+2)^2(x-3)$.

2. Analyze the graph. Find all the critical values, domain, range and describe the end behavior.



3. Draw a graph of the cubic function with solutions of -6 and a repeated root at 1. This function is generally increasing and has a maximum value of 9.

Answers

1. This function is in intercept form. Because the factor, $(x+2)$ is squared, we know it is a repeated root. Therefore, the function should just touch at -2 and not pass through the $x-$axis. There is also a zero at 3. Because the function is negative, it will be generally decreasing. Think of the slope of the line between the two endpoints. It would be negative. Select several points around the zeros to see the behavior of the graph.

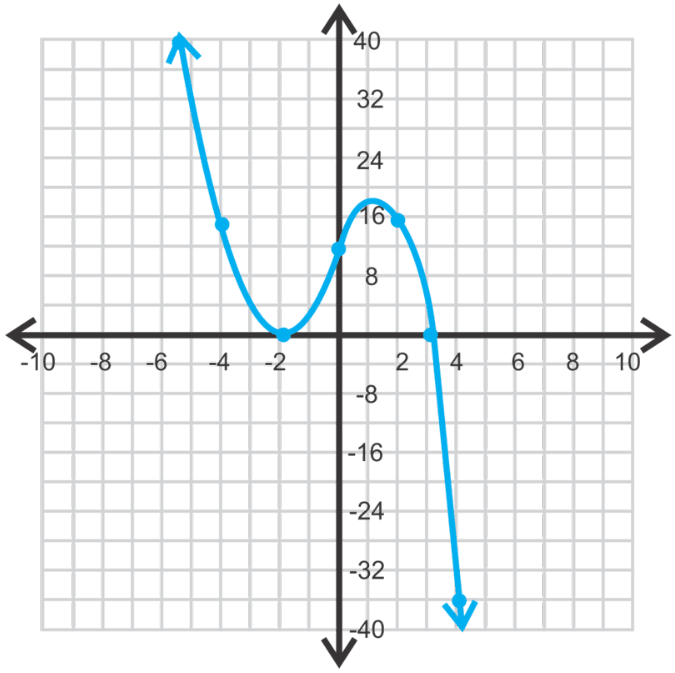
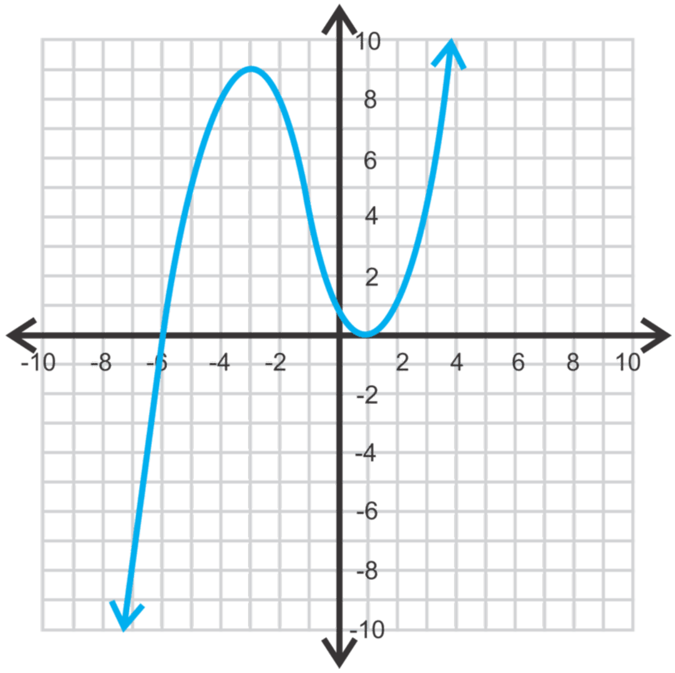


TABLE 1.16:

x	y
-4	14
-2	0
0	12
2	16
3	0
4	-36

2. There are three real zeros at approximately -3.5, 1, and 7. Notice the curve between the zeros 1 and 7. This indicated there are two imaginary zeros, making this at least a fifth-degree polynomial. Think about an imaginary horizontal line at $y = 3$. This line would touch the graph five times, so there should be five solutions. Next, there is an absolute minimum at (-0.5, -7.5), a local maximum at (2.25, 5), a local minimum at (2.25, 2.25) and an absolute maximum at (5, 6). The $y-$intercept is at (0, -6). The domain and range are both all real numbers and the end behavior is mostly decreasing.

3. To say the function is "mostly increasing" means that the slope of the line that connects the two ends (arrows) is positive. Then, the function must pass through (-6, 0) and touch, but not pass through (1, 0). From this information, the maximum must occur between the two zeros and the minimum will be the double root.



Vocabulary

Absolute Maximum/Minimum
: The highest/lowest point of a function. When referring to the absolute maximum/minimum value, use the $y-$value.

Local Maximum/Minimum
: The highest/lowest point relative to the points around it. A function can have multiple local maximums or minimums.

Solutions:

The $x-$intercepts. Also called roots or zeros.

Critical Values: The $x-$intercepts, maximums, minimums, and $y-$intercept.

End Behavior: How the ends of a graph look. End behavior depends on the degree of the function and the leading coefficient.

Parent Graph: The most basic function of a particular type. It has a leading coefficient of 1, no additional $x-$terms, and no constant.

Problem Set

Use the given $x-$values to make a table and graph the functions below.

1.

$$f(x) = x^3 - 7x^2 + 15x - 2$$
$$x = -2, -1, 0, 1, 2, 3, 4$$

2.

$$g(x) = -2x^4 - 11x^3 - 3x^2 + 37x + 35$$
$$x = -5, -4, -3, -2, -1, 0, 1, 2$$

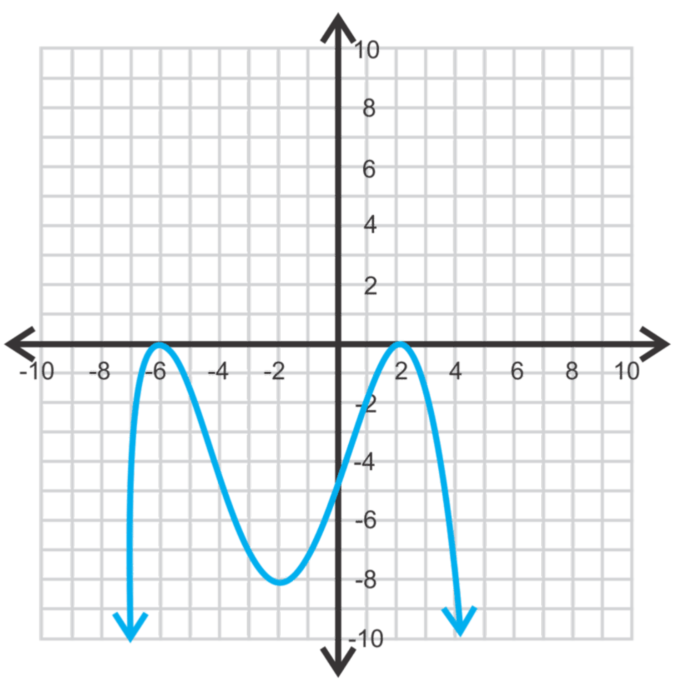
3.

$$y = 2x^3 + 25x^2 + 100x + 125$$
$$x = -7, -6, -5, -4, -3, -2, -1, 0$$

Make your own table and graph the following functions.

4. $f(x) = (x+5)(x+2)(x-1)$
5. $y = x^4$
6. $y = x^5$
7. Analyze the graphs of $y = x^2, y = x^3, y = x^4$, and $y = x^5$. These are all parent functions. What do you think the graph of $y = x^6$ and $y = x^7$ will look like? What can you say about the end behavior of all even functions? Odd functions? What are the solutions to these functions?
8. **Writing** How many repeated roots can one function have? Why?

Analyze the graphs of the following functions. Find all critical values, the domain, range, and end behavior.



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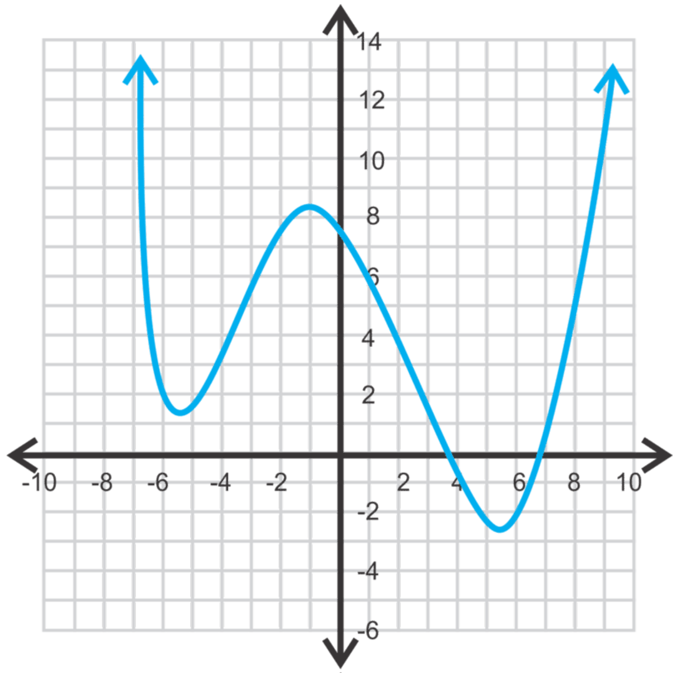

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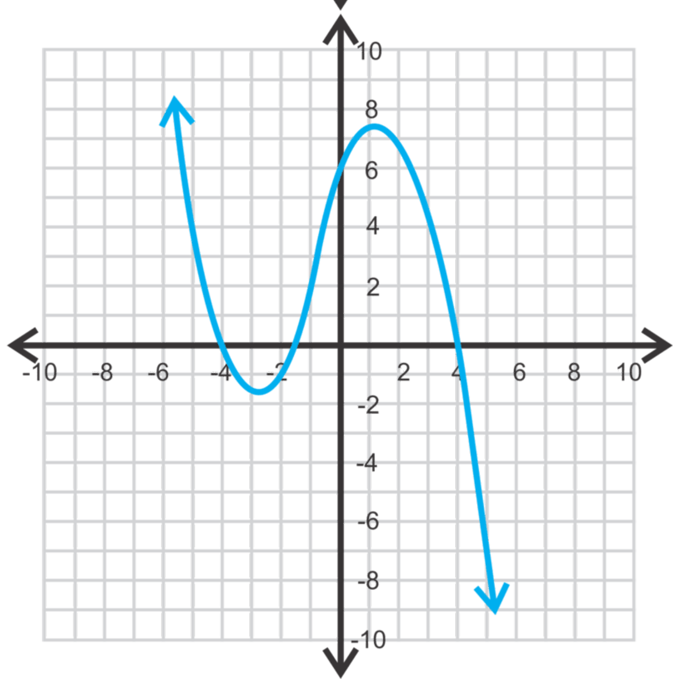
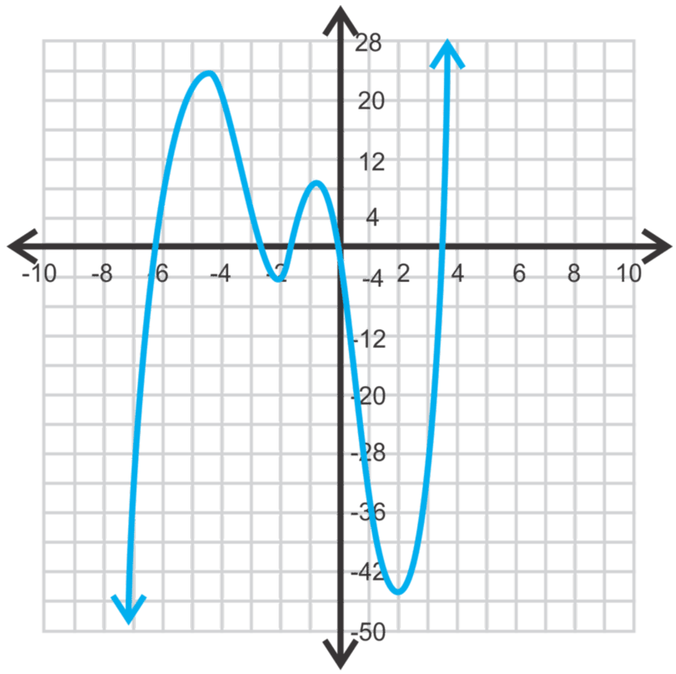


11.

12. 

For questions 13-15, make a sketch of the following real-solution functions.

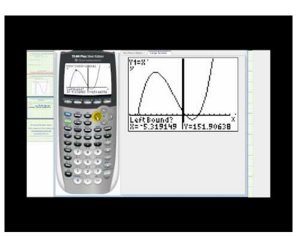
13. Draw **two** different graphs of a cubic function with zeros of -1, 1, and 4.5 and a minimum of -4.
14. A fourth-degree polynomial with roots of -3.2, -0.9, 1.2, and 8.7, positive end behavior, and a local minimum of -1.7.
15. A fourth-degree function with solutions of -7, -4, 1, and 2, negative end behavior, and an absolute maximum at $\left(-\frac{11}{2}, \frac{1755}{128}\right)$.
16. **Challenge** Find the equation of the function from #15.

Graphing Polynomial Functions with a Graphing Calculator

Objective

To graph polynomial functions and find critical values using a graphing calculator.

Watch This



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James Sousa: Ex: Solve a Polynomial Equation Using a Graphing Calculator (Approximate Solutions)

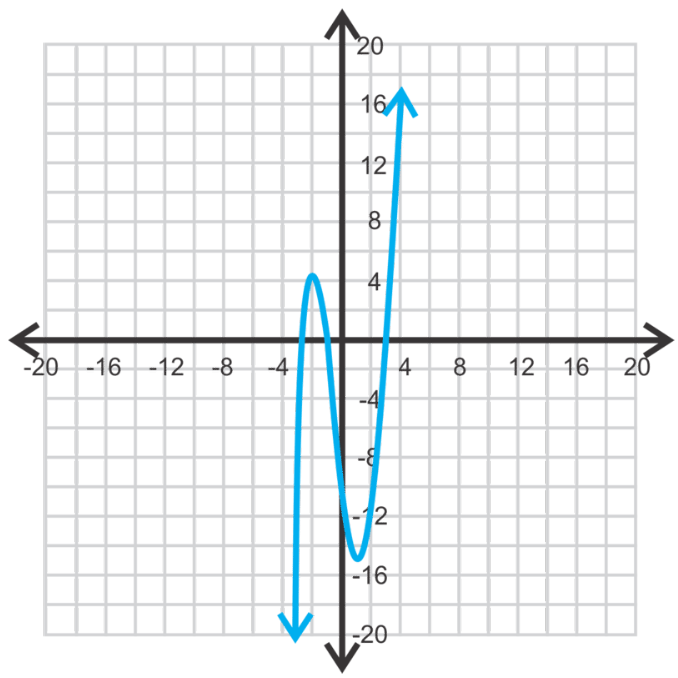
Guidance

In the Quadratic Functions chapter, you used the graphing calculator to graph parabolas. Now, we will expand upon that knowledge and graph higher-degree polynomials. Then, we will use the graphing calculator to find the zeros, maximums and minimums.

Example A

Graph $f(x) = x^3 + x^2 - 8x - 8$ using a graphing calculator.

Solution: *These instructions are for a TI-83 or 84*. First, press $Y =$. If there are any functions in this window, clear them out by highlighting the = sign and pressing **ENTER.** Now, in $Y1$, enter in the polynomial. It should look like: $x^\wedge 3 + x^\wedge 2 - 8x - 8$. Press **GRAPH**.



To adjust the window, press **ZOOM**. To get the typical -10 to 10 screen (for both axes), press **6:ZStandard.** To zoom out, press **ZOOM, 3:ZoomOut, ENTER, ENTER.** For this particular function, the window needs to go from -15 to 15 for both x and y. To manually input the window, press WINDOW and change the $Xmin, Xmax, Ymin,$ and $Ymax$ so that you can see the zeros, minimum and maximum. Your graph should look like the one to the right.

Example B

Find the zeros, maximum, and minimum of the function from Example A.

Solution: To find the zeros, press 2^{nd} **TRACE** to get the **CALC** menu. Select **2:Zero** and you will be asked "Left Bound?" by the calculator. Move the cursor (by pressing the ↑ or ↓) so that it is just to the left of one zero. Press **ENTER.** Then, it will ask "Right Bound?" Move the cursor just to the right of that zero. Press **ENTER.** The calculator will then ask "Guess?" At this point, you can enter in what you think the zero is and press **ENTER** again. Then the calculator will give you the exact zero. For the graph from Example A, you will need to repeat this three times. The zeros are -2.83, -1, and 2.83.

To find the minimum and maximum, the process is almost identical to finding zeros. Instead of selecting **2:Zero**, select **3:min** or **4:max**. The minimum is (1.33, -14.52) and the maximum is (-2, 4).

Example C

Find the $y-$intercept of the graph from Example A.

Solution: If you decide not to use the calculator, plug in zero for x and solve for y.

$$\begin{aligned} f(0) &= 0^3 + 0^2 - 8 \cdot 0 - 8 \\ &= -8 \end{aligned}$$

Using the graphing calculator, press 2^{nd} **TRACE** to get the **CALC** menu. Select **1:value.** $X =$ shows up at the bottom of the screen. If there is a value there, press **CLEAR** to remove it. Then press **0** and **ENTER.** The calculator should then say "$Y = -8$."

Guided Practice

Graph and find the critical values of the following functions.

1. $f(x) = -\frac{1}{3}x^4 - x^3 + 10x^2 + 25x - 4$

2. $g(x) = 2x^5 - x^4 + 6x^3 + 18x^2 - 3x - 8$

3. Find the domain and range of the previous two functions.

4. Describe the types of solutions, as specifically as possible, for question 2.

Answers

Use the steps given in Examples A, B, and C.

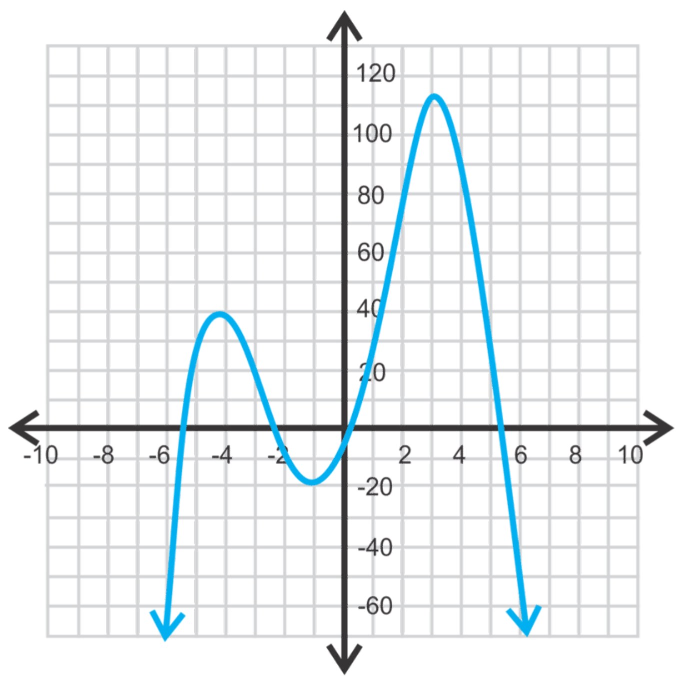
1. zeros: -5.874, -2.56, 0.151, 5.283

$y-$intercept: (0, -4)

minimum: (-1.15, -18.59)

local maximum: (-4.62, 40.69)

absolute maximum: (3.52, 113.12)

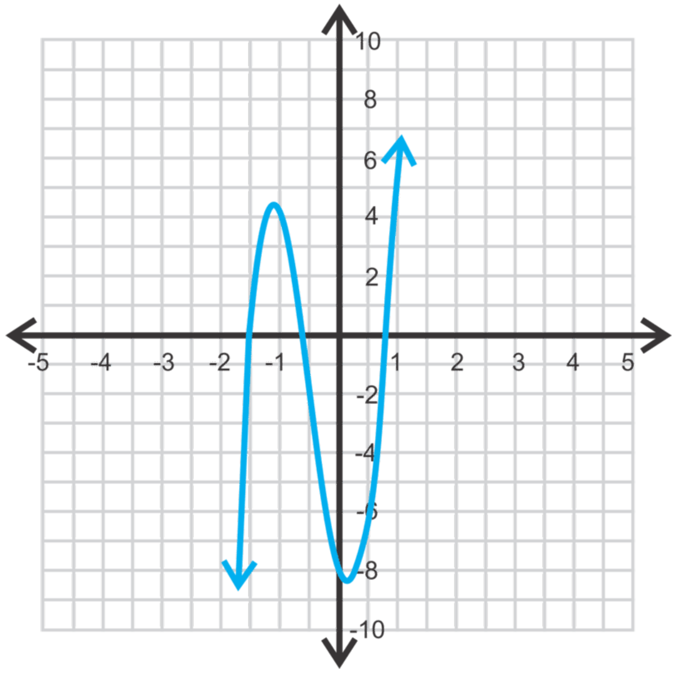


2. zeros: -1.413, -0.682, 0.672

$y-$intercept: (0, -8)

minimum: (-1.11, 4.41)

maximum: (0.08, -8.12)



3. The domain of #1 is all real numbers and the range is all real numbers less than the maximum; $(-\infty, 113.12]$. The domain and range of #2 are all real numbers.

4. There are three irrational solutions and two imaginary solutions.

Problem Set

Graph questions 1-6 on your graphing calculator. Sketch the graph in an appropriate window. Then, find all the critical values, domain, range, and describe the end behavior.

1. $f(x) = 2x^3 + 5x^2 - 4x - 12$
2. $h(x) = -\frac{1}{4}x^4 - 2x^3 - \frac{13}{4}x^2 - 8x - 9$
3. $y = x^3 - 8$
4. $g(x) = -x^3 - 11x^2 - 14x + 10$
5. $f(x) = 2x^4 + 3x^3 - 26x^2 - 3x + 54$
6. $y = x^4 + 2x^3 - 5x^2 - 12x - 6$
7. What are the types of solutions in #2?
8. Find the two imaginary solutions in #3.
9. Find the exact values of the irrational roots in #5.

Determine if the following statements are SOMETIMES, ALWAYS, or NEVER true. Explain your reasoning.

10. The range of an even function is $(-\infty, max]$, where *max* is the maximum of the function.
11. The domain and range of all odd functions are all real numbers.
12. A function can have exactly three imaginary solutions.
13. An n^{th} degree polynomial has n real zeros.
14. **Challenge** The exact value for one of the zeros in #2 is $-4 + \sqrt{7}$. What is the exact value of the other root? Use this information to find the imaginary roots.

1.14 The Unit Circle

Learning Objectives

Here you will use your knowledge of basic triangle trigonometry to identify key points and angles around a circle of radius one centered at the origin. The **unit circle** is a circle of radius one, centered at the origin, that summarizes all the 30-60-90 and 45-45-90 triangle relationships that exist. When memorized, it is extremely useful for evaluating expressions like $\cos(135^\circ)$ or $\sin\left(-\frac{5\pi}{3}\right)$. It also helps to produce the parent graphs of sine and cosine.

How can you use the unit circle to evaluate $\cos(135^\circ)$ and $\sin\left(-\frac{5\pi}{3}\right)$?

The Unit Circle

You already know how to translate between degrees and radians and the triangle ratios for 30-60-90 and 45-45-90 right triangles. In order to be ready to completely fill in and memorize a unit circle, two triangles need to be worked out. Start by finding the side lengths of a 30-60-90 triangle and a 45-45-90 triangle each with hypotenuse equal to 1.

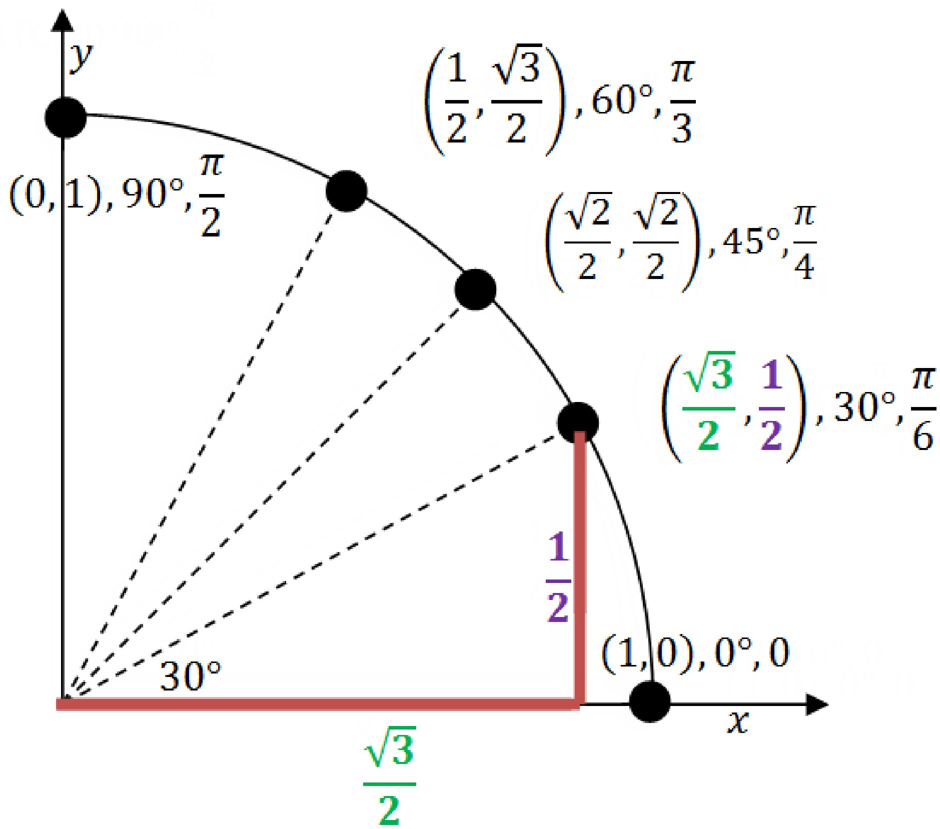
TABLE 1.17:

30°	60°	90°
x	$x\sqrt{3}$	$2x$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

TABLE 1.18:

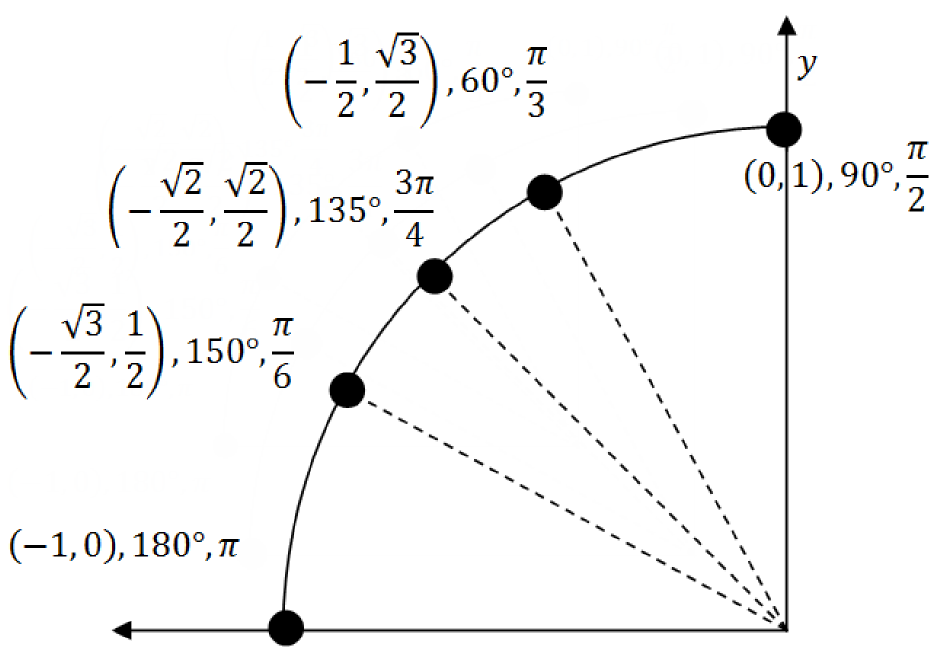
45°	45°	90°
x	x	$x\sqrt{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1

This is enough information to fill out the important points in the first quadrant of the unit circle. The values of the x and y coordinates for each of the key points are shown below. Remember that the x and y coordinates come from the lengths of the legs of the special right triangles, as shown specifically for the 30° angle. Always remember to measure the angle from the positive portion of the x-axis.



Knowing the first quadrant well is the key to knowing the entire unit circle. Every other point on the unit circle can be found using logic and this quadrant.

To use your knowledge of the first quadrant of the unit circle to identify the angles and important points of the second quadrant, notice that the heights are mirrored and equal which correspond to the y values. The x values are all negative.

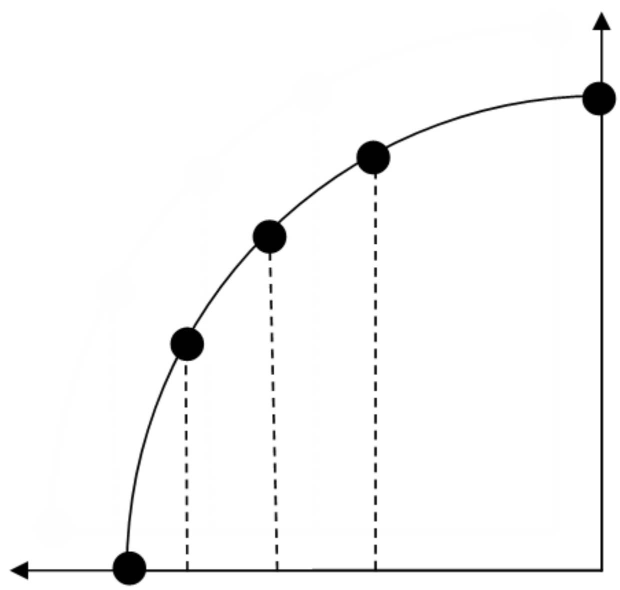


There is a pattern in the heights of the points in the first quadrant that can help you remember the points.

Notice that the heights of the points in the first quadrant are the y-coordinates: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$

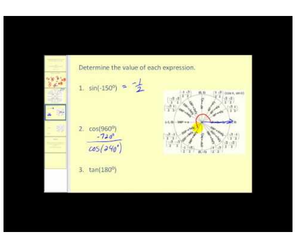
When rewritten, the pattern becomes clear: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

The three points in the middle are the most often mixed up. This pattern illustrates how they increase in size from small $\frac{1}{2}$, to medium $\frac{\sqrt{2}}{2}$, to large $\frac{\sqrt{3}}{2}$. When you fill in the unit circle, look for the heights that are small, medium and large and this will tell you were each value should go. Notice that the heights for these five points in the second quadrant are also $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$.



This technique also works for the widths. This can make memorizing the 16 points of the unit circle a matter of logic and the pattern: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

One last item to note is that **coterminal angles** are sets of angles such as 90º, 450º, and -270º that start at the positive x-axis and end at the same terminal side. Since coterminal angles end at identical points along the unit circle, trigonometric expressions involving coterminal angles are equivalent: $\sin(90) = \sin(450) = \sin(-270)$.



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Examples

Example 1

Earlier, you were asked how you can use the unit circle to evaluate $\cos(135)$and $\sin\left(-\frac{5\pi}{3}\right)$. The x value of a point along the unit circle corresponds to the cosine of the angle. The y value of a point corresponds to the sine of the angle. When the angles and points are memorized, simply recall the x or y coordinate.

When evaluating $\cos(135^\circ)$ your thought process should be something like this:

You know 135° goes with the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and cosine is the x portion. So, $\cos(135^\circ) = -\frac{\sqrt{2}}{2}$.

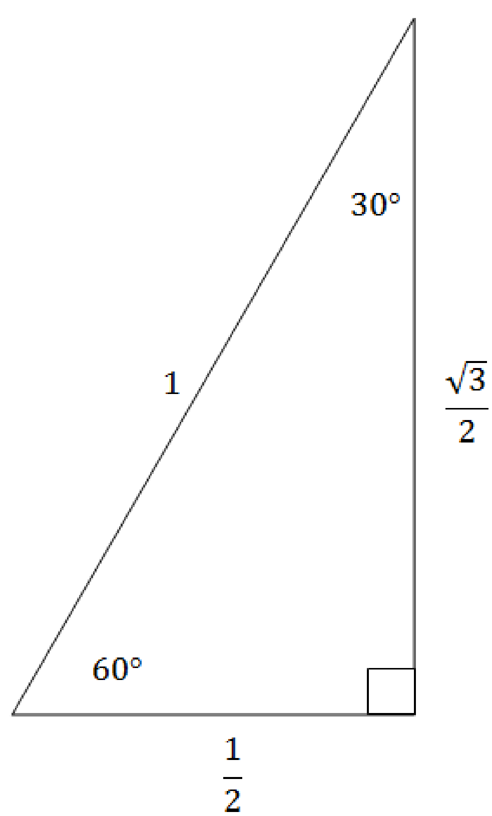
When evaluating $\sin\left(-\frac{5\pi}{3}\right)$ your thought process should be something like this:

You know $-\frac{5\pi}{3}$ goes with the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and sine is the y portion. So, $\sin\left(-\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example 2

Evaluate $\cos 60^\circ$ using the unit circle and right triangle trigonometry. What is the connection between the x coordinate of the point and the cosine of the angle?

The point on the unit circle for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the point is one unit from the origin. This can be represented as a 30-60-90 triangle.

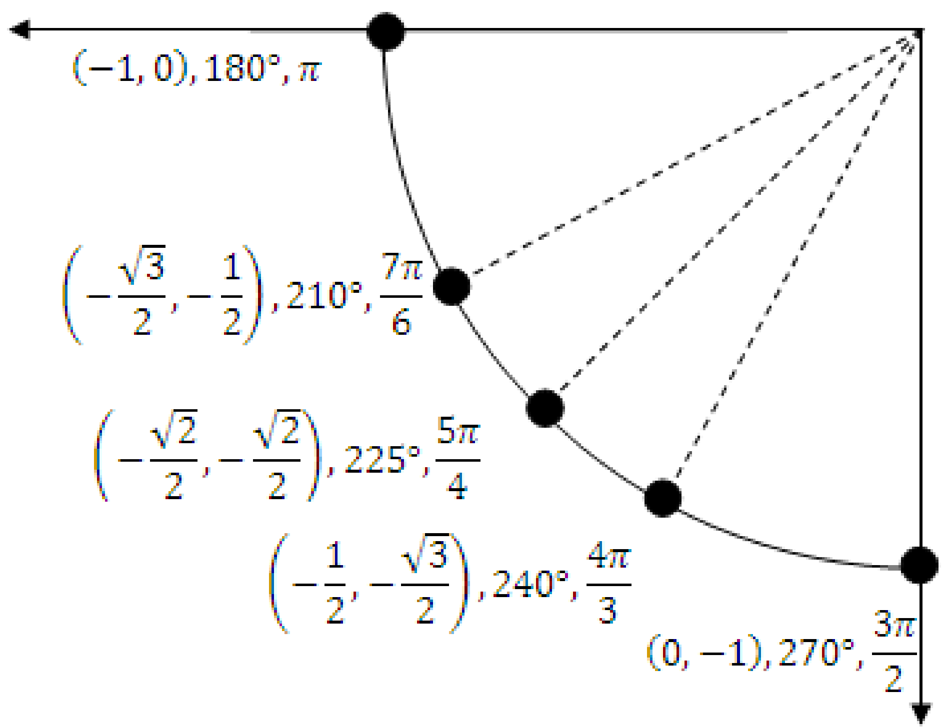


Since cosine is adjacent over hypotenuse, cosine turns out to be exactly the x coordinate $\frac{1}{2}$.

Example 3

Using knowledge of the first quadrant of the unit circle, identify the angles and important points of the third quadrant.

Both the x values and y values are negative and their respective coordinates correspond to those of the other quadrants.



Example 4

For each of the six trigonometric functions, identify the quadrants where they are positive and the quadrants where they are negative.

In quadrant I, the hypotenuse, adjacent and opposite side are all positive. Thus all 6 trigonometric functions are positive.

In quadrant II the hypotenuse and opposite sides are positive and the adjacent side is negative. This means that every trigonometric expression involving an adjacent side is negative. Sine and its reciprocal cosecant are the only two trigonometric functions that do not refer to the adjacent side which makes them the only positive ones.

In quadrant III only the hypotenuse is positive. Thus the only trigonometric functions that are positive are tangent and its reciprocal cotangent because these functions refer to both adjacent and opposite sides which will both be negative.

In quadrant IV the hypotenuse and the adjacent sides are positive while the opposite side is negative. This means that only cosine and its reciprocal secant are positive.

A mnemonic device to remember which trigonometric functions are positive and which trigonometric functions are negative is "All Students Take Calculus." All refers to all the trigonometric functions are positive in quadrant I. The letter S refers to sine and its reciprocal cosecant that are positive in quadrant II. The letter T refers to tangent and its reciprocal cotangent that are positive in quadrant III. The letter C refers to cosine and its reciprocal secant that are positive in quadrant IV.

Example 5

Evaluate the following trigonometric expressions using the unit circle.

1. $\sin \frac{\pi}{2}$

$\sin \frac{\pi}{2} = 1$

2. $\cos 210^\circ$

$\cos 210^\circ = -\frac{\sqrt{3}}{2}$

3. $\tan 315^\circ$

$\tan 315^\circ = -1$

4. $\cot 270^\circ$

$\cot 270^\circ = 0$

5. $\sec \frac{11\pi}{6}$

$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

6. $\csc -\frac{5\pi}{4}$

$\csc -\frac{5\pi}{4} = \frac{1}{\sin -\frac{5\pi}{4}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

Review

Name the angle between 0° and 360° that is coterminal with...

1. -20°

2. 475°

3. -220°

4. 690°

5. -45°

Use your knowledge of the unit circle to help determine whether each of the following trigonometric expressions is positive or negative.

6. $\tan 143^\circ$

7. $\cos \frac{\pi}{3}$

8. $\sin 362^\circ$

9. $\csc \frac{3\pi}{4}$

Use your knowledge of the unit circle to evaluate each of the following trigonometric expressions.

10. $\cos 120^\circ$

11. $\sec \frac{\pi}{3}$
12. $\tan 225^\circ$
13. $\cot 120^\circ$
14. $\sin \frac{11\pi}{6}$
15. $\csc 240^\circ$
16. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta > 0$.
17. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = -\frac{1}{2}$ and $\sec\theta < 0$.
18. Draw the complete unit circle (all four quadrants) and label the important points.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.1.

1.15 The Sinusoidal Function Family

Learning Objectives

Here you will see how the graphs of sine and cosine come from the unit circle.

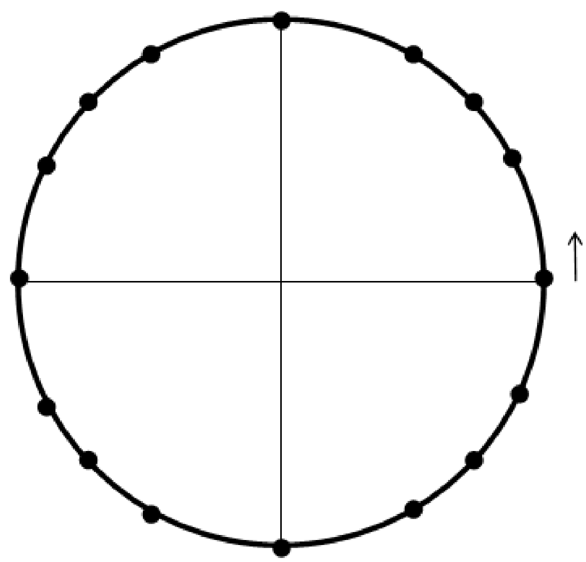
The **cosine function is the** x **coordinates** of the unit circle and the **sine function is the** y **coordinates**. Since the unit circle has radius one and is centered at the origin, both sine and cosine oscillate between positive and negative one.

What happens when the circle is not centered at the origin and does not have a radius of 1?

Graphs of Sinusoidal Functions

The **sinusoidal function** family refers to either sine or cosine waves since they are the same except for a horizontal shift. This function family is also called the periodic function family because the function repeats after a given period of time.

Consider a Ferris wheel that spins evenly with a radius of 1 unit. It starts at (1, 0) or an angle of 0 radians and spins counterclockwise at a rate of one cycle per 2π minutes (so you can use time is equal to radians).



The 16 points around the circle are chosen because they correspond to the key points of the unit circle. Their heights (y-values) and widths (x-values) are already known and can be filled in.

First consider the height at each of the points as you travel around half of the circle from the starting location. Keep track of your work in a table.

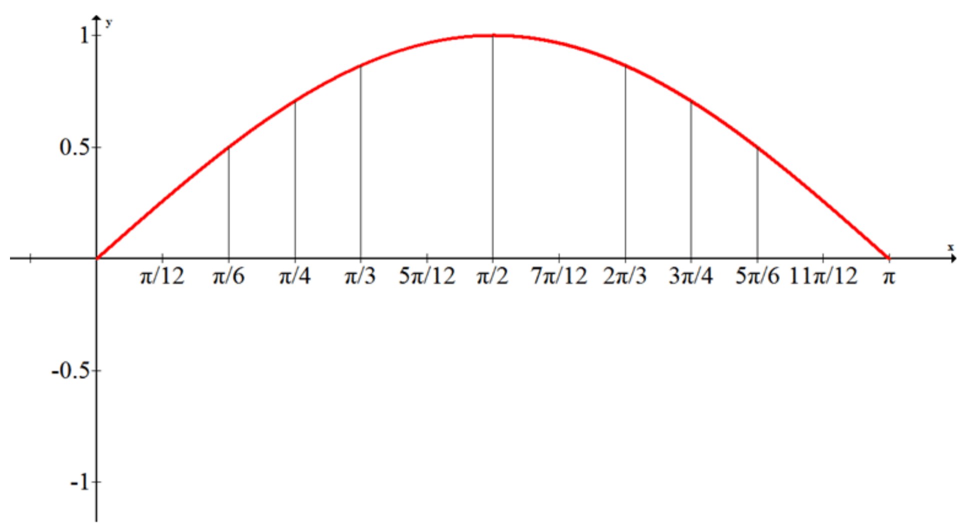
TABLE 1.19:

Angle (radians)	Height (units)
0	0

TABLE 1.19: (continued)

$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

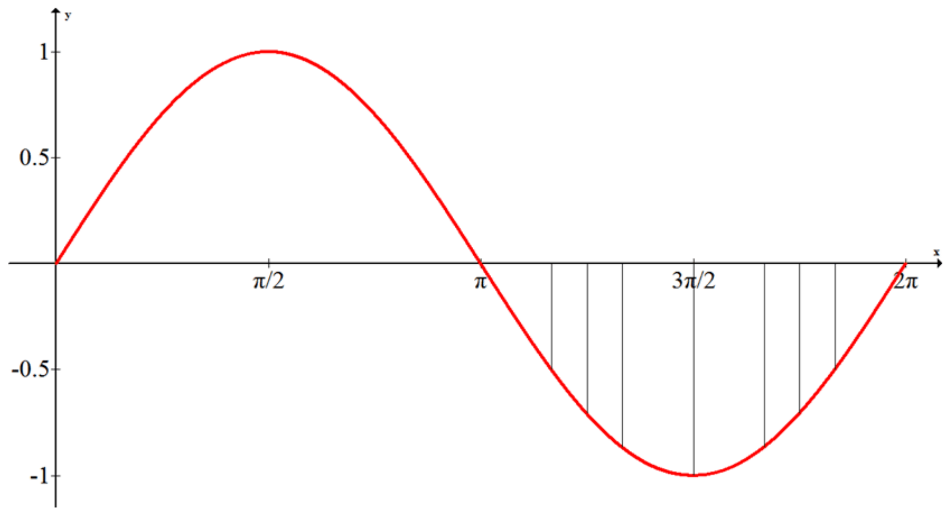
Notice the symmetry of the height around $\frac{\pi}{2}$ and see the rest of the table in the examples. Once the table is finished, you can plot these points on a regular coordinate plane where the x axis is the angle and the y axis is the height. This is the first part of the graph of the sine function.

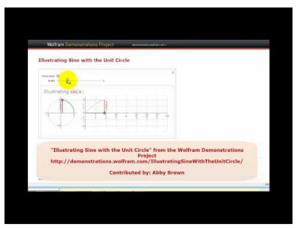


To finish the graph of the sine function, finish the table for heights of the points in quadrants III and IV and draw an entire cycle (known as a period) of the sine function.

TABLE 1.20:

Angle (radians)	**Height (units)**
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0





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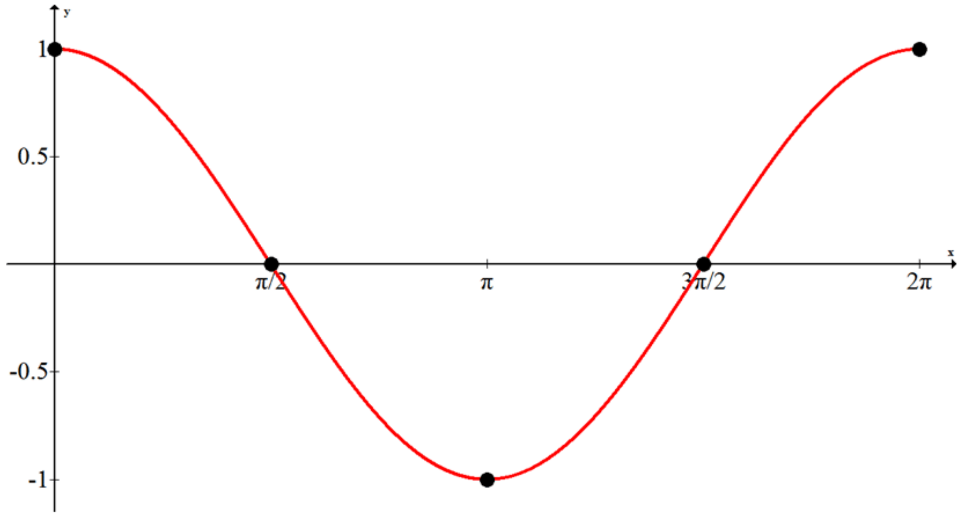
Similar to sine, you can use your knowledge of the angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ on the unit circle to get a complete cycle of the cosine graph. While the sine function uses the x-coordinates, the cosine function is the x-coordinates of the unit circle and measures width. By referring to a unit circle or your memory, you can fill out a much shorter table than before.

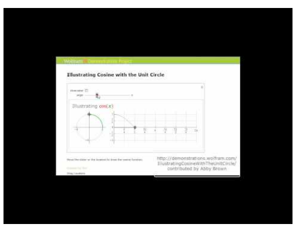
TABLE 1.21:

Angle (radians)	Width (units)
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

First plot these five points and then connect them with a smooth curve. This will produce the cosine graph.



Determining these five main points is the key to graphing sine or cosine graphs even when the graph is shifted or stretched.



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Examples

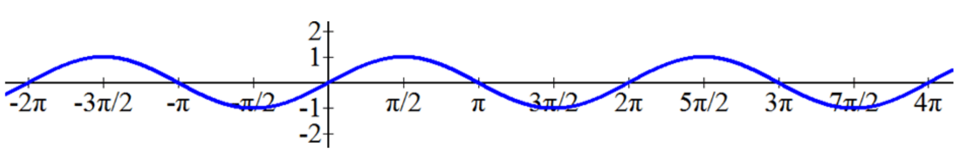
Example 1

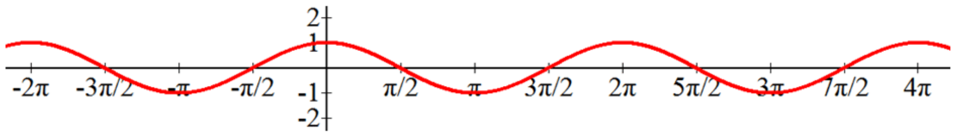
Earlier, you were asked what happens when the circle is not centered at the origin and does not have a radius of 1. The unit circle produces the parent function sine and cosine graphs. When the unit circle is shifted up or down, made wider or narrower, or spun faster or slower in either direction, then the graphs of the sine and cosine functions will be transformed using basic function transformation rules.

Example 2

What happens on either side of the sine and cosine graphs? Can you explain why?

The graphs of the sine (blue) and cosine (red) functions repeat forever in both directions.





If you think about the example with the Ferris wheel, the ride will keep on spinning and has been spinning forever. This is why the same cycle of the graph repeats over and over.

Example 3

How are the sine and cosine graphs the same and how are they different?

The sine graph is the same as the cosine graph offset by $\frac{\pi}{2}$. Besides the shift, both curves are identical due to the perfect symmetry of circles.

Example 4

Where are two maximums and two minimums of the sine graph?

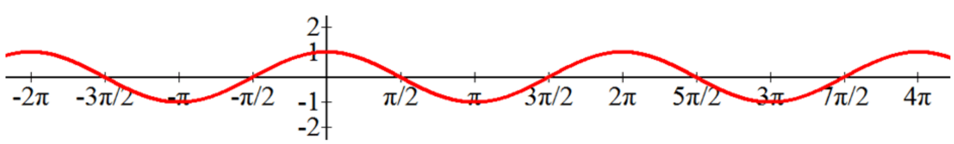
One maximum of the sine graph occurs at $\left(\frac{\pi}{2}, 1\right)$. One minimum occurs at $\left(\frac{3\pi}{2}, -1\right)$. This is one cycle of the sine graph. Since it completes a cycle every 2π, when you add 2π to an x-coordinate you will be on the same point of the cycle giving you another maximum or minimum.

$\left(\frac{5\pi}{2}, 1\right)$ is another maximum. $\left(\frac{7\pi}{2}, -1\right)$ is another minimum.

Example 5

In the interval $[-2\pi, 4\pi)$ where does cosine have zeroes?

Observe where the cosine curve has x-coordinates equal to zero. Note that 4π is excluded from the interval. The values are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.



Review

1. Sketch $p(x) = \sin x$ from memory.

2. Sketch $j(x) = \cos x$ from memory.

3. Where do the maximums of the cosine graph occur?

4. Where do the minimums of the cosine graph occur?

5. Find all the zeroes of the sine function on the interval $\left[-\pi, \frac{5\pi}{2}\right]$.

6. Find all the zeroes of the cosine function on the interval $\left(-\frac{\pi}{2}, \frac{7\pi}{2}\right]$.

7. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $f(x) = 2\cos x$ will look like.

8. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph $g(x) = \cos x + 2$ of will look like.

9. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $h(x) = \cos(x - \pi)$ will look like.

10. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $k(x) = -\cos x$ will look like.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.2.

1.16 Amplitude of Sinusoidal Functions

Learning Objectives

Here you will see how changing the radius of a circle affects the graph of the sine function through a vertical stretch.

The **amplitude** of the sine and cosine functions is the vertical distance between the sinusoidal axis and the maximum or minimum value of the function. In relation to sound waves, amplitude is a measure of how loud something is.

What is the most common mistake made when graphing the amplitude of a sine wave?

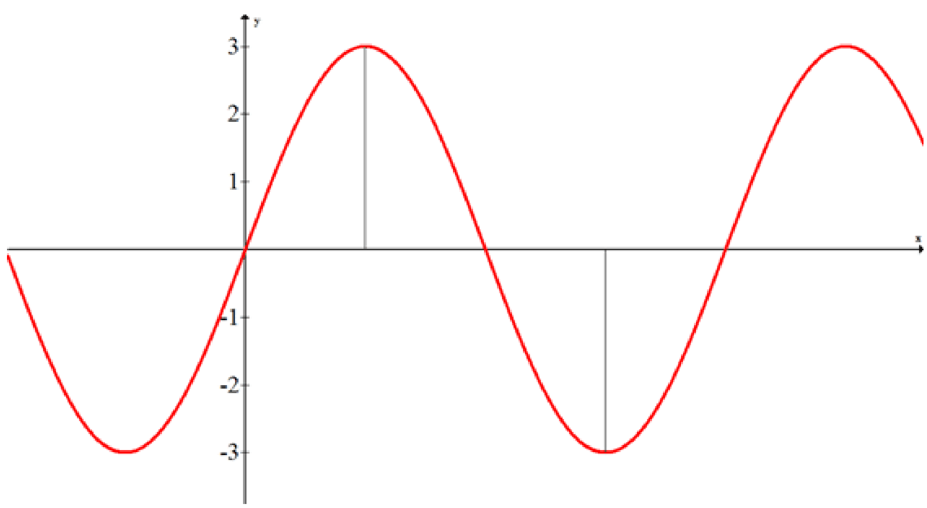
Amplitude of Sinusoidal Functions

The general form a sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

The cosine function can just as easily be substituted and for many problems it will be easier to use a cosine equation. Since both the sine and cosine waves are identical except for a horizontal shift, it all depends on where you see the wave starting.

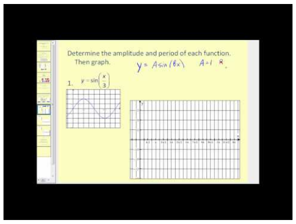
The coefficient a is the amplitude (which fortunately also starts with the letter a). When there is no number present, then the amplitude is 1. The best way to define amplitude is through a picture. Below is the graph of the function $f(x) = 3 \cdot \sin x$, which has an amplitude of 3.



Notice that the amplitude is 3, not 6. This corresponds to the absolute value of the maximum and minimum values of the function. If the function had been $f(x) = -3 \cdot \sin x$, then the whole graph would be reflected across the x axis.

Also notice that the x axis on the graph above is unlabeled. This is to show that amplitude is a vertical distance. The **sinusoidal axis** is the neutral horizontal line that lies between the **crests** and the **troughs** (or peaks and valleys if you prefer). For this function, the sinusoidal axis was just the x axis, but if the whole graph were shifted up, the sinusoidal axis would no longer be the x axis. Instead, it would still be the horizontal line directly between the crests and troughs.

Watch the portion of this video discussing amplitude:



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Examples

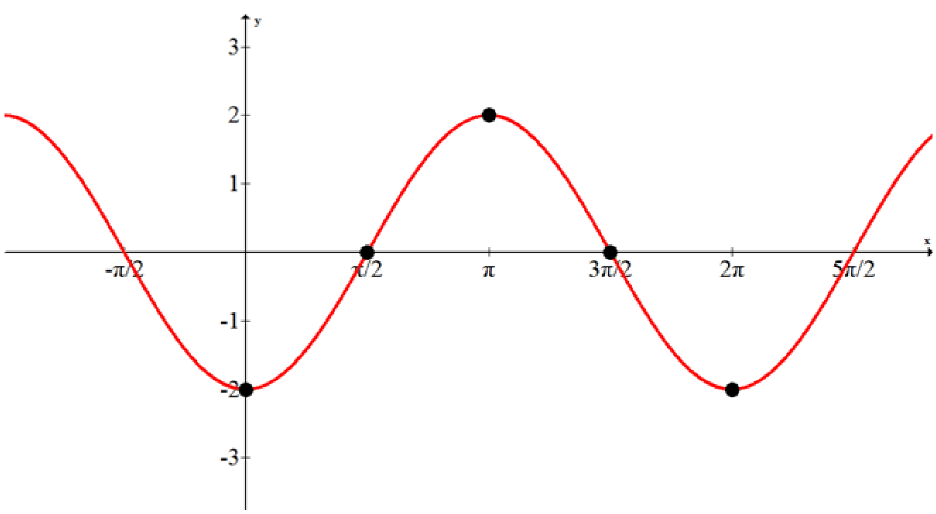
Example 1

Earlier, you were asked about the most common mistake made when graphing the amplitude of a singe wave. The most common mistake is doubling or halving the amplitude unnecessarily. Many people forget that the number a in the equation, like the 3 in $f(x) = 3\sin x$, is the distance from the x axis to both the peak and the valley. It is not the total distance from the peak to the valley.

Example 2

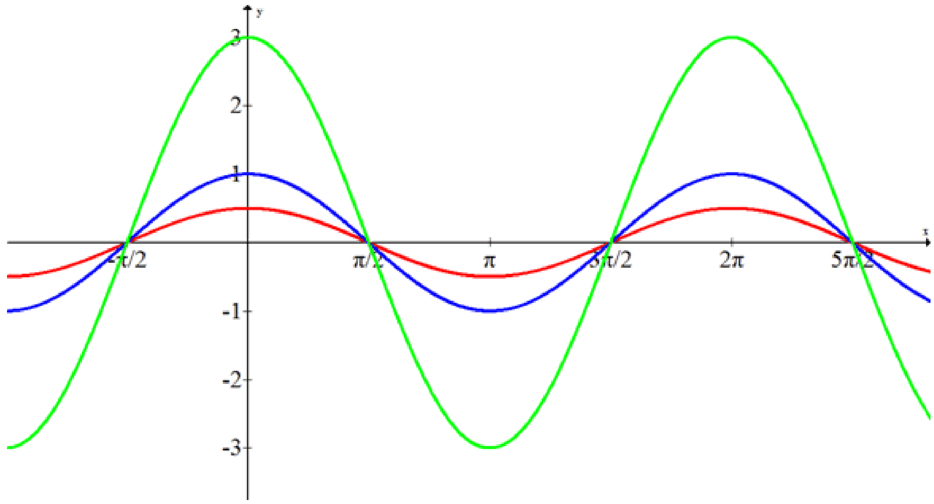
Graph the following function by first plotting main points: $f(x) = -2 \cdot \cos x$.

The amplitude is 2, which means the maximum values will be at 2 and the minimum values will be at -2. Normally with a basic cosine curve the points corresponding to $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ fall above, on or below the line in the following sequence: above, on, below, on, above. The negative sign switches above with below. The whole graph is reflected across the x-axis.



Example 3

Write a cosine equation for each of the following functions.



The amplitudes of the three functions are 3, 1 and $\frac{1}{2}$ and none of them are reflected across the x-axis.

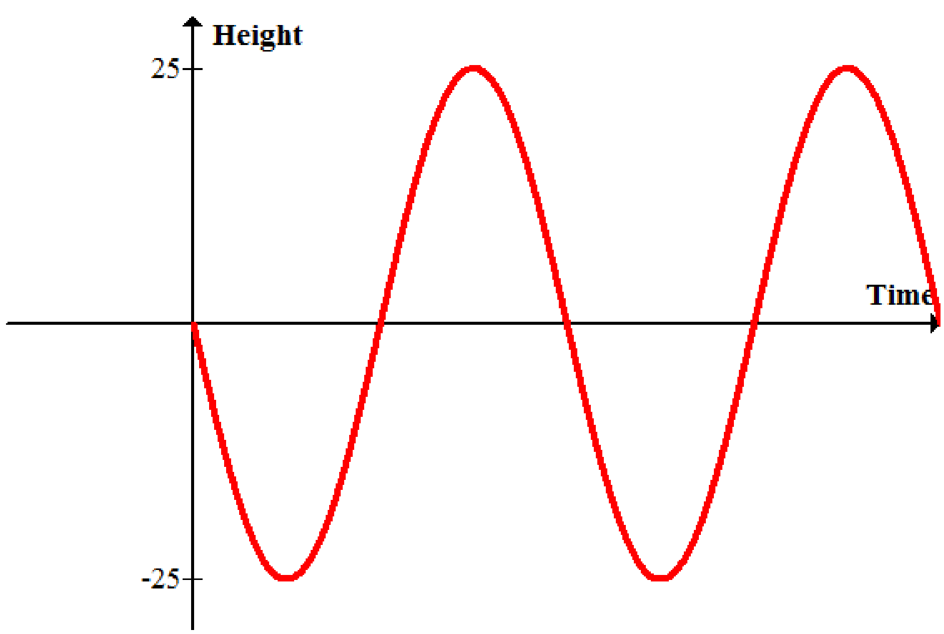
$$f(x) = 3 \cdot \cos x$$
$$h(x) = \cos x$$
$$g(x) = \frac{1}{2} \cdot \cos x$$

Note that amplitude itself is always positive.

Example 4

A Ferris wheel with radius 25 feet sits next to a platform. The ride starts at the platform and travels down to start. Model the height versus time of the ride.

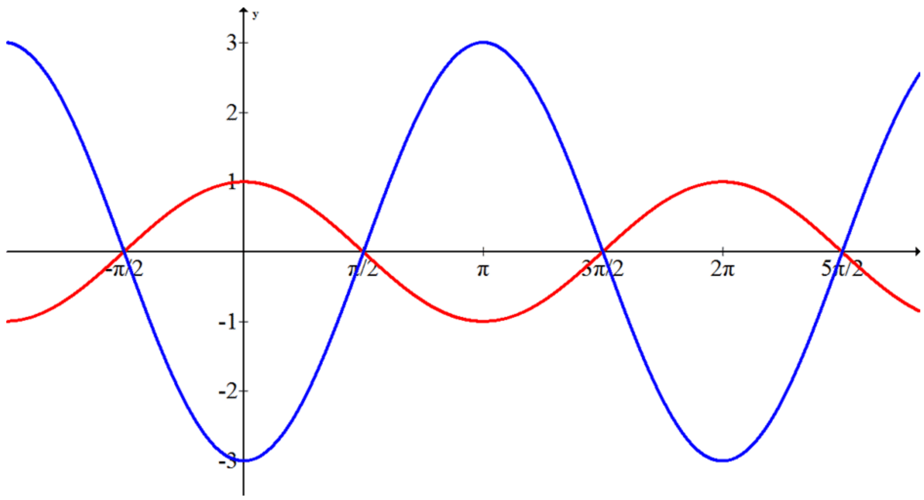
Since no information is given about the time, simply label the x axis as time. At time zero the height is zero. Initially the height will decrease as the ride goes below the platform. Eventually, the wheel will find the minimum and start to increase again all the way until it reaches a maximum.



Example 5

Find the amplitude of the function $f(x) = -3\cos x$ and use the language of transformations to describe how the graph is related to the parent function $y = \cos x$.

The new function is reflected across the x axis and vertically stretched by a factor of 3.



Review

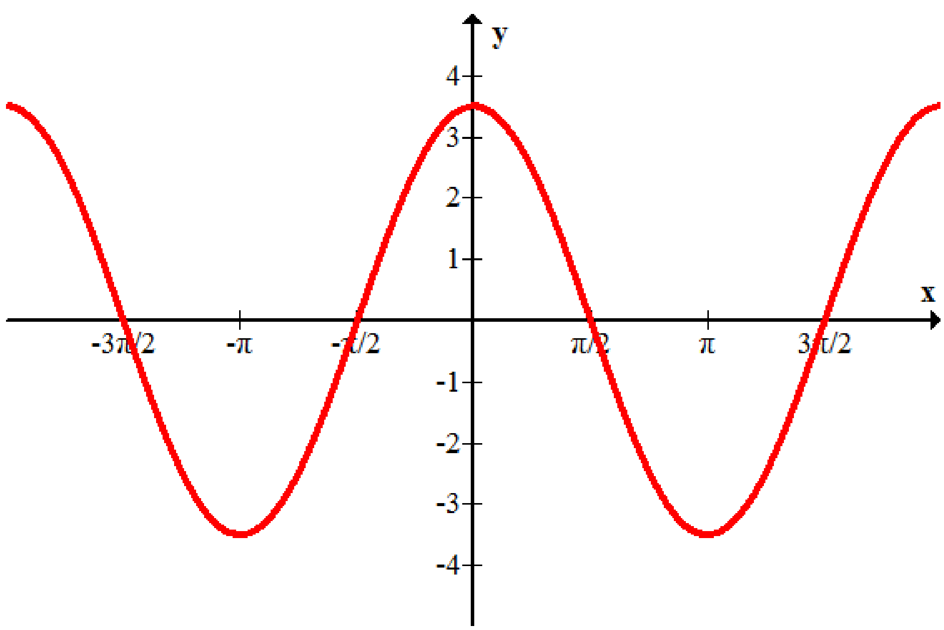
1. Explain how to find the amplitude of a sinusoidal function from its equation.

2. Explain how to find the amplitude of a sinusoidal function from its graph.

Find the amplitude of each of the following functions.

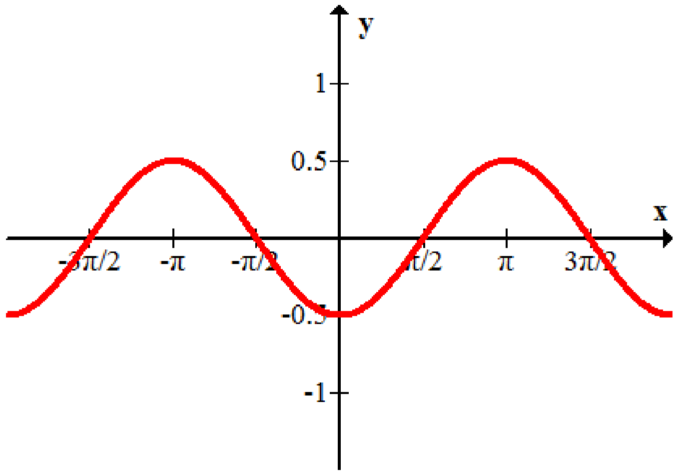
3. $g(x) = -5\cos x$

4.



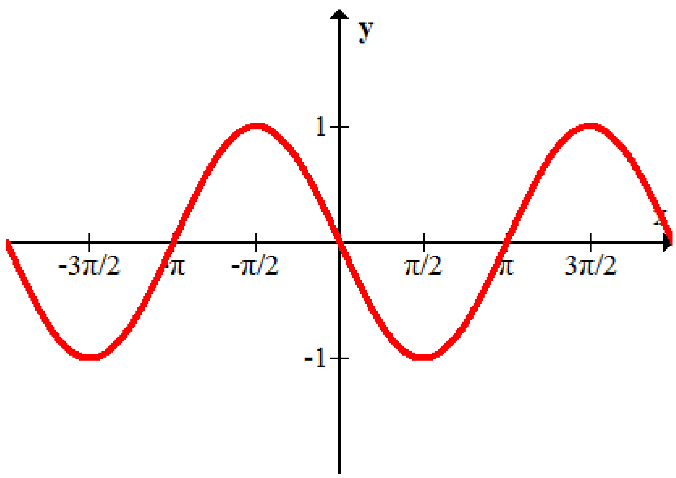
5. $f(x) = \frac{1}{2}\sin x$

6.



7. $j(x) = 3.12 \cos x$

8.



Sketch each of the following functions.

9. $f(x) = 3 \sin x$

10. $g(x) = -4 \cos x$

11. $h(x) = \pi \sin x$

12. $k(x) = -1.2 \cos x$

13. $p(x) = \frac{2}{3} \cos x$

14. $m(x) = -\frac{1}{2} \sin x$

15. Preview: $r(x) = 3 \sin x + 2$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.3.

1.17 Vertical Shift of Sinusoidal Functions

Learning Objectives

Here you will explore the how a vertical shift of a sinusoidal function is represented in an equation and in a graph.

Your knowledge of transformations, specifically vertical shift, apply directly to sinusoidal functions. In practice, sketching shifted sine and cosine functions requires greater attention to detail and more careful labeling than other functions. Can you describe the following transformation in words?

$f(x) = \sin x \rightarrow g(x) = -3\sin x - 4$

In what order do the reflection, stretch and shift occur? Is there a difference?

Vertical Shift of Sinusoidal Functions

The general form of a sinusoidal function is:

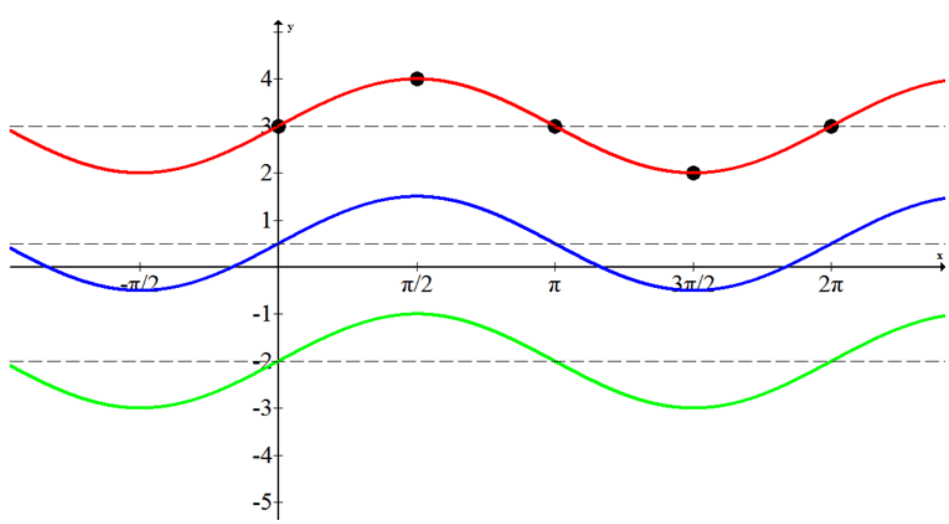
$f(x) = \pm a \cdot \sin(b(x+c)) + d$

Recall that a controls amplitude and the $\pm$ controls reflection. Here you will see how d controls the vertical shift.

The most straightforward way to think about vertical shift of sinusoidal functions is to focus on the **sinusoidal axis**, the horizontal line running through the middle of the sine or cosine wave. At the start of the problem identify the vertical shift and immediately draw the new sinusoidal axis. Then proceed to graph amplitude and reflection about that axis as opposed to the x axis.

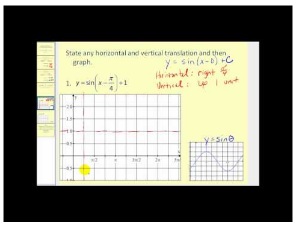
The graphs of the following three functions are shown below:

$$f(x) = \sin x + 3$$
$$g(x) = \sin x - 2$$
$$h(x) = \sin x + \frac{1}{2}$$



To draw these graphs, the new sinusoidal axis for each graph is drawn first. Then, a complete sine wave for each one is drawn. Note the five important points that separate each quadrant to help to get a clear sense of the graph. There are no reflections in these graphs nor and they all have an amplitude of 1. Right now every cycle starts at 0 and ends at 2π but this will not always be the case.

Watch the portions of the following video focused on vertical translations:



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Examples

Example 1

Earlier, you were asked which order vertical shift and reflection should be performed in and if it matters. The following transformation can be described in basically two ways.

$f(x) = \sin x \rightarrow g(x) = -3\sin x - 4$

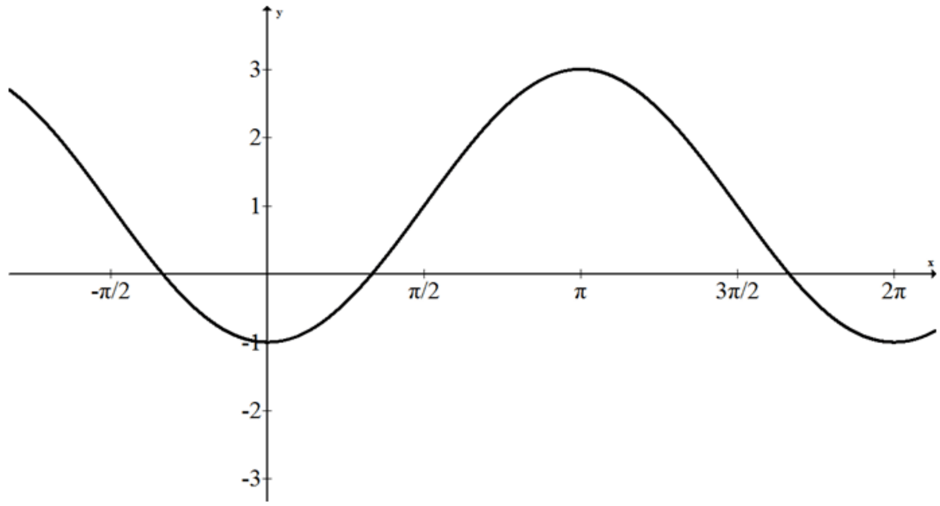
The first is to describe the stretching and reflecting first and then the vertical shift. This is the most logical way to discuss the transformation verbally because then the numbers like 3 and -4 can be explicitly identified in the graph.

The second way to describe the transformation is to attempt to say the vertical shift first. In this case the vertical shift would initially be $-\frac{4}{3}$, and then the vertical stretch would magnify this distance from the x-axis. This is significantly less intuitive. If a description showed the vertical shift to be -4 initially followed by a stretch by a factor of 3, the sinusoidal axis would move to $y = 12$ which is incorrect.

The order in describing the transformation matters. When describing vertical transformations it is most intuitive to simply describe the transformations in the same order as the order of operations.

Example 2

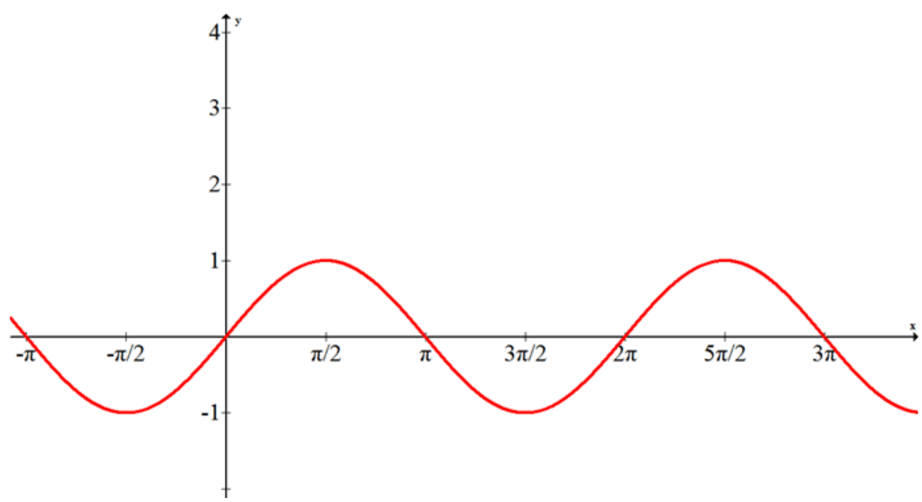
Identify the equation of the following transformed cosine graph.



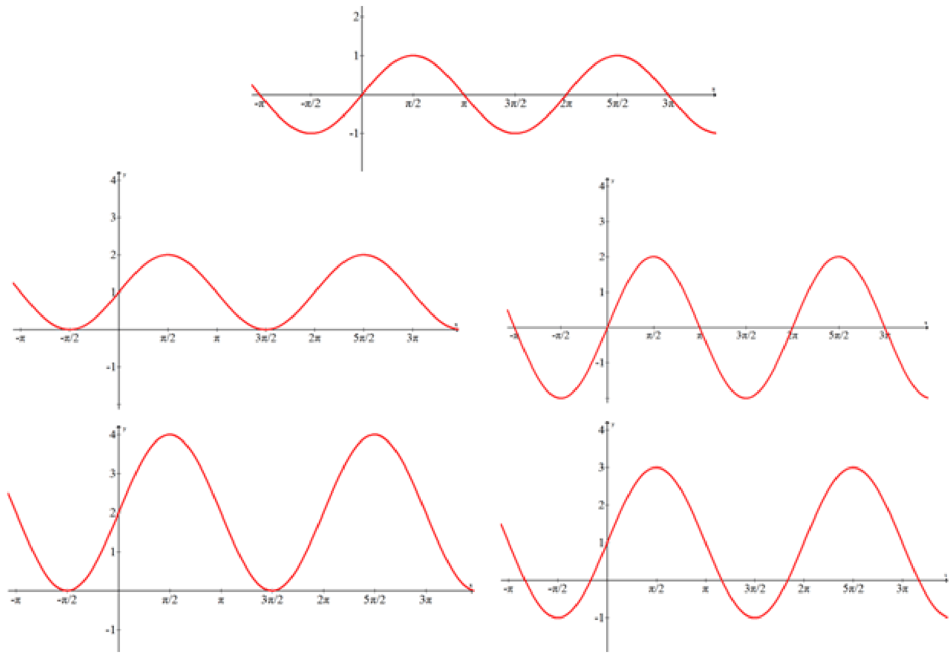
Since there is no sinusoidal axis given, you must determine the vertical shift, stretch and reflection. The peak occurs at $(\pi, 3)$ and the trough occurs at (0, -1) so the horizontal line directly between +3 and -1 is $y = 1$. Since the sinusoidal axis has been shifted up by one unit $d = 1$. From this height, the graph goes two above and two below which means that the amplitude is 2. Since this cosine graph starts its cycle at (0, -1) which is a lowest point, it is a negative cosine. The function is $f(x) = -2\cos x + 1$.

Example 3

Transform the following sine graph in two ways. First, transform the sine graph by shifting it vertically up 1 unit and then stretching it vertically by a factor of 2 units. Second, transform the sine graph by stretching it vertically by a factor of 2 units and then shifting it vertically up 1 unit.

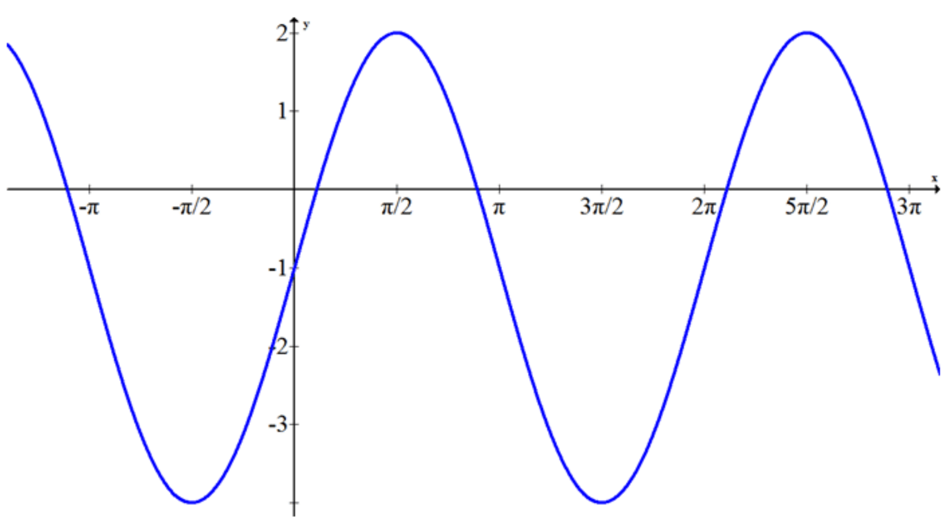


When doing ordered transformations it is good to show where you start and where you end up so that you can effectively compare and contrast the outcomes. See how both transformations start with a regular sine wave. The two columns represent the sequence of transformations that produce different outcomes.



Example 4

What equation models the following graph?

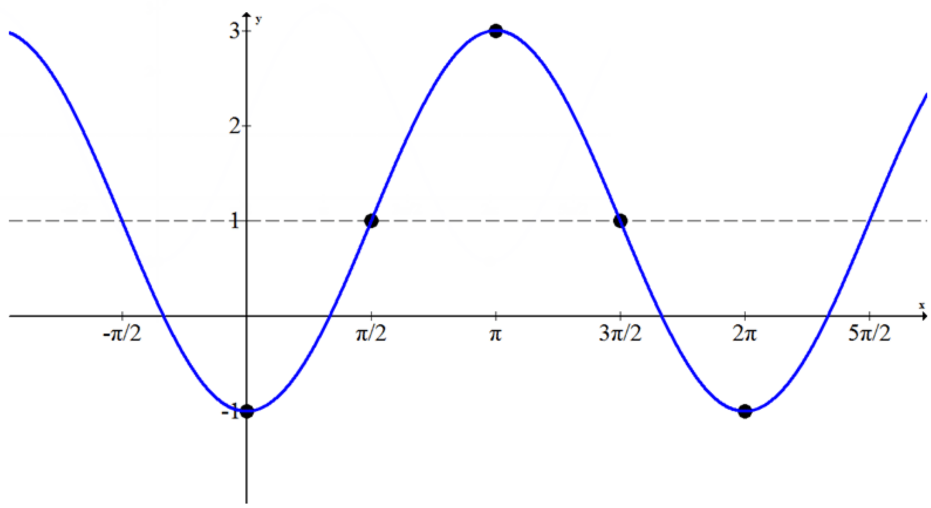


$f(x) = 3 \cdot \sin x - 1$

Example 5

Graph the following function: $f(x) = -2 \cdot \cos x + 1$.

First draw the horizontal sinusoidal axis and identify the five main points for the cosine wave. Be careful to note that the amplitude is 2 and the cosine wave starts and ends at a low point because of the negative sign.



Review

Graph each of the following functions that have undergone a vertical stretch, reflection, and/or a vertical shift.

1. $f(x) = -2\sin x + 4$

2. $g(x) = \frac{1}{2}\cos x - 1$

3. $h(x) = 3\sin x + 2$

4. $j(x) = -1.5\cos x + \frac{1}{2}$

5. $k(x) = \frac{2}{3}\sin x - 3$

Find the minimum and maximum values of each of the following functions.

6. $f(x) = -3\sin x + 1$

7. $g(x) = 2\cos x - 4$

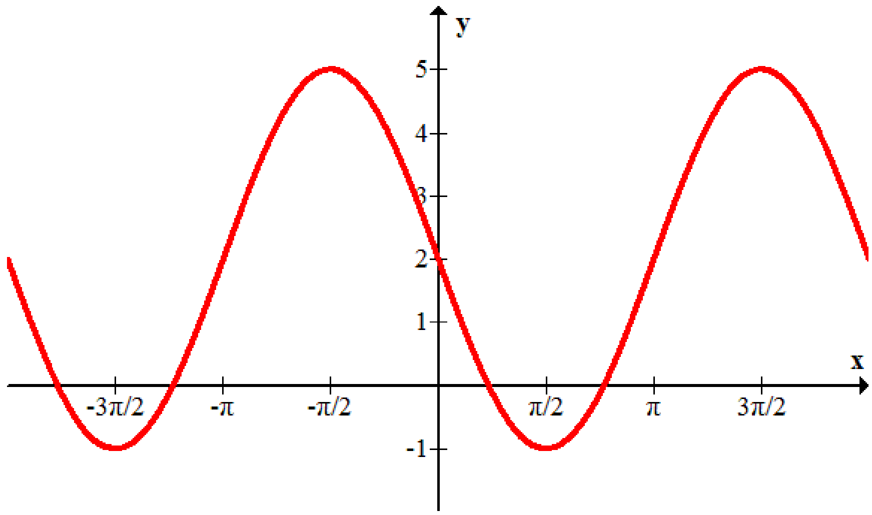
8. $h(x) = \frac{1}{2}\sin x + 1$

9. $j(x) = -\cos x + 5$

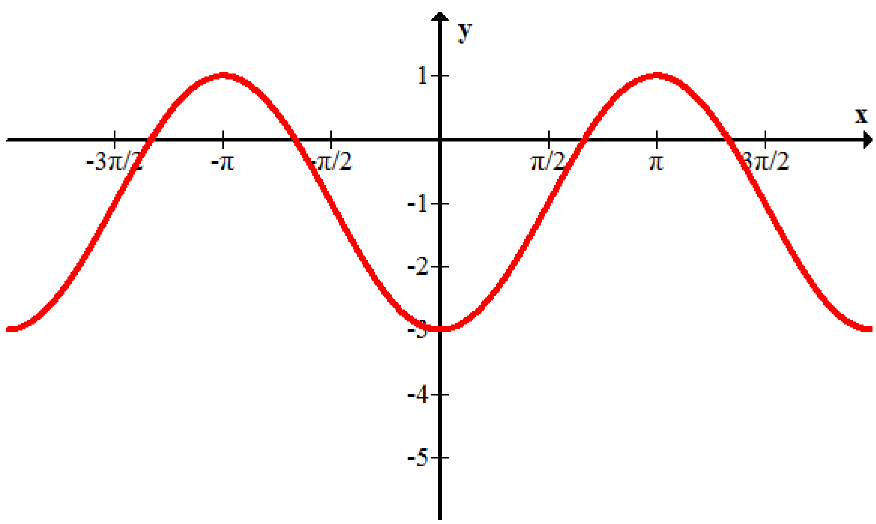
10. $k(x) = \sin(x) - 1$

Give the equation of each function graphed below.

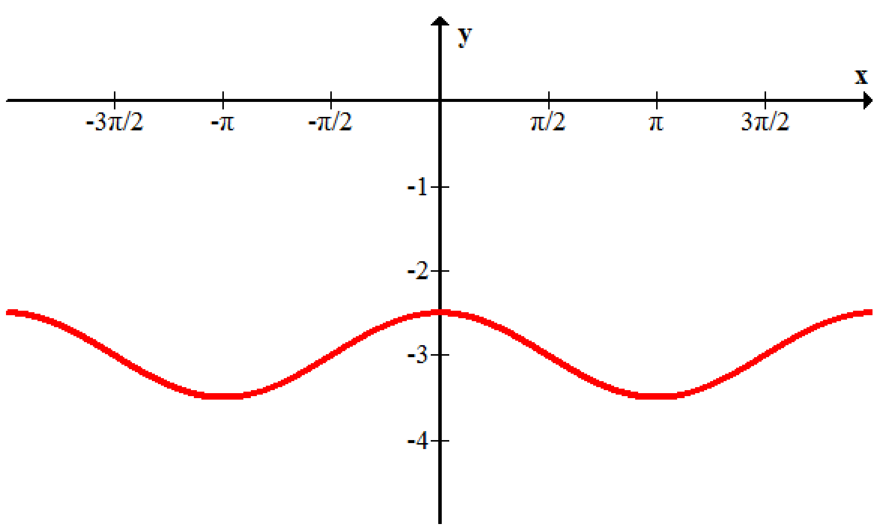
11.


y
5
4
3
2
1
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1

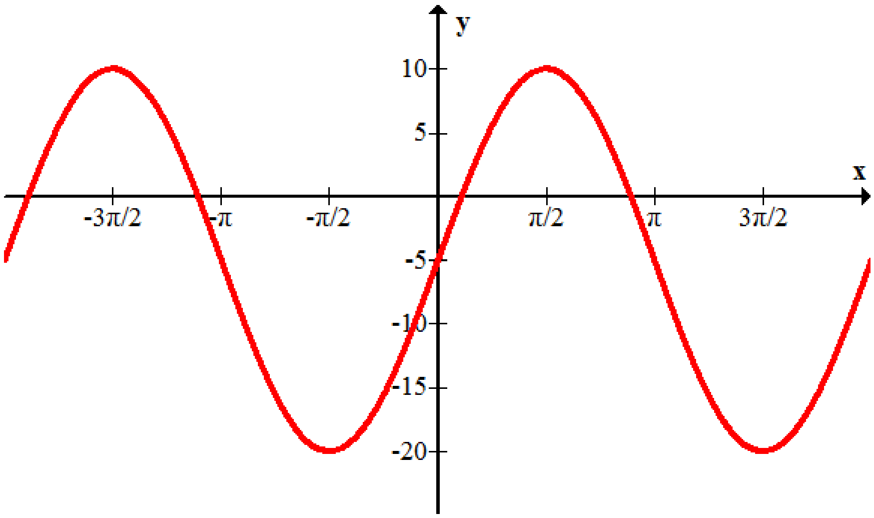
12,


y
1
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1
-2
-3
-4
-5

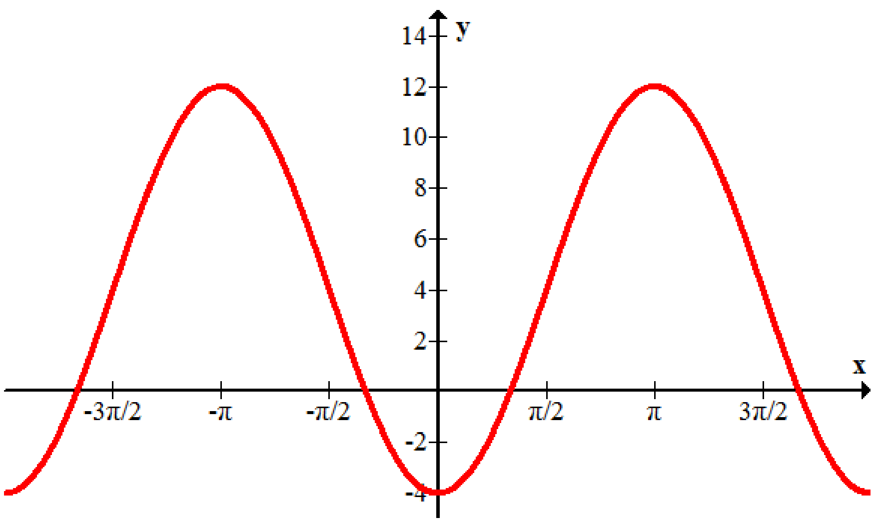
13.


y
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1
-2
-3
-4

14.



15.



Review (Answers)

To see the Review answers, open this PDF file and look for section 5.4.

1.18 Frequency and Period of Sinusoidal Functions

Learning Objectives

Here you will apply your knowledge of horizontal stretching transformations to sine and cosine functions.

The transformation rules about horizontal stretching and shrinking directly apply to sine and cosine graphs. If a sine graph is horizontally stretched by a factor of $\frac{1}{2}$ that is the same as a horizontal compression by a factor of 2.

How does the equation change when a sine or cosine graph is stretched by a factor of 3?

Period and Frequency of Sinusoidal Functions

The general equation for a sinusoidal function is:

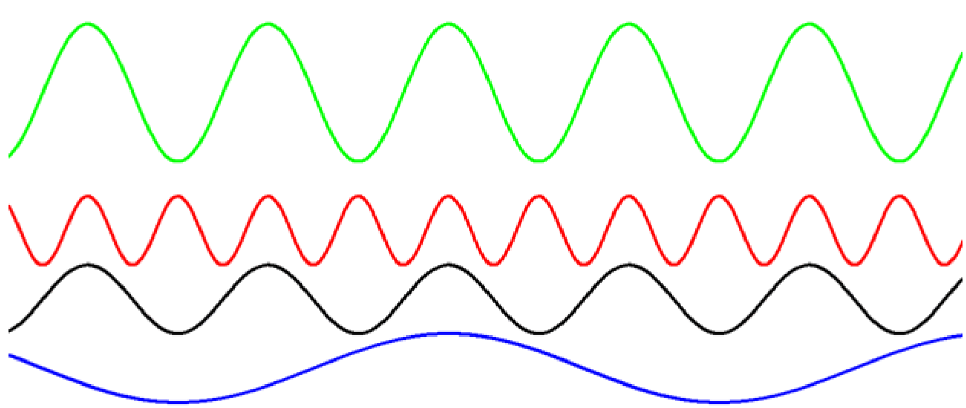
$f(x) = \pm a \cdot \sin(b(x+c)) + d$

The $\pm$ controls the reflection across the x-axis. The coefficient a controls the amplitude. The constant d controls the vertical shift. Here you will see that the coefficient b controls the horizontal stretch.

Period

Horizontal stretch is measured for sinusoidal functions as their periods. This is why this function family is also called the periodic function family. The **period** of a sinusoidal is the length of a complete cycle. For basic sine and cosine functions, the period is 2π. This length can be measured in multiple ways. In word problems and in other tricky circumstances, it may be most useful to measure from peak to peak.

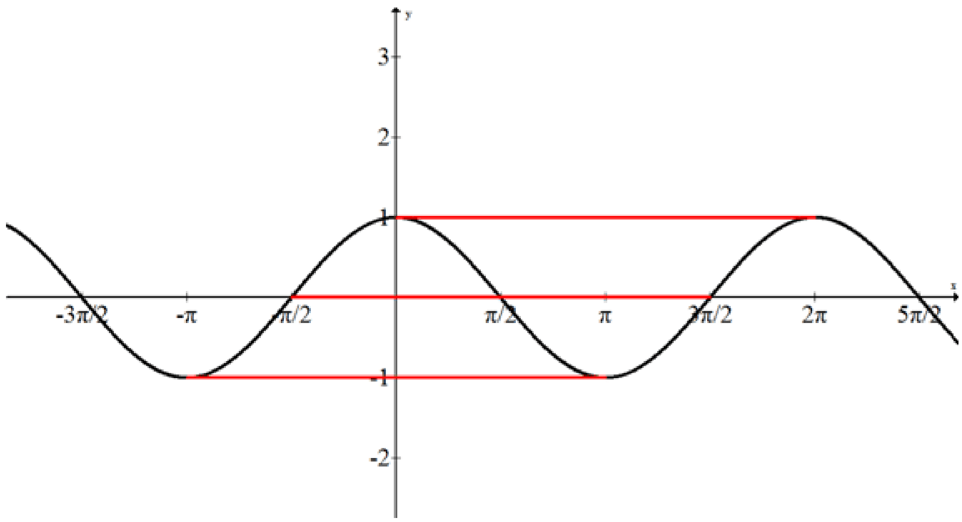
The following three waves have different periods. To rank each wave by period from shortest to longest, look at he distance between each peak.



The red wave has the shortest period.

The green and black waves have equal periods. A common mistake is to see that the green wave has greater amplitude and confuse that with greater periods.

The blue wave has the longest period.



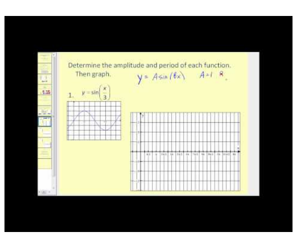
The ability to measure the period of a function in multiple ways allows different equations to model an identical graph. In the image above, the top red line would represent a regular cosine wave. The center red line would represent a regular sine wave with a horizontal shift. The bottom red line would represent a negative cosine wave with a horizontal shift. This flexibility in perspective means that many of the examples, guided practice and practice problems may have multiple solutions. For now, try to always choose the function that has a period starting at $x = 0$.

Frequency

Frequency is a different way of measuring horizontal stretch. For sound, frequency is known as pitch. With sinusoidal functions, **frequency** is the number of cycles that occur in 2π. A shorter period means more cycles can fit in 2π and thus a higher frequency. Period and frequency are inversely related by the equation:

$\text{period} = \frac{2\pi}{\text{frequency}}$

The equation of a basic sine function is $f(x) = \sin x$. In this case b, the frequency, is equal to 1 which means one cycle occurs in 2π. This relationship is a common mistake in graphing sinusoidal functions. Students find $b = \frac{1}{2}$ and then mistakenly conclude that the period is $\frac{1}{2}$ when it is in fact stretched to 4π.



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Examples

Example 1

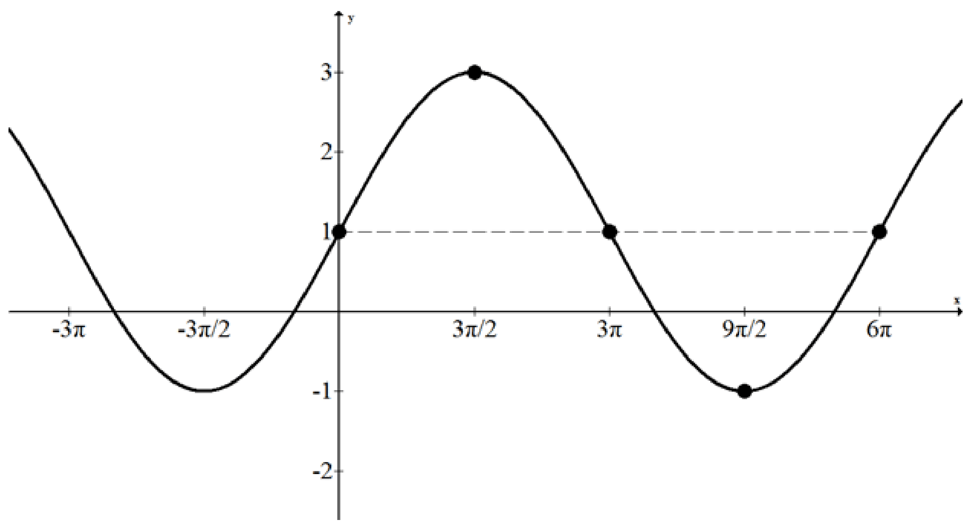
Earlier, you were asked how an equation changes when a sine or cosine graph is stretched by a factor of 3. If a sine graph is horizontally stretched by a factor of 3 then the general equation has $b = \frac{1}{3}$. This is because b is the frequency and counts the number (or fraction) of a period that fits in a normal period of 2π. Graphically, the sine wave will make a complete cycle in 6π. Similarly, a cosine graph will have $b = \frac{1}{3}$ and will have a period of 6π.

Example 2

Identify the amplitude, vertical shift, period and frequency of the following function. Then graph the function.

$f(x) = 2\sin\left(\frac{x}{3}\right) + 1$

$a = 2, b = \frac{1}{3}, d = 1$. Since $b = \frac{1}{3}$ (frequency), then the period must be 6π.



Often the most challenging part of graphing periodic functions is labeling the axes. Since the period is 6π, start by drawing the sinusoidal axis shifted appropriately. Then divide the 6π into four parts so that the 5 guiding points of the sine graph can be plotted with the amplitude and reflection in mind. The very last thing to do is to draw and extend the curve. Many students try to draw the curve too early and end up having to redo their work.

Example 3

A measuring stick on a dock measures high tide to be 18 feet and low tide to be 6 feet. It takes about 6 hours for the tide to switch between low and high tides. Determine a graphical and algebraic model for the tides knowing that at $t = 0$ there is a high tide.

Usually the best course of action for word problems is to identify information, plot points, sketch and then finally come up with an equation.

From the given information you can deduce the following points. Notice how the sinusoidal axis can be assumed to be the average of the high and low tides.

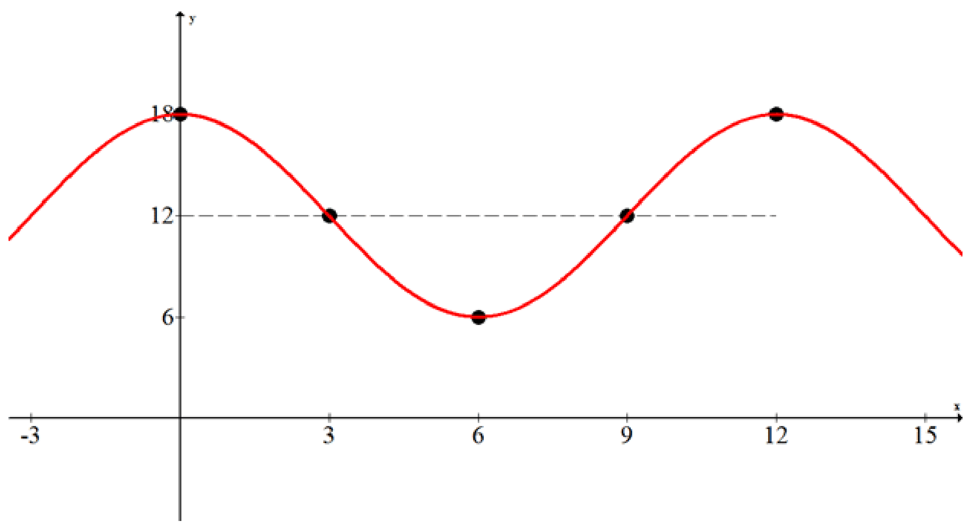
TABLE 1.22:

Time (hours)	Water level (feet)

TABLE 1.22: (continued)

0	18
6	6
12	18
3	$\frac{18+6}{2} = 12$
9	12

By plotting those points and filling in the sinusoidal axis you can observe a cosine graph.



The amplitude is 6 so $a = 6$. There is no vertical reflection. Since the period is 12 you can determine the frequency b:

$12 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{6}$

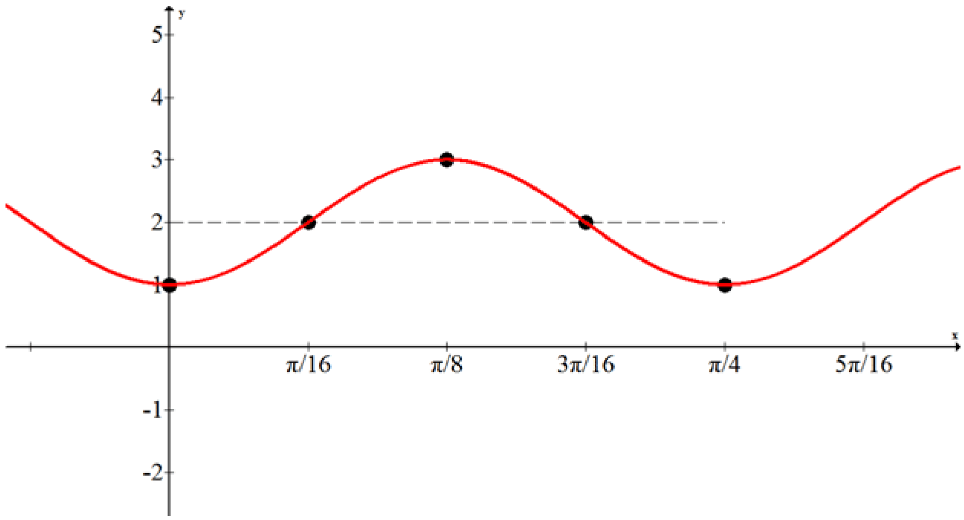
The vertical shift is 12 so $d = 12$. Thus you have all the pieces to make an algebraic model:

$f(x) = +6 \cdot \cos\left(\frac{\pi}{6}x\right) + 12$

Example 4

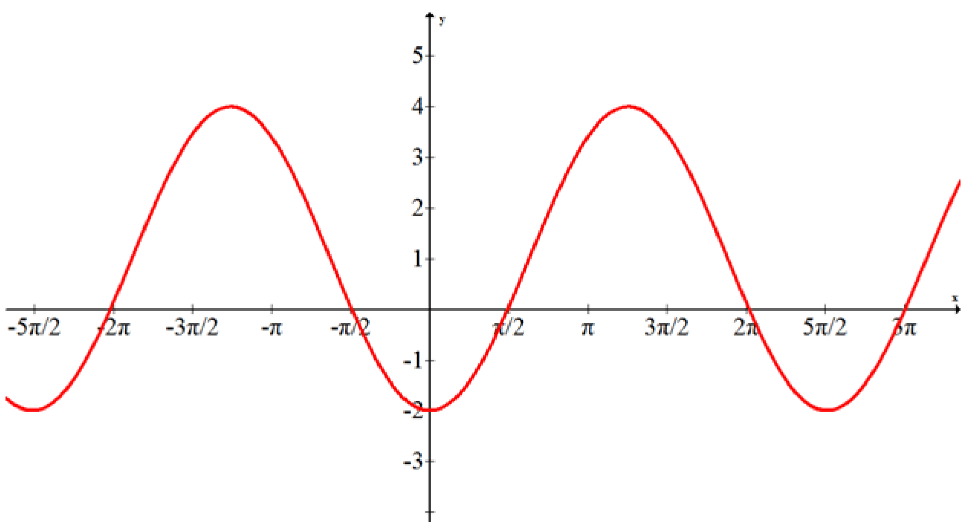
Graph the following function: $g(x) = -\cos(8x) + 2$.

The labeling is the most important and challenging part of this problem. The amplitude is 1. The shape is a negative cosine. The vertical shift is 2. The period is $\frac{2\pi}{8} = \frac{\pi}{4}$. Working with this small period may be challenging at first, but remember that halving fractions is as simple as doubling the denominator.



Example 5

Given the following graph, identify the amplitude, period, and frequency and create an algebraic model.



The amplitude is 3. The shape is a negative cosine. The period is $\frac{5\pi}{2}$ which implies that $b = \frac{4}{5}$. The vertical shift is 1. $f(x) = -3 \cdot \cos\left(\frac{4}{5}x\right) + 1$.

Review

Find the frequency and period of each function below.

1. $f(x) = \sin(4x) + 1$
2. $g(x) = -3\cos(2x)$
3. $h(x) = \cos\left(\frac{1}{2}x\right) + 2$
4. $k(x) = -2\sin\left(\frac{3}{4}x\right) + 1$
5. $j(x) = 4\cos(3x) - 1$

Graph each of the following functions.

6. $f(x) = 3\sin(2x) + 1$

7. $g(x) = 2.5\cos(\pi x) - 4$

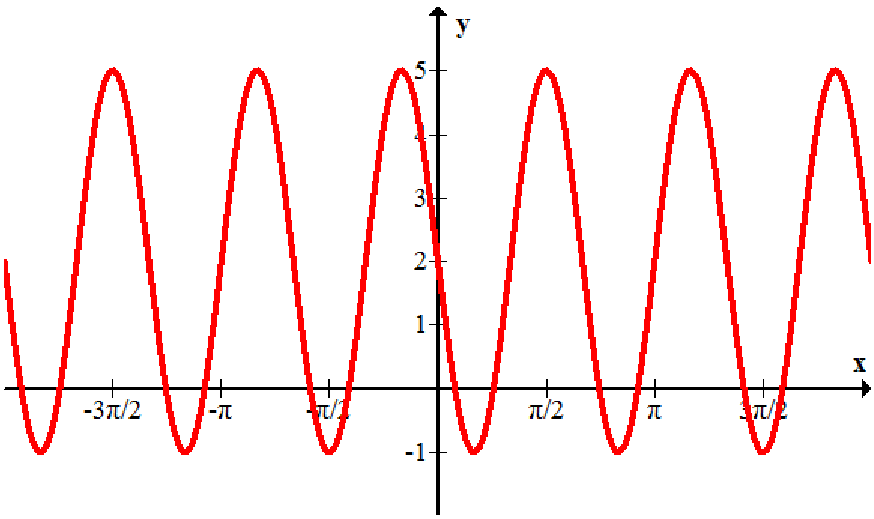
8. $h(x) = -\sin(4x) - 3$

9. $k(x) = \frac{1}{2}\cos(2x)$

10. $j(x) = -2\sin\left(\frac{3}{4}x\right) - 1$

Create an algebraic model for each of the following graphs.

11.



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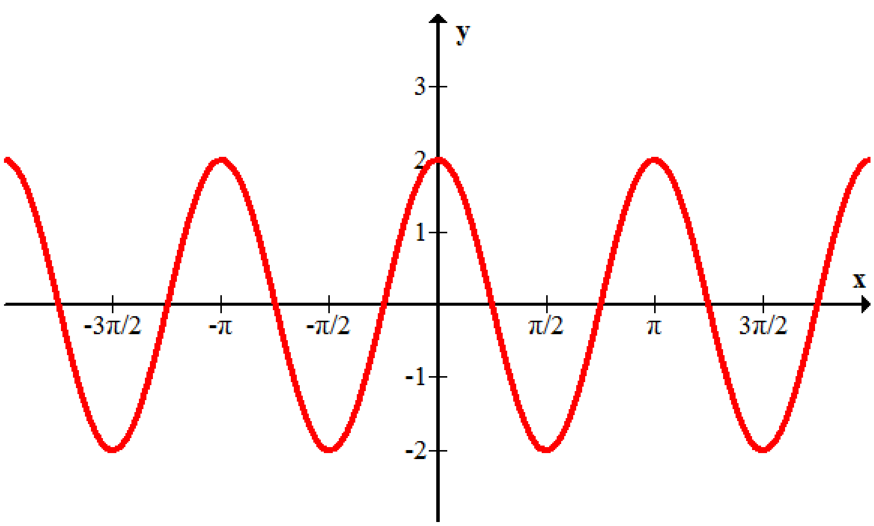

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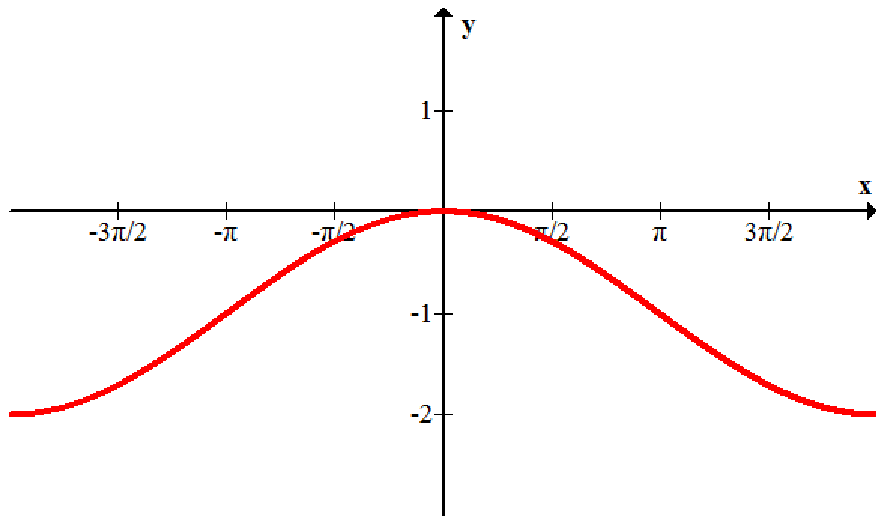


14. At time 0 it is high tide and the water at a certain location is 10 feet high. At low tide 6 hours later, the water is 2 feet high. Given that tides can be modeled by sinusoidal functions, find a graph that models this scenario.

15. Find the equation that models the scenario in the previous problem.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.5.

CHAPTER 2 Algebraic Representations of Functions

Chapter Outline

2.1 Point Notation and Function Notation

Learning Objectives

Here you will learn about the notation conventions involved with transformations.

When performing multiple transformations, it is very easy to make a small error. This is especially true when you try to do every step mentally. Point notation is a useful tool for concentrating your efforts on a single point and helps you to avoid making small mistakes.

What would $f(3x)+7$ look like in point notation and why is it useful?

Using Function Notation and Point Notation

A transformation can be written in function notation and in point notation. Function notation is very common and practical because it allows you to graph any function using the same basic thought process it takes to graph a parabola in vertex form.

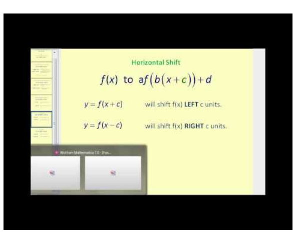
Another way to graph a function is to transform each point one at a time. This method works well when a table of x, y values is available or easily identified from the graph.

Essentially, it takes each coordinate (x, y) and assigns a new coordinate based on the transformation.

$(x, y) \rightarrow (new\ x, new\ y)$

This notation is called **point notation**. The new y coordinate is straightforward and is directly from what takes place outside $f(x)$ because $f(x)$ is just another way to write y. For example, $f(x) \rightarrow 2f(x)-1$ would have a new y coordinate of $2y-1$.

The new x coordinate is trickier. It comes from undoing the operations that affect x. For example, $f(x) \rightarrow f(2x-1)$ would have a new x coordinate of $\frac{x+1}{2}$.



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The function notation and point notation representations of the transformation "Horizontal shift right three units, vertical shift up 4 units" are

$f(x) \rightarrow f(x-3)+4$

$(x, y) \rightarrow (x+3, y+4)$

Notice that the operations with the x are different.

Apply the transformation above to the following table of points.

TABLE 2.1:

x	y
1	3
2	5
8	-11

x	y
1	3
2	5
8	-11

$\rightarrow$

x	y
4	7
5	9
11	-7

Notice that point notation greatly reduces the mental visualization required to keep all the transformations straight at once.

Examples

Example 1

Earlier, you were asked what the function $f(3x)+7$ would be when written in point notation. When written in point notation, it would be written as $(x,y) \rightarrow \left(\frac{x}{3}, y+7\right)$. This is useful because it becomes obvious that the x values are all divided by three and the y values all increase by 7.

Example 2

Convert the following function in point notation to words and then function notation.

$(x,y) \rightarrow (3x+1, -y+7)$

Horizontal stretch by a factor of 3 and then a horizontal shift right one unit. Vertical reflection over the x axis and then a vertical shift 7 units up.

$f(x) \rightarrow -f\left(\frac{1}{3}x - \frac{1}{3}\right) + 7$

Example 3

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

$f(x) \rightarrow -2f(x-1)+4$

Vertical reflection across the x axis. Vertical stretch by a factor of 2. Vertical shift 4 units. Horizontal shift right one unit.

$(x,y) \rightarrow (x+1, -2y+4)$

x	y
0	5
1	6
2	7

$\longrightarrow$

x	y
1	-6
2	-8
3	-10

Example 4

Convert the following function notation into point notation and apply it to the included table of points

$f(x) \to \frac{1}{4}f(-x-3)-1$

TABLE 2.2:

x	y
0	0
1	4
2	8

The y component can be directly observed. For the x component you need to undo the argument. $(x,y) \to \left(-x-3, \frac{1}{4}y-1\right)$

x	y
0	0
1	4
2	8

$\longrightarrow$

x	y
-3	-1
-4	0
-5	1

Example 5

Convert the following point notation to words and to function notation and then apply the transformation to the included table of points.

$(x+3, y-1) \to (2x+6, -y)$

TABLE 2.3:

x	y
10	8
12	7
14	6

This problem is different because it seems like there is a transformation happening to the original left point. This is an added layer of challenge because the transformation of interest is just the difference between the two points. Notice that the x coordinate has simply doubled and the y coordinate has gotten bigger by one and turned negative. This problem can be rewritten as:

$(x,y) \to (2x, -(y+1)) = (2x, -y-1)$

$f(x) \to -f\left(\frac{x}{2}\right) - 1$

x	y
10	8
12	7
14	6

$\longrightarrow$

x	y
5	-9
6	-8
7	-7

Review

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

TABLE 2.4:

x	y
0	5
1	6
2	7

1. $f(x) \to -\frac{1}{2}f(x+1)$
2. $g(x) \to 2g(3x)+2$
3. $h(x) \to -h(x-4)-3$
4. $j(x) \to 3j(2x-4)+1$
5. $k(x) \to -k(x-3)$

Convert the following functions in point notation to function notation.

6. $(x,y) \to \left(\frac{1}{2}x+3, y-4\right)$
7. $(x,y) \to (2x+4, -y+1)$
8. $(x,y) \to (4x, 3y-5)$
9. $(2x,y) \to (4x, -y+1)$
10. $(x+1, y-2) \to (3x+3, -y+3)$

Convert the following functions in function notation to point notation.

11. $f(x) \to 3f(x-2)+1$
12. $g(x) \to -4g(x-1)+3$
13. $h(x) \to \frac{1}{2}h(2x+2)-5$

14. $j(x) \rightarrow 5j\left(\frac{1}{2}x-2\right)-1$
15. $k(x) \rightarrow \frac{1}{4}k(2x-4)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.3.

2.2 Function Composition

Learning Objectives

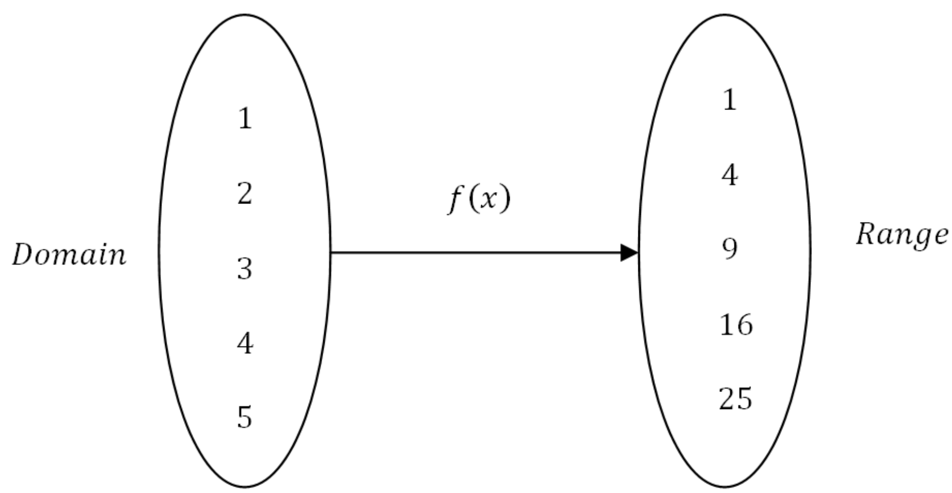
Here you will learn a new type of transformation called composition. Composing functions means having one function inside the argument of another function. This creates a brand new function that may not look like a regular transformation of any of the basic functions.

Functions can be added, subtracted, multiplied and divided creating new functions and graphs that are complicated combinations of the various original functions. One important way to transform functions is through function composition. Function composition allows you to line up two or more functions that act on an input in tandem.

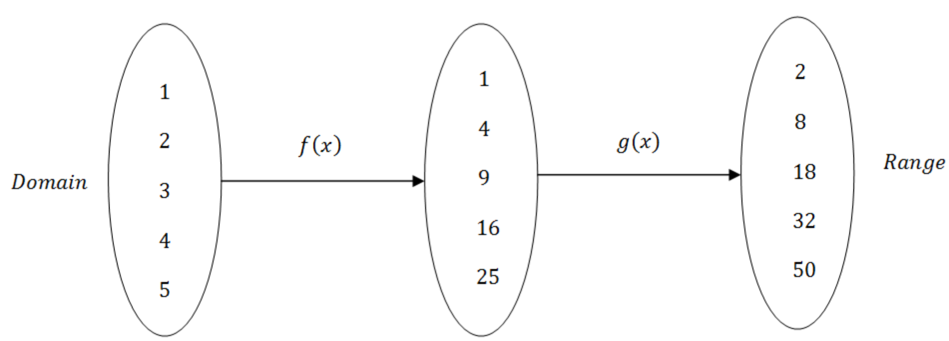
Is function composition essentially the same as multiplying the two functions together?

Composition of Functions

A common way to describe functions is a mapping from the domain space to the range space:



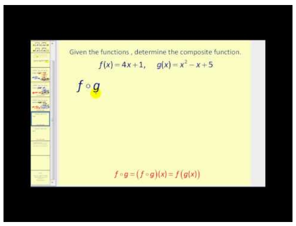
Function composition means that you have two or more functions and the range of the first function becomes the domain of the second function.



There are two notations used to describe function composition. In each case the order of the functions matters because arithmetically the outcomes will be different. Squaring a number and then doubling the result will be different from doubling a number and then squaring the result. In the diagram above, $f(x)$ occurs first and $g(x)$ occurs second. This can be written as:

$g(f(x))$ or $(g \circ f)(x)$

You should read this "g of f of x." In both cases notice that the f is closer to the x and operates on the x values first.



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Examples

Example 1

Earlier, you were asked if function composition is the same as multiplying two functions together. Function composition is not the same as multiplying two functions together. With function composition there is an outside function and an inside function. Suppose the two functions were doubling and squaring. It is clear just by looking at the example input of the number 5 that 50 (squaring then doubling) is different from 100 (doubling then squaring). Both 50 and 100 are examples of function composition, while 250 (five doubled multiplied by five squared) is an example of the product of two separate functions happening simultaneously.

For the next two examples, use the functions below:

$f(x) = x^2 - 1$

$h(x) = \frac{x-1}{x+5}$

$g(x) = 3e^x - x$

$j(x) = \sqrt{x+1}$

Example 3

Show $f(h(x)) \neq h(f(x))$

$f(h(x)) = f\left(\frac{x-1}{x+5}\right) = \left(\frac{x-1}{x+5}\right)^2 - 1$

$h(f(x)) = h(x^2-1) = \frac{(x^2-1)-1}{(x^2-1)+5} = \frac{x^2-2}{x^2+4}$

In order to truly show they are not equal it is best to find a specific counter example of a number where they differ. Sometimes algebraic expressions may look different, but are actually the same. You should notice that $f(h(x))$is undefined when $x = -5$ because then there would be zero in the denominator. $h(f(x))$ on the other hand is defined at $x = -5$. Since the two function compositions differ, you can conclude:

$f(h(x)) \neq h(f(x))$

Example 4

What is $f\Big(j(h(g(x)))\Big)$?

These functions are *nested* within the arguments of the other functions. Sometimes functions simplify significantly when composed together, as f and j do in this case. It makes sense to evaluate those two functions first together and keep them on the outside of the argument.

$f(x) = x^2 - 1; h(x) = \frac{x-1}{x+5}; g(x) = 3e^x - x; j(x) = \sqrt{x+1}$

$f(j(y)) = f\left(\sqrt{y+1}\right) = \left(\sqrt{y+1}\right)^2 - 1 = y+1-1 = y$

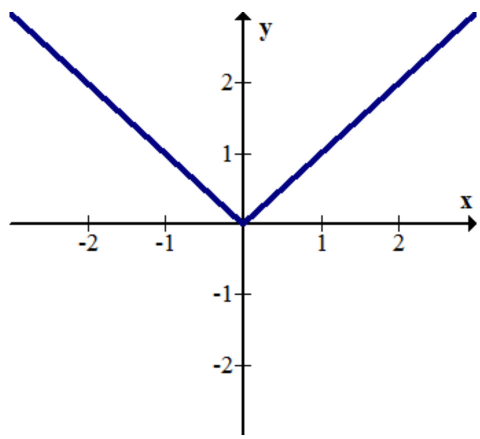
Notice how the composition of f and j produced just the argument itself?

Thus,

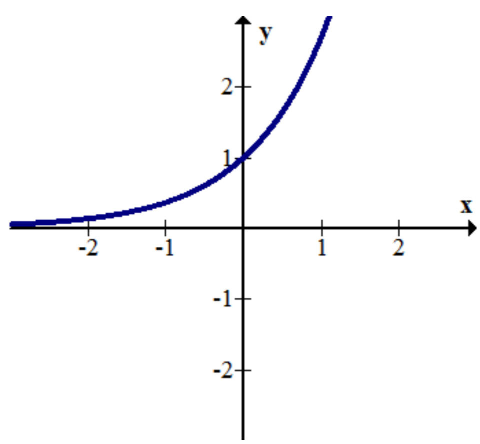
$$\begin{aligned} f\Big(j(h(g(x)))\Big) &= h(g(x)) = h(3e^x - x) \\ &= \frac{(3e^x - x) - 1}{(3e^x - x) + 5} \\ &= \frac{3e^x - x - 1}{3e^x - x + 5} \end{aligned}$$

For the next two examples, use the graphs shown below:

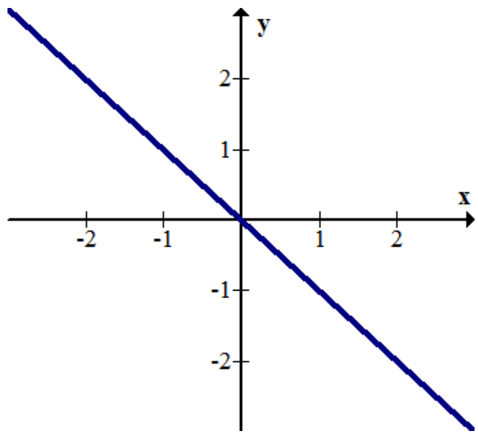
$f(x) = |x|$



$g(x) = e^x$



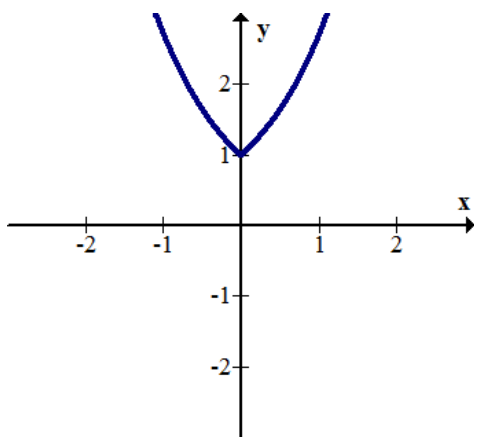
$h(x) = -x$



Example 4

Compose $g(f(x))$ and graph the result. Describe the transformation.

$g(f(x)) = g(|x|) = e^{|x|}$

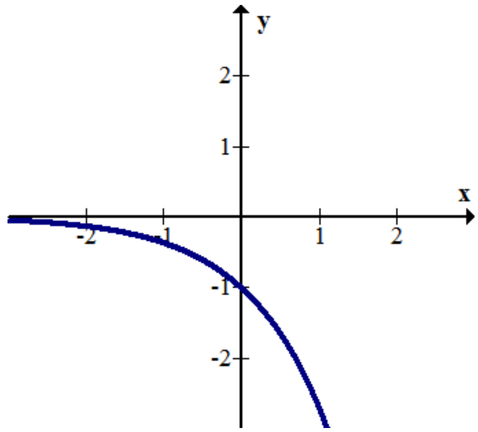


The positive portion of the exponential graph has been mirrored over the y axis and the negative portion of the exponential graph has been entirely truncated.

Example 5

Compose $h(g(x))$ and graph the result. Describe the transformation.

$h(g(x)) = h(e^x) = -e^x$



The exponential graph has been reflected over the x-axis.

Review

For questions 1-9, use the following three functions: $f(x) = |x|, h(x) = -x, g(x) = (x-2)^2 - 3$.

1. Graph $f(x), h(x)$ and $g(x)$.

2. Find $f(g(x))$ algebraically.

3. Graph $f(g(x))$ and describe the transformation.

4. Find $g(f(x))$ algebraically.

5. Graph $g(f(x))$ and describe the transformation.

6. Find $h(g(x))$ algebraically.

7. Graph $h(g(x))$ and describe the transformation.

8. Find $g(h(x))$ algebraically.

9. Graph $g(h(x))$ and describe the transformation.

For 10-16, use the following three functions: $j(x) = x^2, k(x) = |x|, m(x) = \sqrt{x}$.

10. Graph $j(x), k(x)$ and $m(x)$.

11. Find $j(k(x))$ algebraically.

12. Graph $j(k(x))$ and describe the transformation.

13. Find $k(m(x))$ algebraically.

14. Graph $k(m(x))$ and describe the transformation.

15. Find $m(k(x))$ algebraically.

16. Graph $m(k(x))$ and describe the transformation.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.11.

2.3 Inverses of Functions

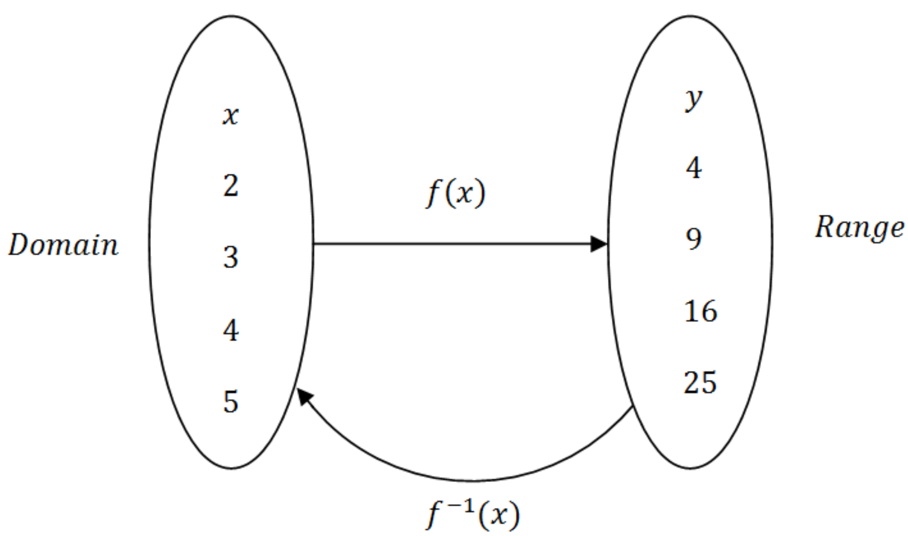
Learning Objectives

Here you will learn about inverse functions and what they look like graphically and numerically. You will also learn how to find and verify inverse functions algebraically.

Functions are commonly known as rules that take inputs and produce outputs. An **inverse function** does exactly the reverse, undoing what the original function does. How can you tell if two functions are inverses?

Finding Inverses of Functions

A function is written as $f(x)$ and its inverse is written as $f^{-1}(x)$. A common misconception is to see the -1 and interpret it as an exponent and write $\frac{1}{f(x)}$, but this is not correct. Instead, $f^{-1}(x)$ should be viewed as a new function from the range of $f(x)$ back to the domain.

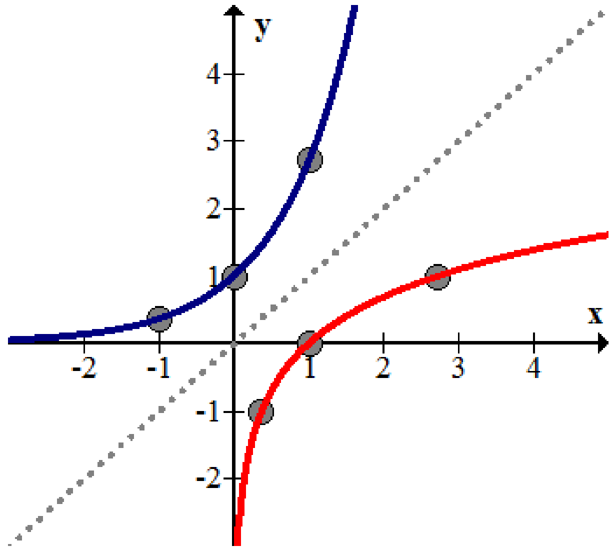


It is important to see the cycle that starts with x, becomes y and then goes back to x. In order for two functions to truly be inverses of each other, this cycle must hold algebraically.

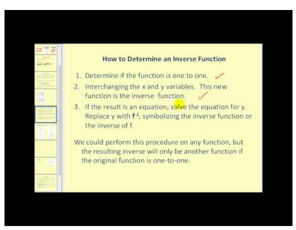
$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

When given a function there are two steps to take to find its inverse. In the original function, first switch the variables x and y. Next, solve the function for y. This will give you the inverse function. After finding the inverse, it is important to check both directions of compositions to make sure that together the function and its inverse produce the value x. In other words, verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Graphically, inverses are reflections across the line $y = x$. Below you see inverses $y = e^x$ and $y = \ln x$. Notice how the (x, y) coordinates in one graph become (y, x) coordinates in the other graph.



In order to decide whether an inverse function is also actually a function you can use the vertical line test on the inverse function like usual. You can also use the horizontal line test on the original function. The horizontal line test is exactly like the vertical line test except the lines simply travel horizontally.



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Examples

Example 1

Earlier, you were asked how you can tell that two functions are inverses. You can tell that two functions are inverses if each undoes the other, always leaving the original x.

Example 2

Find the inverse, then verify the inverse algebraically. $f(x) = y = (x+1)^2 + 4$

To find the inverse, switch x and y then solve for y.

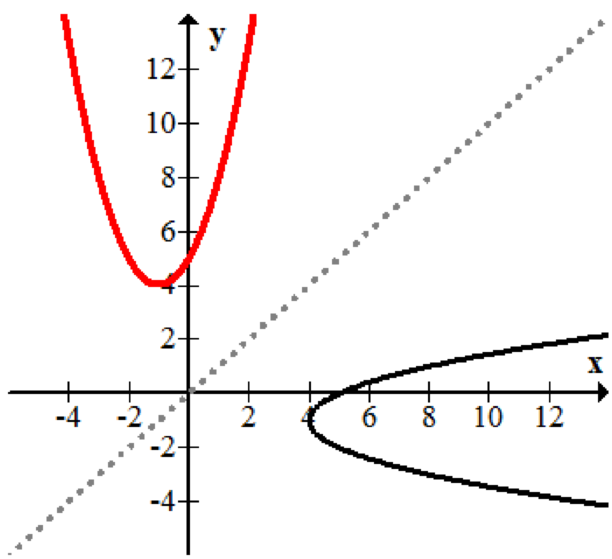
$$\begin{aligned} x &= (y+1)^2 + 4 \\ x - 4 &= (y+1)^2 \\ \pm\sqrt{x-4} &= y + 1 \\ -1 \pm \sqrt{x-4} &= y = f^{-1}(x) \end{aligned}$$

To verify algebraically, you must show $x = f(f^{-1}(x)) = f^{-1}(f(x))$:

$$\begin{aligned}
f(f^{-1}(x)) &= f\left(-1 \pm \sqrt{x-4}\,\right) \\
&= ((-1 \pm \sqrt{x-4}\,)+1)^2+4 \\
&= \left(\pm\sqrt{x-4}\,\right)^2+4 \\
&= x-4+4 = x
\end{aligned}$$

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}((x+1)^2+4) \\
&= -1 \pm \sqrt{((x+1)^2+4)-4} \\
&= -1 \pm \sqrt{(x+1)^2} \\
&= -1+x+1 = x
\end{aligned}$$

As you can see from the graph, the $\pm$ causes the inverse to be a relation instead of a function. This can be observed in the graph because the original function does not pass the horizontal line test and the inverse does not pass the vertical line test.



Example 3

Find the inverse of the function and then verify that $x = f(f^{-1}(x)) = f^{-1}(f(x))$.

$f(x) = y = \frac{x+1}{x-1}$

Sometimes it is quite challenging to switch x and y and then solve for y. You must be careful with your algebra.

$$\begin{aligned} x &= \frac{y+1}{y-1} \\ x(y-1) &= y+1 \\ xy - x &= y+1 \\ xy - y &= x+1 \\ y(x-1) &= x+1 \\ y &= \frac{x+1}{x-1} \end{aligned}$$

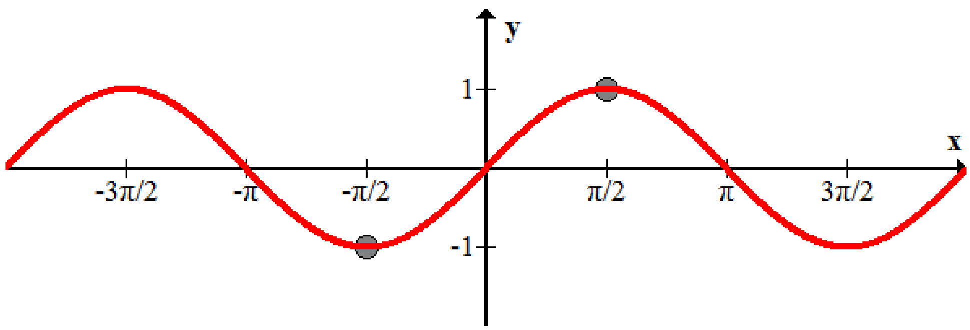
This function turns out to be its own inverse. Since they are identical, you only need to show that $x = f(f^{-1}(x))$.

$f\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{x+1+x-1}{x+1-(x-1)} = \frac{2x}{2} = x$

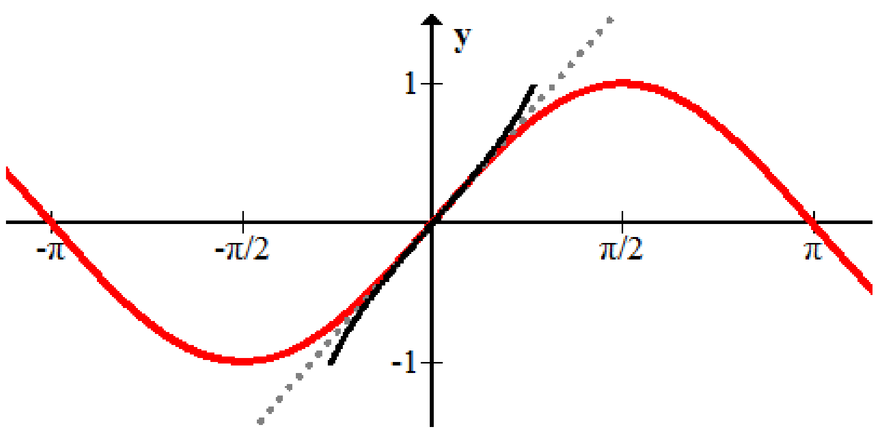
Example 4

What is the inverse of $f(x) = y = \sin x$?

The sine function does not pass the horizontal line test and so its true inverse is not a function.



However, if you restrict the domain to just the part of the x-axis between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ then it will pass the horizontal line test and the inverse will be a function.



The inverse of the sine function is called the arcsine function, $f(x) = \sin^{-1}(x)$, and is shown in black. It is truncated so that it only inverts a part of the whole sine wave. You will study periodic functions and their inverses in more detail later.

Example 5

Determine if $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ are inverses of one another.

Even though $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ have some inverted pieces, they are not inverses of each other. In order to show this, you must show that the composition does not simplify to x.

$\frac{3}{7}\left(\frac{7}{3}x + 21\right) - 21 = x + 9 - 21 = x - 12 \neq x$

Review

Consider $f(x) = x^3$.

1. Sketch $f(x)$ and $f^{-1}(x)$.

2. Find $f^{-1}(x)$ algebraically. It is actually a function?

3. Verify algebraically that $f(x)$ and $f^{-1}(x)$ are inverses.

Consider $g(x) = \sqrt{x}$.

4. Sketch $g(x)$ and $g^{-1}(x)$.

5. Find $g^{-1}(x)$ algebraically. It is actually a function?

6. Verify algebraically that $g(x)$ and $g^{-1}(x)$ are inverses.

Consider $h(x) = |x|$.

7. Sketch $h(x)$ and $h^{-1}(x)$.

8. Find $h^{-1}(x)$ algebraically. It is actually a function?

9. Verify graphically that $h(x)$ and $h^{-1}(x)$ are inverses.

Consider $j(x) = 2x - 5$.

10. Sketch $j(x)$ and $j^{-1}(x)$.

11. Find $j^{-1}(x)$ algebraically. It is actually a function?

12. Verify algebraically that $j(x)$ and $j^{-1}(x)$ are inverses.

13. Use the horizontal line test to determine whether or not the inverse of $f(x) = x^3 - 2x^2 + 1$ is also a function.

14. Are $g(x) = \ln(x+1)$ and $h(x) = e^{x-1}$ inverses? Explain.

15. If you were given a table of values for a function, how could you create a table of values for the inverse of the function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.12.

2.4 Factoring Review

Learning Objectives

Here you will review factoring techniques from Algebra 1 and 2 in preparation for more advanced factoring techniques. The review will include factoring out a greatest common factor, factoring into binomials and the difference of squares.

To **factor** means to write an expression as a product instead of a sum. Factoring is particularly useful when solving equations set equal to zero because then logically at least one factor must be equal to zero. In PreCalculus, you should be able to factor even when there is no obvious greatest common factor or the difference is not between two perfect squares.

How do you use the difference of perfect squares factoring technique on polynomials that don't contain perfect squares and why would this be useful?

Factoring Functions

A **polynomial** is a sum of a finite number of terms. Each term consists of a constant that multiplies a variable. The variable may only be raised to a non-negative exponent. The letters $a, b, c\dots$ in the following general polynomial expression stand for regular numbers like $0, 5, -\frac{1}{4}, \sqrt{2}$ and the x represents the variable.

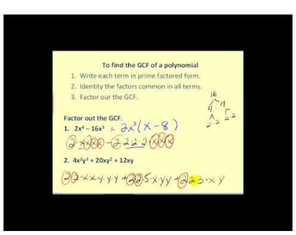
$ax^n + bx^{n-1} + \dots + fx^2 + gx + h$

You have already learned many properties of polynomials. For example, you know the commutative property which states that terms of a polynomial can be rearranged to create an equivalent polynomial. When two polynomials are added, subtracted or multiplied the result is always a polynomial. This means polynomials are closed under addition, and is one of the properties that makes the factoring of polynomials possible. Polynomials are not closed under division because dividing two polynomials could result in a variable in the denominator, which is a rational expression (not a polynomial).

There are three methods for factoring that are essential to master.

Greatest Common Factor Method

The first method you should always try is to factor out the greatest common factor (GCF) of the expression.



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To factor the following expression, first apply the GCF method:

$-\frac{1}{2}x^4 + \frac{7}{2}x^2 - 6$

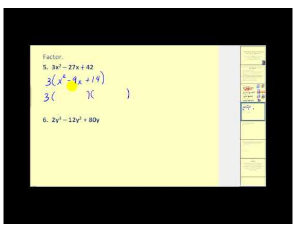
Many students just learning factoring may conclude that the three terms share no factors besides one. However, the name GCF is deceiving because this expression has an infinite number of equivalent expressions many of which are more useful. In order to find these alternative expressions you must strategically factor numbers that are neither the greatest factor nor common to all three terms. In this case, $-\frac{1}{2}$ is an excellent choice.

$-\frac{1}{2}x^4+\frac{7}{2}x^2-6=-\frac{1}{2}(x^4-7x^2+12)$

In order to check to see that this is an equivalent expression, you need to distribute the $-\frac{1}{2}$. When you distribute, the first coefficient matches because it just gets multiplied by 1, the second term becomes $\frac{7}{2}$ and the third term becomes -6. Note that this expression is not completely factored yet but it is simplified as much as it can be with just the GCF method.

Factoring Into Binomials Method

The second method you should implement after factoring out a GCF is to see if you can factor the expression into the product of two binomials. This type of factoring is usually recognizable as a trinomial where x^2 has a coefficient of 1.



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To continue factoring the expression from the Greatest Common Factor section, factor the following expression into the product of two binomials and a constant:

$-\frac{1}{2}(x^4-7x^2+12)$

Many students familiar with basic factoring may be initially stuck on a problem like this. However, you should recognize that beneath the 4^{th} degree and the $-\frac{1}{2}$ the problem boils down to being able to factor $u^2-7u+12$ which is just $(u-3)(u-4)$.

Start by rewriting the problem: $-\frac{1}{2}(x^4-7x^2+12)$

Then choose a temporary substitution: Let $u=x^2$.

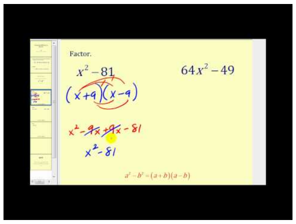
Then substitute and factor away. Remember to substitute back at the end.

$$\begin{aligned}
-\frac{1}{2}(u^2-7u+12) &= -\frac{1}{2}(u-3)(u-4)\\
&= -\frac{1}{2}(x^2-3)(x^2-4)
\end{aligned}$$

This type of temporary substitution that enables you to see the underlying structure of an expression is very common in calculus. The expression is still not completely factored and since there are no more trinomials, you must apply the last method.

Difference of Squares Method

The third method of basic factoring is the difference of squares. It is recognizable as one square monomial being subtracted from another square monomial.



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To finish factoring the resulting expression from the Factoring Into Binomials section, factor the expression into four linear factors and a constant:

$-\frac{1}{2}(x^2-3)(x^2-4)$

Many students may recognize that x^2-4 immediately factors by the difference of squares method to be $(x-2)(x+2)$. This problem asks for more because sometimes the difference of squares method can be applied to expressions like x^2-3 where each term is not a perfect square. The number 3 actually is a square.

$3=\left(\sqrt{3}\right)^2$

So the fully factored expression would be:

$-\frac{1}{2}\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)(x-2)(x+2)$

Examples

Example 1

Earlier, you were asked how you use the difference of perfect squares factoring technique on polynomials that don't contain perfect squares and why it would be useful. One reason why it might be useful to completely factor an expression like $-\frac{1}{2}(x^4-7x^2+12)$ into linear factors is if you wanted to find the roots of the function $f(x)=-\frac{1}{2}(x^4-7x^2+12)$. The roots are $x=\pm\sqrt{3},\pm 2$.

You should recognize that x^2-3 can still be thought of as the difference of perfect squares because the number 3 can be expressed as $\left(\sqrt{3}\right)^2$. Rewriting the number 3 to fit a factoring pattern that you already know is an example of using the basic factoring techniques at a PreCalculus level.

Example 2

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$\frac{x^5}{3}-\frac{11x^3}{3}+6x$

$\frac{x^5}{3} - \frac{11x^3}{3} + 6x$

$$\begin{aligned}
&= \frac{1}{3}x(x^4 - 11x^2 + 18) \\
&= \frac{1}{3}x(x^2 - 2)(x^2 - 9) \\
&= \frac{1}{3}x(x + \sqrt{2})(x - \sqrt{2})(x + 3)(x - 3)
\end{aligned}$$

Example 3

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$-\frac{2}{7}x^4 + \frac{74}{63}x^2 - \frac{8}{63}$

For $-\frac{2}{7}x^4 + \frac{74}{63}x^2 - \frac{8}{63}$, let $u = x^2$.

$$\begin{aligned}
&= -\frac{2}{7}u^2 + \frac{74}{63}u - \frac{8}{63} \\
&= -\frac{2}{7}\left(u^2 - \frac{37}{9}u + \frac{4}{9}\right)
\end{aligned}$$

Factoring through fractions like this can be extremely tricky. You must recognize that $-\frac{1}{9}$ and -4 sum to $-\frac{37}{9}$ and multiply to $\frac{4}{9}$.

$$\begin{aligned}
&= -\frac{2}{7}\left(u - \frac{1}{9}\right)(u - 4) \\
&= -\frac{2}{7}\left(x^2 - \frac{1}{9}\right)(x^2 - 4) \\
&= -\frac{2}{7}\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)(x - 2)(x + 2)
\end{aligned}$$

Example 4

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$x^4 + x^2 - 72$

$x^4 + x^2 - 72 = (x^2 - 8)(x^2 + 9)$

Notice that $(x^2 - 8)$ can be written as the difference of perfect squares because $8 = \left(\sqrt{8}\right)^2 = \left(2\sqrt{2}\right)^2$. On the other hand, $x^2 + 9$ cannot be written as the difference between squares because the x^2 and the 9 are being added not subtracted. This polynomial cannot be factored into strictly linear factors.

$x^4 + x^2 - 72 = (x - 2\sqrt{2})(x + 2\sqrt{2})(x^2 + 9)$

Review

Factor each polynomial into strictly linear factors if possible. If not possible, explain why not.

1. $x^2 + 5x + 6$
2. $x^4 + 5x^2 + 6$

3. x^4-16
4. $2x^2-20$
5. $3x^2+9x+6$
6. $\frac{x^4}{2}-5x^2+\frac{9}{2}$
7. $\frac{2x^4}{3}-\frac{34x^2}{3}+\frac{32}{3}$
8. $x^2-\frac{1}{4}$
9. $x^4-\frac{37x^2}{4}+\frac{9}{4}$
10. $\frac{3}{4}x^4-\frac{87}{4}x^2+75$
11. $\frac{1}{2}x^4-\frac{29}{2}x^2+50$
12. $\frac{x^4}{2}-\frac{5x^2}{9}+\frac{1}{18}$
13. $x^4-\frac{13}{36}x^2+\frac{1}{36}$
14. How does the degree of a polynomial relate to the number of linear factors?
15. If a polynomial does not have strictly linear factors, what does this imply about the type of roots that the polynomial has?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.1.

2.5 Advanced Factoring

Learning Objectives

Here you will be exposed to a variety of factoring techniques for special situations. Additionally, you will see alternatives to trial and error for factoring.

The difference of perfect squares can be generalized as a factoring technique. By extension, any difference between terms that are raised to an even power like $a^6 - b^6$ can be factored using the difference of perfect squares technique. This is because even powers can always be written as perfect squares: $a^6 - b^6 = (a^3)^2 - (b^3)^2$.

What about the sum or difference of terms with matching odd powers? How can those be factored?

More Factoring Techniques

Factoring a trinomial of the form $ax^2 + bx + c$ is much more difficult when $a \neq 1$. There are four techniques that can be used to factor such expressions.

Guess and Check

The educated guess and check method can be time consuming but if the first and last coefficient only have a few factors, there are a finite number of possibilities. Take the expression:

$6x^2 - 13x - 28$

The 6 can be factored into the following four pairs:

1, 6

2, 3

-1, -6

-2, -3

The -28 can be factored into the following twelve pairs:

1, -28 or -28, 1

-1, 28 or 28, -1

2, -14 or -14, 2

-2, 14 or 14, -2

4, -7 or -7, 4

-4, -7 or -7, -4

The correctly factored expression will need a pair from the top list and a pair from the bottom list. This is 48 possible combinations to try.

If you try the first pair from each list and multiply out you will see that the first and the last coefficients are correct but the b coefficient does not.

$(1x+1)(6x-28) = 6x - 28x + 6x - 28$

A systematic approach to every one of the 48 possible combinations is the best way to avoid missing the correct pair. In this case it is:

$(2x-7)(3x+4) = 6x^2+8x-21x-28 = 6x^2-13x-28$

This method can be extremely long and rely heavily on good guessing which is why other methods are preferable.

Factoring by Grouping

The next factoring technique is factoring by grouping. Suppose you start with an expression already in factored form:

$12x^2+4xz+3xy+yz$

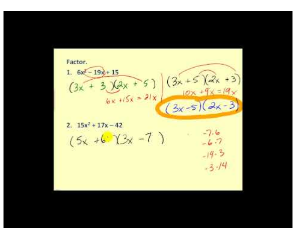
Notice that the first two terms are divisible by both 4 and x and the last two terms are divisible by y. First, factor out these common factors and then notice that there emerges a second layer of common factors. The binomial $(3x+z)$ is now common to both terms and can be factored out just as before.

$$\begin{aligned} 12x^2+4xz+3xy+yz &= 4x(3x+z)+y(3x+z) \\ &= (3x+z)(4x+y) \end{aligned}$$

To check your work, multiply the binomials and compare it with the original expression.

$(4x+y)(3x+z) = 12x^2+4xz+3xy+yz$

Usually when you multiply the factored form of a polynomial, two terms can be combined because they are like terms. In this case, there are no like terms that can be combined.



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Quadratic Formula

An alternative method to the guess and check method and factoring by grouping is the quadratic formula as a clue even though this is an expression and not an equation set equal to zero.

$6x^2-13x-28$

$a=6, b=-13, c=-28$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{13\pm\sqrt{169-4\cdot 6\cdot -28}}{2\cdot 6} = \frac{13\pm 29}{12} = \frac{42}{12} \; or - \frac{16}{12} = \frac{7}{2} \; or \; -\frac{4}{3}$

This means that when set equal to zero, this expression is equivalent to

$\left(x-\frac{7}{2}\right)\left(x+\frac{4}{3}\right)=0$

Multiplying by 2 and multiplying by 3 only changes the left hand side of the equation because the right hand side will remain 0. This has the effect of shifting the coefficient from the denominator of the fraction to be in front of the x.

$6x^2-13x-28=(2x-7)(3x+4)$

Factoring Algorithm

Another useful and efficient technique is the procedural factoring algorithm. The proof of the algorithm is beyond the scope of this book, but is a reliable technique for getting a handle on tricky factoring questions of the form: $6x^2-13x-28$

We will factor $6x^2-13x-28$ using the factoring algorithm to introduce it to you.

First, multiply the first coefficient with the last coefficient and set the first coefficient to 1:

$x^2-13x-168$

Second, factor as you normally would with $a=1$:

$(x-21)(x+8)$

Third, divide the second half of each binomial by the coefficient that was multiplied in step 1:

$\left(x-\frac{21}{6}\right)\left(x+\frac{8}{6}\right)$

Fourth, simplify each fraction completely:

$\left(x-\frac{7}{2}\right)\left(x+\frac{4}{3}\right)$

Lastly, move the denominator of each fraction to become the coefficient of x:

$(2x-7)(3x+4)$

When you compare the computational difficulty of the three methods mentioned above, you will see that the factoring algorithm is the most efficient.

Sum or Difference of Matching Odd Powers

The last method of advanced factoring does not involve expressions of the form ax^2+bx+c. Instead, it involves the patterns that arise from factoring the sum or difference of terms with matching odd powers. The patterns are:

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

This method is shown in the examples below and the pattern is fully explored in the Review.

Examples

Example 1

Earlier, you were asked how the sum and difference of terms with matching odd powers can be factored. The sum or difference of terms with matching odd powers can be factored in a precise pattern because when multiplied out, all intermediate terms cancel each other out.

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$$

When a is distributed: $a^5-a^4b+a^3b^2-a^2b^3+ab^4$

When b is distributed: $+a^4b-a^3b^2+a^2b^3-ab^4+b^5$

Notice all the inside terms cancel: a^5+b^5

Example 2

Show that $a^3 - b^3$ factors into the result given in the Sum or Difference of Matching Odd Powers section.

Factoring,

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3 \end{aligned}$$

Example 3

Show that $a^3 + b^3$ factors into the result given in the Sum or Difference of Matching Odd Powers section.

Factoring,

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2-ab+b^2) \\ &= a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3 \\ &= a^3 + b^3 \end{aligned}$$

Example 4

Factor the following expression without using the quadratic formula or trial and error:

$8x^2 + 30x + 27$

Using the factoring algorithm:

$$\begin{aligned} 8x^2 + 30x + 27 &\to x^2 + 30x + 216 \\ &\to (x+12)(x+18) \\ &\to \left(x + \frac{12}{8}\right)\left(x + \frac{18}{8}\right) \\ &\to \left(x + \frac{3}{2}\right)\left(x + \frac{9}{4}\right) \\ &\to (2x+3)(4x+9) \end{aligned}$$

Review

Factor each expression completely.

1. $2x^2 - 5x - 12$
2. $12x^2 + 5x - 3$
3. $10x^2 + 13x - 3$
4. $18x^2 + 9x - 2$
5. $6x^2 + 7x + 2$

6. $8x^2+34x+35$

7. $5x^2+23x+12$

8. $12x^2-11x+2$

Expand the following expressions. What do you notice?

9. $(a+b)(a^8-a^7b+a^6b^2-a^5b^3+a^4b^4-a^3b^5+a^2b^6-ab^7+b^8)$

10. $(a-b)(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$

11. Describe in words the pattern of the signs for factoring the difference of two terms with matching odd powers.

12. Describe in words the pattern of the signs for factoring the sum of two terms with matching odd powers.

Factor each expression completely.

13. $27x^3-64$

14. x^5-y^5

15. $32a^5-b^5$

16. $32x^5+y^5$

17. $8x^3+27$

18. $2x^2+2xy+x+y$

19. $8x^3+12x^2+2x+3$

20. $3x^2+3xy-4x-4y$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.2.

2.6 Exponential Functions

Learning Objectives

Here you will explore exponential functions as a way to model a special kind of growth or decay and you will learn more about the number e. Exponential growth is one of the most powerful forces in nature. A famous legend states that the inventor of chess was asked to state his own reward from the king. The man asked for a single grain of rice for the first square of the chessboard, two grains of rice for the second square and four grains of rice for the third. He asked for the entire 64 squares to be filled in this way and that would be his reward. Did the man ask for too little, or too much?

Exponential Functions

Exponential functions take the form $f(x) = a \cdot b^x$ where a and b are constants. a is the starting amount when $x = 0$. b tells the story about the growth. If the growth is doubling then b is 2. If the growth is halving (which would be decay), then b is $\frac{1}{2}$. If the growth is increasing by 6% then b is 1.06.

Exponential growth is everywhere. Money grows exponentially in banks. Populations of people, bacteria and animals grow exponentially when their food and space aren't limited.

Radioactive isotopes like Carbon 14 have something called a **half-life** that indicates how long it takes for half of the molecules present to decay into other more stable molecules. It takes about 5,730 years for this process to occur which is how scientists can date artifacts of ancient humans.

Let's say a mummified animal is found preserved on the slopes of an ice covered mountain. After testing, you see that exactly one fourth of the carbon-14 has yet to decay and you want to find out how long ago was this animal alive. How would you do that?

Since this problem does not give specific amounts of carbon, it can be inferred that the time will not depend on the specific amounts. One technique that makes the problem easier to work with could be to create an example scenario that fits the one fourth ratio. Suppose 60 units were present when the animal was alive at time zero. This means that 15 units must be present today.

$$15 = a \cdot \left(\frac{1}{2}\right)^x$$
$$60 = a \cdot \left(\frac{1}{2}\right)^0$$

The second equation yields $a = 60$ and then the first equation becomes:

$15 = 60 \cdot \left(\frac{1}{2}\right)^x$

Although you may not yet have the algebraic tools to solve for x, you should still be able to see that x is 2. This does not mean that two years ago the animal was alive, it means that two half life cycles ago the animal was alive. The half life cycle for carbon 14 is 5,730 years so this animal was alive over 11,000 years ago.

Now, suppose you invested \$100 the day you were born and it grew by 6% every year until you were 100 years old. How would you use an exponential function to determine how much would this investment be worth then?

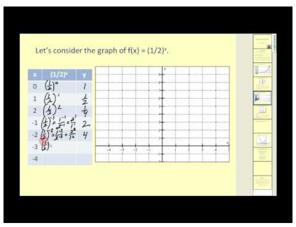
The starting amount is 100 and the growth is 1.06 because it grows by 6% each year. This is enough information to write an exponential function. The x stands for time in years and the $f(x)$ stands for the amount of money in the account. Plugging in to the formula for an exponential function, you will get the equation:

$f(x) = 100 \cdot 1.06^x$

Then, you need to substitute in 100 years for x.

$$\begin{aligned} f(x) &= 100 \cdot 1.06^x \\ f(x) &= 100 \cdot 1.06^{100} \\ f(x) &\approx 33,930.21 \end{aligned}$$

After a century, there will be almost \$34,000 in the account. Interest has greatly increased the \$100 initial investment.



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Examples

Example 1

Earlier you were asked if the man asked for too little or too much rice if he gets one grain for the first square, two grains for the second, four grains for the third and so on. The number of grains of rice on the last square, the 64th square, would be almost ten quintillion (million million million). That is more rice than is produced in the world in an entire year.

$2^{63} = 9,223,372,036,854,775,808$

Example 2

Suppose forty rabbits are released on an island. The rabbits mate once every four months and produce up to 4 offspring who also produce more offspring four months later. Estimate the number of rabbits on the island in 3 years if their population grows exponentially. Assume half the population is female.

Even though parts of this problem are unrealistic, it serves to illustrate how quickly exponential growth works. Forty is the initial amount so $a = 40$. At the end of the first 4 month period 20 female rabbits could have their litters and up to 80 newborn rabbits could be born. The population has grown from 40 to 120 which means tripled. Thus, $b = 3$. The last thing to remember is that the time period is in 4 month periods. Three years must be 9 periods.

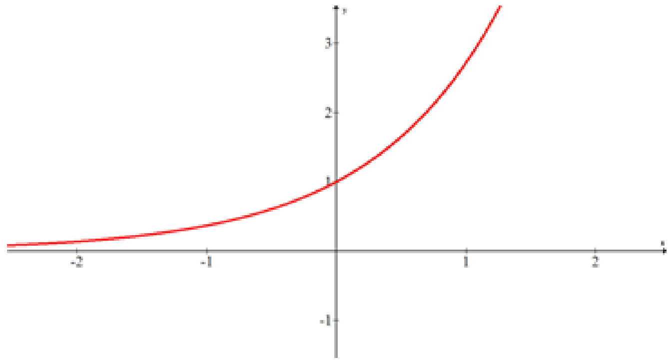
$f(x) = 40 \cdot 3^9 = 787,320$

So after three years, there could be up to 787,320 rabbits!

Example 3

Completely analyze the following exponential function.

$f(x) = e^x$



Analyze in this context means to define all the characteristics of a function.

Domain: $x\ \varepsilon\ (-\infty, \infty)$

Range: $y\ \varepsilon\ (0, \infty)$

Increasing: $x\ \varepsilon\ (-\infty, \infty)$

Decreasing: NA

Zeroes: None

Intercepts: (0, 1)

Maximums: None

Minimums: None

Asymptotes: $y = 0$ as x gets infinitely small

Holes: None

Example 4

Identify which of the following functions are exponential functions and which are not.

1. $y = x^6$
2. $y = 5^x$
3. $y = 1^x$
4. $y = x^x$
5. $y = x^{\frac{1}{2}}$

Exponential functions are of the form $y = a \cdot b^x$

a. $y = x^6$ is not an exponential function because x is not in the exponent.

b. $y = 5^x$ Exponential function.

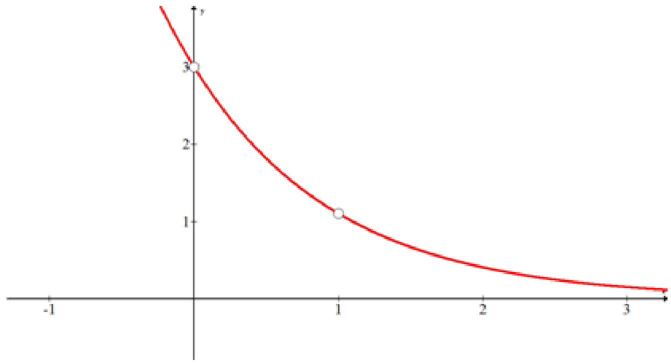
c. $y = 1^x$ Not a true exponential function because y is always 1 which is a constant function.

d. $y = x^x$ Not an exponential function because x is both the base and power of the exponent.

e. $y = x^{\frac{1}{2}}$ Not an exponential function.

Example 5

Write the exponential function that passes through the following points: $(0, 3), \left(1, \frac{3}{e}\right)$.



The starting number is $a = 3$. This number is changed by a factor of $\frac{1}{e}$ which is b.

$$f(x) = 3\left(\frac{1}{e}\right)^x = 3e^{-x}$$

Review

1. Explain what makes a function an exponential function. What does its equation look like?

2. Is the domain for all exponential functions all real numbers?

3. How can you tell from its equation whether or not the graph of an exponential function will be increasing?

4. How can you tell from its equation whether or not the graph of an exponential function will be decreasing?

5. What type of asymptotes do exponential functions have? Explain.

6. Suppose you invested \$4,500 and it grew by 4% every year for 30 years. How much would this investment be worth after 30 years?

7. Suppose you invested \$10,000 and it grew by 12% every year for 40 years. How much would this investment be worth after 40 years?

Write the exponential function that passes through the following points.

8. (0, 5) and (1, 25)

9. (0, 2) and (1, 8)

10. (0, 16) and (2, 144)

11. (1, 4) and (3, 36)

12. (0, 16) and (3, 2)

13. (0, 81) and (2, 9)

14. (1, 144) and (3, 12)

15. Explain why for exponential functions of the form $y = a \cdot b^x$ the y-intercept is always the value of a.

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.1.

2.7 Properties of Exponents

Learning Objectives

Here you will learn how exponents interact in a variety of algebraic situations including addition, subtraction, multiplication and exponentiation.

It is important to quickly and effectively manipulate algebraic expressions involving exponents. One simplification that comes up often is that expressions and numbers raised to the 0 power are always equal to 1. Why is this true and is it always true?

Exponent Properties

Consider the following exponential expressions with the same base and what happens through the algebraic operations. You should feel comfortable with all of these types of manipulations. Let b^y, b^x be exponential terms.

Addition and Subtraction

$b^x \pm b^y = b^x \pm b^y$

Only in the special case when $x = y$ can the terms be combined. This is a basic property of combining like terms.

Multiplication

$b^x \cdot b^y = b^{x+y}$

When the bases are the same then exponents can be added.

Division

$\frac{b^x}{b^y} = b^{x-y}$

The division rule is an extension of the multiplication rule with the possibility of a negative in the exponent.

Negative exponent

$b^{-x} = \frac{1}{b^x}$

A negative exponent means reciprocal.

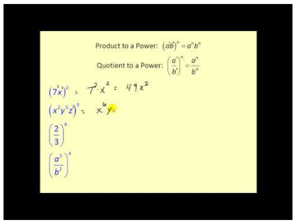
Fractional exponent

$(b)^{\frac{1}{x}} = \sqrt[x]{b}$

Square roots are what most people think of when they think of roots, but roots can be taken with any real number using fractional exponents.

Powers of Powers

$(b^x)^y = b^{x \cdot y}$



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Examples

Example 1

Earlier, you were asked why expressions and numbers raised to the 0 power are always equal to 1. Consider the following pattern and decide what the next term in the sequence should be:

16, 8, 4, 2, ___

It makes sense that the next term is 1 because each successive term is half that of the previous term. These numbers correspond to powers of 2.

$2^4, 2^3, 2^2, 2^1,$ ___

In this case you could decide that the next term must be 2^0. This is a useful technique for remembering what happens when a number is raised to the 0 power.

One question that extends this idea is what is the value of 0^0? People have argued about this for centuries. Euler argued that it should be 1 and many other mathematicians like Cauchy and Möbius argued as well. If you search today you will still find people discussing what makes sense.

In practice, it is convenient for mathematicians to rely on $0^0 = 1$.

Example 2

Simplify the following expression until all exponents are positive.

$$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0}$$

$$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0} = \frac{a^6b^{-9}}{ab^2 \cdot 1} = \frac{a^5}{b^{11}}$$

Example 3

Simplify the following expression until all exponents are positive.

$$(2x)^5 \cdot \frac{4^2}{2^{-3}} \cdot \frac{a^3b^2c^4}{a^2b^{-4}c^0}$$

$$(2x)^5 \cdot \frac{4^2}{2^{-3}} = \frac{2^5x^52^4}{2^{-3}} = \frac{2^9x^5}{2^{-3}} = 2^{12}x^5$$

Example 4

Simplify the following expression using positive exponents.

$$\frac{(2^6 \cdot 8^3)^{-3}}{4^2\left(\frac{1}{2}\right)^4 64^{\frac{1}{3}}}$$

Rewrite every exponent as a power of 2.

For example $8^3 = (2^3)^3 = 2^9$ and $64^{\frac{1}{3}} = (2^6)^{\frac{1}{3}} = 2^2$

$$\frac{(2^6 \cdot 8^3)^{-3}}{4^2\left(\frac{1}{2}\right)^4 64^{\frac{1}{3}}} = \frac{(2^6 \cdot 2^9)^{-3}}{2^42^{-4}2^2} = \frac{(2^{15})^{-3}}{2^2} = \frac{2^{-45}}{2^2} = \frac{1}{2^{47}}$$

Example 5

Solve the following equation using properties of exponents.

$(32^{0.6})^2 = x^3$

First work with the left hand side of the equation.

$$\begin{aligned}
(32^{0.6})^2 = \left((2^5)^{\frac{3}{5}}\right)^2 &= 2^6 \\
2^6 &= x^3 \\
(2^6)^{\frac{1}{3}} &= (x^3)^{\frac{1}{3}} \\
2^2 &= x \\
4 &= x
\end{aligned}$$

Review

Simplify each expression using positive exponents.

1. $81^{-\frac{1}{4}}$
2. $64^{\frac{2}{3}}$
3. $\left(\frac{1}{32}\right)^{-\frac{2}{5}}$
4. $(-125)^{\frac{1}{3}}$
5. $(4x^3y)(3x^5y^2)^4$

6. $(5x^3y^2)^2(7x^3y)^2$
7. $\frac{8a^3b^{-2}}{(-4a^2b^4)^{-2}}$
8. $\frac{5x^2y^{-3}}{(-2x^3y^2)^{-4}}$
9. $\left(\frac{3m^3n^{-4}}{2m^{-5}n^{-2}}\right)^{-4}$
10. $\left(\frac{4m^{-3}n^{-4}}{5m^5n^{-4}}\right)^{-3}$
11. $\left(\frac{a^{-1}b}{a^5b^4}\right)^{-3}$
12. $\frac{15c^{-2}d^{-6}}{3c^{-4}d^{-2}}$
13. $\frac{12e^5f}{(-2ef^3)^{-2}}$

Solve the following equations using properties of exponents.

14. $(81^{0.75})^2 = x^3$
15. $\left(64^{\frac{1}{6}}\right)^{-3} = x^3$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.2.

2.8 Properties of Logs

Learning Objectives

Here you will be introduced to logarithmic expressions and will learn how they can be combined using properties of arithmetic.

Log functions are inverses of exponential functions. This means the domain of one is the range of the other. This is extremely helpful when solving an equation and the unknown is in an exponent. Before solving equations, you must be able to simplify expressions containing logs. The rules of exponents are applied, but in non-obvious ways. In order to get a conceptual handle on the properties of logs, it may be helpful to continually ask, what does a log expression represent? For example, what does $\log_{10} 1,000$ represent?

Log Properties

Exponential and **logarithmic** expressions have the same 3 components. They are each written in a different way so that a different variable is isolated. The following two equations are equivalent to one another.

$b^x = a \leftrightarrow \log_b a = x$

The exponential equation on the left is read "b to the power x is a." The logarithmic equation on the right is read "log base b of a is x".

The two most common bases for logs are 10 and e. At the PreCalculus level *log* by itself implies log base 10 and *ln* implies base e. *ln* is called the **natural log**. One important restriction for all log functions is that they must have strictly positive numbers in their arguments. So, if you press log -2 or log 0 on your calculator, it will give an error.

There are three basic properties of logs that correlate to properties of exponents.

Addition/Multiplication

$\log_b x + \log_b y = \log_b (x \cdot y)$

$b^{w+z} = b^w \cdot b^z$

Subtraction/Division

$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$

$b^{w-z} = \frac{b^w}{b^z}$

Exponentiation

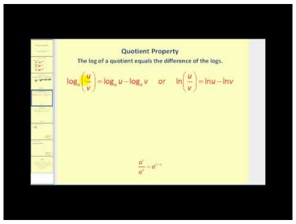
$\log_b (x^n) = n \cdot \log_b x$

$(b^w)^n = b^{w \cdot n}$

There are also a few standard results that should be memorized and should serve as baseline reference tools.

- $\log_b 1 = 0$

- $\log_b b = 1$
- $\log_b(b^x) = x$
- $b^{\log_b x} = x$



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Example

Example 1

Earlier you were asked what $\log_{10} 1,000$ represents. A log expression represents an exponent. The expression $\log_{10} 1,000$ represents the number 3. The reason to keep this in mind is that it can solidify the properties of logs. For example, adding exponents implies bases are multiplied. Thus adding logs means the bases of the exponents are multiplied.

Example 2

Write the expression as a logarithm of a single argument.

$\log_2 12 + \log_4 6 - \log_2 24$

Note that the center expression is of a different base. First change it to base 2 by switching back to exponential form.

$$\begin{aligned}\log_4 6 = x &\leftrightarrow 4^x = 6\\ 2^{2x} = 6 &\leftrightarrow \log_2 6 = 2x\\ x &= \frac{1}{2}\log_2 6 = \log_2 6^{\frac{1}{2}}\end{aligned}$$

Thus the expression with the same base is:

$$\begin{aligned}\log_2 12 + \log_2 6^{\frac{1}{2}} - \log_2 24 &= \log_2\left(\frac{12\cdot\sqrt{6}}{24}\right)\\ &= \log_2\left(\frac{\sqrt{6}}{2}\right)\end{aligned}$$

Example 3

Prove the following log identity:

$\log_a b = \frac{1}{\log_b a}$

Start by letting the left side of the equation be equal to x. Then, rewrite in exponential form, manipulate, and rewrite back in logarithmic form until you get the expression from the left side of the equation.

$$\begin{aligned}\log_a b &= x\\ b^x &= a\\ b &= a^{\frac{1}{x}}\\ \log_b a &= \frac{1}{x}\\ x &= \frac{1}{\log_b a}\end{aligned}$$

Therefore, $\frac{1}{\log_b a} = \log_a b$ because both expressions are equal to x.

Example 4

Rewrite the following expression under a single log.

$\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$

$\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$

$$\begin{aligned}&= \ln\left(\frac{e}{4x}\right) + 2x \cdot \ln 5\\ &= \ln\left(\frac{e}{4x}\right) + \ln(5^{2x})\\ &= \ln\left(\frac{e \cdot 5^{2x}}{4x}\right)\end{aligned}$$

Example 5

True or false:

$(\log_3 4x) \cdot (\log_3 5y) = \log_3(4x + 5y)$

False. It is true that the log of a product is the sum of logs. It is not true that the product of logs is the log of a sum.

Note that it may be very tempting to make errors in this type of problem.

Review

Decide whether each of the following statements are true or false. Explain.

1. $\frac{\log x}{\log y} = \log\left(\frac{x}{y}\right)$
2. $(\log x)^n = n \log x$
3. $\log x + \log y = \log xy$

Rewrite each of the following expressions under a single log and simplify.

4. $\log 4x + \log(2x+4)$
5. $5\log x + \log x$
6. $4\log_2 x + \frac{1}{2}\log_2 9 - \log_2 y$
7. $6\log_3 z^2 + \frac{1}{4}\log_3 y^8 - 2\log_3 z^4 y$

Expand the expression as much as possible.

8. $\log_4\left(\frac{2x^3}{5}\right)$
9. $\ln\left(\frac{4xy^2}{15}\right)$
10. $\log\left(\frac{x^2(yz)^3}{3}\right)$

Translate from exponential form to logarithmic form.

11. $2^{x+1} + 4 = 14$

Translate from logarithmic form to exponential form.

12. $\log_2(x-1) = 12$

Prove the following properties of logarithms.

13. $\log_{b^n} x = \frac{1}{n}\log_b x$
14. $\log_{b^n} x^n = \log_b x$
15. $\log_{\frac{1}{b}} \frac{1}{x} = \log_b x$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.4.

2.9 Change of Base

Learning Objectives

Here you will extend your knowledge of log properties to a simple way to change the base of a logarithm. While it is possible to change bases by always going back to exponential form, it is more efficient to find out how to change the base of logarithms in general. Since there are only base e and base 10 logarithms on a calculator, how would you evaluate an expression like $\log_3 12$?

Changing the Base of Logarithms

The **change of base property** states:

$\log_b a = \frac{\log_x a}{\log_x b}$

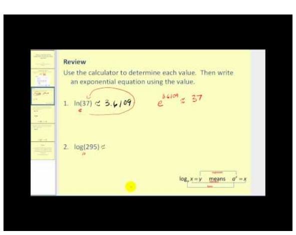
You can derive this formula by converting $\log_b a$ to exponential form and then taking the log base x of both sides. This is shown below.

$$\begin{aligned} \log_b a &= y \\ \rightarrow b^y &= a \\ \rightarrow \log_x b^y &= \log_x a \\ \rightarrow y \log_x b &= \log_x a \\ \rightarrow y &= \frac{\log_x a}{\log_x b} \end{aligned}$$

Therefore, $\log_b a = \frac{\log_x a}{\log_x b}$.

If you were to evaluate $\log_3 4$ using your calculator, you would need to use the change of base formula since a calculator only has base 10 or base e . The result would be:

$\log_3 4 = \frac{\log_{10} 4}{\log_{10} 3} = \frac{\ln 4}{\ln 3} \approx 1.262$



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Examples

Example 1

Earlier, you were asked how to use a calculator to evaluate an expression like $\log_3 12$. In order to evaluate an expression like $\log_3 12$ you have two options on your calculator:

$\frac{\ln 12}{\ln 3} = \frac{\log 12}{\log 3} \approx 2.26$

Example 2

Prove the following log identity.

$\log_a b = \frac{1}{\log_b a}$

$\log_a b = \frac{\log_x b}{\log_x a} = \frac{1}{\frac{\log_x a}{\log_x b}} = \frac{1}{\log_b a}$

Example 3

Simplify to an exact result: $(\log_4 5) \cdot (\log_3 4) \cdot (\log_5 81) \cdot (\log_5 25)$

$\frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 3^4}{\log 5} \cdot \frac{\log 5^2}{\log 5} = \frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{4 \cdot \log 3}{\log 5} \cdot \frac{2 \cdot \log 5}{\log 5} 4 \cdot 2 = 8$

Example 4

Evaluate: $\log_2 48 - \log_4 36$

$\log_2 48 - \log_4 36$

$$\begin{aligned}
&= \frac{\log 48}{\log 2} - \frac{\log 36}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{\log 6^2}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{2 \cdot \log 6}{2 \cdot \log 2} \\
&= \frac{\log 48 - \log 6}{\log 2} \\
&= \frac{\log \left(\frac{48}{6}\right)}{\log 2} \\
&= \frac{\log 8}{\log 2} \\
&= \frac{\log 2^3}{\log 2} \\
&= \frac{3 \cdot \log 2}{\log 2} \\
&= 3
\end{aligned}$$

Example 5

Given $\log_3 5 \approx 1.465$ find $\log_{25} 27$ without using a log button on the calculator.

$\log_{25} 27 = \frac{\log 3^3}{\log 5^2} = \frac{3}{2} \cdot \frac{1}{\left(\frac{\log 5}{\log 3}\right)} = \frac{3}{2} \cdot \frac{1}{\log_3 5} \approx \frac{3}{2} \cdot \frac{1}{1.465} = 1.0239$

Review

Evaluate each expression by changing the base and using your calculator.

1. $\log_6 15$

2. $\log_9 12$

3. $\log_5 25$

Evaluate each expression.

4. $\log_8(\log_4(\log_3 81))$

5. $\log_2 3 \cdot \log_3 4 \cdot \log_6 16 \cdot \log_4 6$

6. $\log 125 \cdot \log_9 4 \cdot \log_4 81 \cdot \log_5 10$

7. $\log_5(5^{\log_5 125})$

8. $\log(\log_6(\log_2 64))$

9. $10^{\log_{100} 9}$

10. $(\log_4 x)(\log_x 16)$

11. $\log_{49} 49^5$

12. $3\log_{24} 24^8$

13. $4^{\log_2 3}$

Prove the following properties of logarithms.

14. $(\log_a b)(\log_b c) = \log_a c$

15. $(\log_a b)(\log_b c)(\log_c d) = \log_a d$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.5.

2.10 Exponential Equations

Learning Objectives

Here you will apply the new algebraic techniques associated with logs to solve equations.

When you were first learning equations, you learned the rule that whatever you do to one side of an equation, you must also do to the other side so that the equation stays in balance. The basic techniques of adding, subtracting, multiplying and dividing both sides of an equation worked to solve almost all equations up until now. With logarithms, you have more tools to isolate a variable. Consider the following equation and ask yourself: why is $x = 3$? Logically it makes sense that if the bases match, then the exponents must match as well, but how can it be shown?

$$1.79898^{2x} = 1.79898^6$$

Solving Exponential Equations

A common technique for solving equations with unknown variables in exponents is to take the log of the desired base of both sides of the equation. Then, you can use properties of logs to simplify and solve the equation.

Take the following equation. To solve for t, you should first simplify the expression as much as possible and then take the natural log of both sides.

$9,000 = 300 \cdot \frac{(1.06)^t - 1}{0.06}$

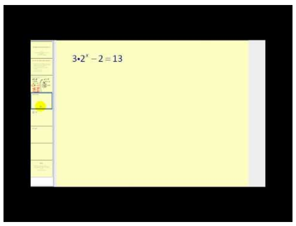
$$\begin{aligned} 30 &= \frac{(1.06)^t - 1}{0.06} \\ 1.8 &= (1.06)^t - 1 \\ 2.8 &= 1.06^t \\ \ln 2.8 &= \ln(1.06^t) = t \cdot \ln(1.06) \\ t &= \frac{\ln 2.8}{\ln 1.06} \approx 17.67 \textit{ years} \end{aligned}$$

It does not matter what base you use in this situation as long as you use the same base on both sides. Choosing natural log allows you to use a calculator to finish the problem.

Note that this type of equation is common in financial mathematics. The equation above represents the unknown amount of time it will take you to save \$9,000 in a savings account if you save \$300 at the end of each year in an account that earns 6% annual compound interest.

The other good base to use is base 10. When solving the following equation for x: $16^x = 25$, you will need to use a calculator to get the final answer and your calculator can handle base 10 as well. First take the log of both sides. Then, use log properties and your calculator to help.

$$\begin{aligned}
16^x &= 25 \\
\log 16^x &= \log 25 \\
x \log 16 &= \log 25 \\
x &= \frac{\log 25}{\log 16} \\
x &= 1.61
\end{aligned}$$



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Examples

Example 1

Earlier, you were asked how to show that if the bases match in an equation, the exponents should match. In the equation, logs can be used to reduce the equation to $2x = 6$.

$1.79898^{2x} = 1.79898^6$

Take the log of both sides and use the property of exponentiation of logs to bring the exponent out front.

$$\begin{aligned}
\log 1.79898^{2x} &= \log 1.79898^6 \\
2x \cdot \log 1.79898 &= 6 \cdot \log 1.79898 \\
2x &= 6 \\
x &= 3
\end{aligned}$$

Example 2

Solve the following equation for all possible values of x: $(x+1)^{x-4} - 1 = 0$

$(x+1)^{x-4} - 1 = 0$

$(x+1)^{x-4} = 1$

Case 1 is that $x+1$ is positive in which case you can take the log of both sides.

$$\begin{aligned}
&(x-4) \cdot \log(x+1) = 0 \\
&x = 4, 0
\end{aligned}$$

Note that $\log 1 = 0$

Case 2 is that $(x+1)$ is negative 1 and raised to an even power. This happens when $x = -2$.

The reason why this exercise is included is because you should not fall into the habit of assuming that you can take the log of both sides of an equation. It is only valid when the argument is strictly positive.

Example 3

Light intensity as it travels at specific depths of water in a swimming pool can be described by the relationship between i for intensity and d for depth in feet. What is the intensity of light at 10 feet?

$\log\left(\frac{i}{12}\right) = -0.0145 \cdot d$

Given $d = 10$, solve for i measured in lumens.

$$\begin{aligned}
\log\left(\frac{i}{12}\right) &= -0.0145 \cdot d \\
\log\left(\frac{i}{12}\right) &= -0.0145 \cdot 10 \\
\log\left(\frac{i}{12}\right) &= -0.145 \\
\left(\frac{i}{12}\right) &= e^{-0.145} \\
i &= 12 \cdot e^{-0.145} \approx 10.380
\end{aligned}$$

Example 4

Solve the following equation for all possible values of x.

$$\frac{e^x - e^{-x}}{3} = 14$$

First solve for e^x,

$$\begin{aligned}
\frac{e^x - e^{-x}}{3} &= 14 \\
e^x - e^{-x} &= 42 \\
e^{2x} - 1 &= 42e^x \\
(e^x)^2 - 42e^x - 1 &= 0
\end{aligned}$$

Let $u = e^x$.

$$\begin{aligned}
&u^2 - 42u - 1 = 0 \\
&u = \frac{-(-42) \pm \sqrt{(-42)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{42 \pm \sqrt{1768}}{2} \approx 42.023796, -0.0237960
\end{aligned}$$

Note that the negative result is extraneous so you only proceed in solving for x for one result.

$$
\begin{aligned}
e^x &\approx 42.023796 \\
x &\approx \ln 42.023796 \approx 3.738
\end{aligned}
$$

Example 5

Solve the following equation for all possible values of x: $(\log_2 x)^2 - \log_2 x^7 = -12$.

In calculus it is common to use a small substitution to simplify the problem and then substitute back later. In this case let $u = \log_2 x$ after the 7 has been brought down and the 12 brought over.

$$
\begin{aligned}
(\log_2 x)^2 - 7\log_2 x + 12 &= 0 \\
u^2 - 7u + 12 &= 0 \\
(u-3)(u-4) &= 0 \\
u &= 3, 4
\end{aligned}
$$

Now substitute back and solve for x in each case.

$$
\begin{aligned}
\log_2 x = 3 &\leftrightarrow 2^3 = x = 8 \\
\log_2 x = 4 &\leftrightarrow 2^4 = x = 16
\end{aligned}
$$

Review

Solve each equation for x. Round each answer to three decimal places.

1. $4^x = 6$
2. $5^x = 2$
3. $12^{4x} = 1020$
4. $7^{3x} = 2400$
5. $2^{x+1} - 5 = 22$
6. $5x + 12^x = 5x + 7$
7. $2^{x+1} = 2^{2x+3}$
8. $3^{x+3} = 9^{x+1}$
9. $2^{x+4} = 5^x$
10. $13 \cdot 8^{0.2x} = 546$
11. $b^x = c + a$
12. $32^x = 0.94 - .12$

Solve each log equation by using log properties and rewriting as an exponential equation.

13. $\log_3 x + \log_3 5 = 2$
14. $2\log x = \log 8 + \log 5 - \log 10$
15. $\log_9 x = \frac{3}{2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.6.

2.11 Simplifying Algebraic Expressions and Equations

Objective

To solve and simplify algebraic expressions and equations for a variable or number.

Review Queue

Use the Order of Operations to simplify the following expressions.

1. $9+6\cdot 2+6^2 \div 4-5$
2. $(9+6)\cdot 2+6^2 \div (4-5)$
3. $7(3x-4)+x$
4. $x+5-(4x+3)$

Evaluating Algebraic Expressions and Equations

Objective

To solve an expression or equation for a given value of a variable.

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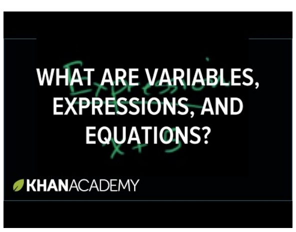

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Khan Academy: Variables Expressions and Equations

Guidance

In #3 and #4 in the Review Queue above, we introduce a letter into a mathematical expression. These letters, also called **variables,** represent an unknown number. One of the goals of algebra is to solve various equations for a variable. Typically, x is used to represent the unknown number, but any letter can be used.

To evaluate an expression or equation, that means, a value for the variable will be given and we need to plug it into the expression or equation and test it. In order for the given value to be true for an equation, the two sides of the equation must simplify to the same number.

Example A

Evaluate $2x^2-9$ for $x=-3$.

Solution: We know that $2x^2-9$ is an expression because it does not have an equals sign. Therefore, to evaluate this expression, plug in -3 for x and simplify using the Order of Operations.

$$\begin{aligned}
&2(-3)^2-9 \rightarrow (-3)^2 = -3\cdot -3 = 9\\
&2(9)-9\\
&18-9\\
&9
\end{aligned}$$

You will need to remember that when squaring a negative number, the answer will always be positive. There are three different ways to write multiplication: $2\times 9, 2\cdot 9$, and 2(9).

Example B

Determine if $x = 5$ is a solution to $3x - 11 = 14$.

Solution: Even though the directions are different, this problem is almost identical to Example A. However, this is an equation because of the equals sign. Both sides of an equation must be equal to each other in order for it to be true. Plug in 5 everywhere there is an x. Then, determine if both sides are the same.

$$\begin{aligned}
3(5)-11 &\overset{?}{=} 14\\
15-11 &\neq 14\\
4 &\neq 14
\end{aligned}$$

Because $4 \neq 14$, this is not a true equation. Therefore, 5 is not a solution.

Example C

Determine if $t = -2$ is a solution to $7t^2 - 9t - 10 = 36$.

Solution: Here, t is the variable and it is listed twice in this equation. Plug in -2 everywhere there is a t and simplify.

$$\begin{aligned}
7(-2)^2-9(-2)-10 &\overset{?}{=} 36\\
7(4)+18-10 &\overset{?}{=} 36\\
28+18-10 &\overset{?}{=} 36\\
36 &= 36
\end{aligned}$$

-2 is a solution to this equation.

Guided Practice

1. Evaluate $s^3 - 5s + 6$ for $s = 4$.

2. Determine if $a = -1$ is a solution to $4a - a^2 + 11 = -2 - 2a$.

Answers

1. Plug in 4 everywhere there is an s.

$$
\begin{aligned}
&4^3 - 5(4) + 6 \\
&64 - 20 + 6 \\
&50
\end{aligned}
$$

2. Plug in -1 for *a* and see if both sides of the equation are the same.

$$
\begin{aligned}
4(-1) - (-1)^2 + 11 &\stackrel{?}{=} -2 - 2(-1) \\
-4 - 1 + 11 &\stackrel{?}{=} -2 + 2 \\
6 &\neq 0
\end{aligned}
$$

Because the two sides are not equal, -1 is not a solution to this equation.

Vocabulary

Variable
A letter used to represent an unknown value.

Expression
A group of variables, numbers, and operators.

Equation
Two expressions joined by an equal sign.

Solution
A numeric value that makes an equation true.

Problem Set

Evaluate the following expressions for $x = 5$.

1. $4x - 11$
2. $x^2 + 8$
3. $\frac{1}{2}x + 1$

Evaluate the following expressions for the given value.

4. $-2a + 7; a = -1$
5. $3t^2 - 4t + 5; t = 4$
6. $\frac{2}{3}c - 7; c = -9$
7. $x^2 - 5x + 6; x = 3$
8. $8p^2 - 3p - 15; p = -2$
9. $m^3 - 1; m = 1$

Determine if the given values are solutions to the equations below.

10. $x^2 - 5x + 4 = 0; x = 4$
11. $y^3 - 7 = y + 3; x = 2$
12. $7x - 3 = 4; x = 1$
13. $6z + z - 5 = 2z + 12; z = -3$
14. $2b - 5b^2 + 1 = b^2; b = 6$
15. $-\frac{1}{4}g + 9 = g + 15; g = -8$

Find the value of each expression, given that $a = -1, b = 2, c = -4$, and $d = 0$.

16. $ab - c$
17. $b^2 + 2d$
18. $c + \frac{1}{2}b - a$
19. $b(a + c) - d^2$

For problems 20-25, use the equation $y^2 + y - 12 = 0$.

20. Is $y = 4$ a solution to this equation?
21. Is $y = -4$ a solution to this equation?
22. Is $y = 3$ a solution to this equation?
23. Is $y = -3$ a solution to this equation?
24. Do you think there are any other solutions to this equation, other than the ones found above?
25. **Challenge** Using the solutions you found from problems 20-23, find the sum of these solutions and their product. What do you notice?

Simplifying Algebraic Expressions

Objective

To combine like terms within an expression.

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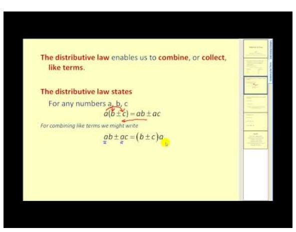

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James Sousa: Combining Like Terms

Guidance

You might have noticed from the previous concept, that sometimes variables and numbers can be repeated within an expression. If the same variable is in an expression more than once, they can be combined by addition or subtraction. This process is called **combining like terms.**

Example A

Simplify $5x - 12 - 3x + 4$.

Solution: Reorganize the expression to group together the x's and the numbers. You can either place the like terms next to each together or place parenthesis around the like terms.

$$\begin{aligned}
&5x-12-3x+4\\
&5x-3x-12+4 \text{ } or \text{ } (5x-3x)+(-12+4)\\
&2x-8
\end{aligned}$$

Notice that the Greatest Common Factor (GCF) for $2x$ and 8 is 2. Therefore, you can and use the Distributive Property to pull out the GCF to get $2(x-4)$.

Example B

Simplify $6a-5b+2a-10b+7$

Solution: Here there are two different variables, a and b. Even though they are both variables, they are *different* variables and cannot be combined. Group together the like terms.

$$\begin{aligned}
&6a-5b+2a-10b+7\\
&(6a+2a)+(-5b-10b)+7\\
&(8a-15b+7)
\end{aligned}$$

There is only one number term, called the **constant**, so we leave it at the end. Also, in general, list the variables in alphabetical order.

Example C

Simplify $w^2+9-4w^2+3w^4-7w-11$.

Solution: Here we have one variable, but there are different powers (exponents). Like terms must have the same exponent in order to combine them.

$$\begin{aligned}
&w^2+9-4w^2+3w^4-7w-11\\
&3w^4+(w^2-4w^2)-7w+(9-11)\\
&3w^4-3w^2-7w-2
\end{aligned}$$

When writing an expression with different powers, list the powers from greatest to least, like above.

Guided Practice

Simplify the expressions below.

1. $6s-7t+12t-10s$

2. $7y^2-9x^2+y^2-14x+3x^2-4$

Answers

1. Combine the s's and the t's.

$$\begin{aligned}
&6s-7t+12t-10s\\
&(6s-10s)+(-7t+12t)\\
&-4s+5t
\end{aligned}$$

2. Group together the like terms.

$$7y^2 - 9x^2 + y^2 - 14x + 3x^2 - 4$$
$$(-9x^2 + 3x^2) + (7y^2 + y^2) - 14x - 4$$
$$-6x^2 + 8y^2 - 14x - 4$$

Notice in #1, we did not write $(6s - 10s) - (7t + 12t)$ in the second step. This would lead us to an incorrect answer. Whenever grouping together like terms and one is negative (or being subtracted), always change the operator to addition and make the number negative.

In #2, we can also take out the Greatest Common Factor of -2 from each term using the Distributive Property. This would reduce to $-2(3x^2 - 4y^2 + 7x + 2)$. In this case, we take out a -2 so that the first term is positive.

Vocabulary

Constant
A number that is added or subtracted within an expression. In the expression $3x^2 - 8x - 15$, -15 is the constant.

Greatest Common Factor (GCF)
The largest number or variable that goes into a set of numbers.

Problem Set

Simplify the following expressions as much as possible. If the expression cannot be simplified, write "cannot be simplified."

1. $5b - 15b + 8d + 7d$
2. $6 - 11c + 5c - 18$
3. $3g^2 - 7g^2 + 9 + 12$
4. $8u^2 + 5u - 3u^2 - 9u + 14$
5. $2a - 5f$
6. $7p - p^2 + 9p + q^2 - 16 - 5q^2 + 6$
7. $20x - 6 - 13x + 19$
8. $8n - 2 - 5n^2 + 9n + 14$

Find the GCF of the following expressions and use the Distributive Property to simplify each one.

9. $6a - 18$
10. $9x^2 - 15$
11. $14d + 7$
12. $3x - 24y + 21$

Challenge We can also use the Distributive Property and GCF to pull out common variables from an expression. Find the GCF and use the Distributive Property to simplify the following expressions.

13. $2b^2 - 5b$
14. $m^3 - 6m^2 + 11m$
15. $4y^4 - 12y^3 - 8y^2$

Solving Algebraic Equations for a Variable

Objective

To isolate the variable in an equation or formula.

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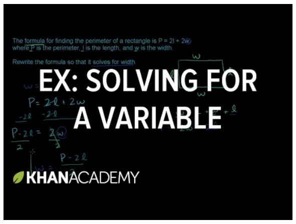

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Khan Academy: Example of Solving for a Variable

Guidance

Solving an algebraic equation for a variable can be tricky, but it can also be very useful. This technique can be used to go back and forth between to different units of measurement. To solve for, or **isolate**, a variable within an equation, you must undo the operations that are in the equation.

Example A

Solve for x in the equation $3x-4y=12$.

Solution: To solve for x, you need to move the $4y$ to the other side of the equation. In order to do this, we must do the opposite operation. Since the $4y$ is being subtracted, we must add it to the other side of the equation.

$$\begin{aligned} 3x-4y&=12\\ \underline{+4y}&\underline{=+4y}\\ 3x&=4y+12 \end{aligned}$$

Now we need to get x alone. $3x$ is the same as "3 multiplied by x." Therefore, to undo the multiplication we must divide both sides of the equation by 3.

$$\begin{aligned} \frac{3x}{3}&=\frac{4y}{3}+\frac{12}{3}\\ x&=\frac{4}{3}y+4 \end{aligned}$$

When undoing multiplication, you must divide *everything* in the equation by 3.

A few things to note:

1. To undo an operation within an equation, do the opposite.
2. Perform the opposite operations in the reverse order of the Order of Operations.
3. Combine all like terms on the same side of the equals sign before doing #1.

Example B

Given the equation $\frac{2b}{3}+6a-4=8$, find b when $a=1$ and $a=-2$.

Solution: This example combines what was learned in the last section with what we did in the previous example. First, isolate b. Move both the $6a$ and the 4 over to the other side by doing the opposite operation.

$$\begin{aligned}\frac{2b}{3}+6a-4&=8\\ -6a+4&=+4-6a\\ \hline \frac{2b}{3}&=12-6a\end{aligned}$$

Now, we have a fraction multiplied by b. To undo this we must multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$. This means, we must multiply everything in the equation by the reciprocal.

$$\begin{aligned}\frac{2b}{3}&=12-6a\\ \frac{3}{2}\left(\frac{2b}{3}\right.&=\left.12-6a\right)\\ \frac{3}{2}\cdot\frac{2b}{3}&=\frac{3}{2}\cdot 12-\frac{3}{2}\cdot 6a\\ b&=18-9a\end{aligned}$$

Even though we know that a 9 can be pulled out of the 18 and the 9 in the above equation, it is not a necessary step in solving for b. Now, we can do what the question asks, find b when $a = 1$ and -2. Plug in these values for a.

$$a=1: b=18-9(1)=18-9=9 \qquad a=-2: 18-9(-2)=18+18=36$$

Example C

The area of a triangle is $A=\frac{1}{2}bh$, where b is the base of the triangle and h is the height. You know that the area of a triangle is 60 in^2 and the base is 12 in. Find the height.

Solution: First solve the equation for h.

$$\begin{aligned}A&=\frac{1}{2}bh\\ 2\cdot A&=2\cdot\frac{1}{2}bh && \text{Multiply both sides by 2.}\\ 2A&=bh && \text{Divide both side by } b.\\ \frac{2A}{b}&=h\end{aligned}$$

Now, plug in what we know to find h. $\frac{2(60)}{12}=\frac{120}{12}=10$. The height is 10 inches.

This process is helpful when solving for any variable in any equation or formula. Some formulas you may need to know for this text are listed below.

TABLE 2.5: Helpful Formulas

Distance	$d=rt$	$d=$ distance, $r=$ rate, $t=$ time
Temperature	$F=\frac{9}{5}C+32$	F = degrees in Fahrenheit, C = degrees in Celsius
Area of a Triangle	$A=\frac{1}{2}bh$	$A=$ area, $b=$ base, $h=$ height

TABLE 2.5: (continued)

Distance	$d = rt$	$d = \text{distance}, r = \text{rate}, t = \text{time}$
Area of a Rectangle	$A = bh$	$A = \text{area}, b = \text{base}, h = \text{height}$
Area of a Circle	$A = \pi r^2$	$A = \text{area}, r = \text{radius}$
Area of a Trapezoid	$A = \frac{1}{2}h(b_1 + b_2)$	$A = \text{area}, h = \text{height}, b_1 = \text{one base}, b_2 = \text{other base}$
Perimeter of a Rectangle	$P = 2l + 2w$	$P = \text{perimeter}, l = \text{length}, w = \text{width}$
Circumference of a Circle	$C = 2\pi r$	$C = \text{circumference}, r = \text{radius}$

Guided Practice

1. Solve $\frac{3}{4}x - 2y = 15$ for x.

2. If the temperature is $41^\circ F$, what is it in Celsius?

Answers

1. First add $2y$ to both sides, then multiply both sides by the reciprocal of $\frac{3}{4}$.

$$\begin{aligned} \frac{3}{4}x - 2y &= 15 \\ +2y \quad &+2y \\ \hline \frac{3}{4}x &= 2y + 15 \\ x &= \frac{4}{3} \cdot 2y + \frac{4}{3} \cdot 15 \\ x &= \frac{8}{3}y + 20 \end{aligned}$$

2. You can solve this problem two different ways: First, plug in 41 for F and then solve for C or solve for C first and then plug in 41 for F. We will do the second option.

$$\begin{aligned} F &= \frac{9}{5}C + 32 \\ F - 32 &= \frac{9}{5}C \\ \frac{5(F-32)}{9} &= C \end{aligned}$$

Now, plug in 41 for F.

$$\begin{aligned} C &= \frac{5(41-32)}{9} \\ C &= \frac{5 \cdot 9}{9} \\ C &= 5^\circ \end{aligned}$$

.

Problem Set

Solve the following equations or formulas for the indicated variable.

1. $6x-3y=9$; solve for y.
2. $4c+9d=16$; solve for c.
3. $5f-6g=14$; solve for f.
4. $\frac{1}{3}x+5y=1$; solve for x.
5. $\frac{4}{5}m+\frac{2}{3}n=24$; solve for m.
6. $\frac{4}{5}m+\frac{2}{3}n=24$; solve for n.
7. $P=2l+2w$; solve for w.
8. $F=\frac{9}{5}C+32$; solve for C.

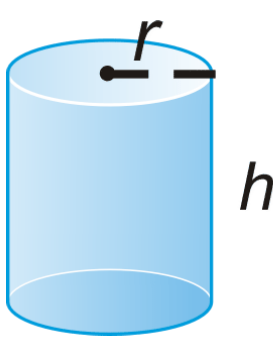
Find the value of y given the value of x.

9. $4x-8y=2; x=-1$
10. $2y-5x=12; x=16$
11. $3x+\frac{1}{2}y=-5; x=7$
12. $\frac{1}{4}x+\frac{2}{3}y-18=0; x=-24$

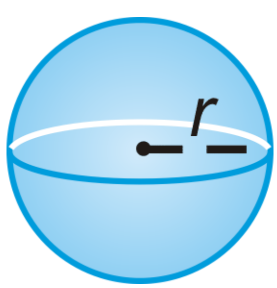
For questions 13-15, use the formulas in the chart above to answer the following questions.

13. If the temperature is $86°F$, what is it in Celsius?
14. If the area of a circle is $36\pi\ cm^2$, what is the radius?
15. The area of a trapezoid is $72\ ft^2$, the height is 8 ft and b_1 is 6, what is the length of the other base?

For questions 16-17, use the equation for the surface area of a cylinder, $SA=2\pi r^2+2\pi rh$, where r is the radius and h is the height.



16. Solve the equation for h.
17. Find h if the surface area is $120\pi\ cm^2$ and the radius is 6 cm.
18. **Challenge** The formula for the volume of a sphere is $V=\frac{4}{3}\pi r^3$. Solve the equation for r, the radius.



2.12 Solving Linear Equations

Objective

To solve equations in one variable using one, two, or multiple steps.

Review Queue

1. Find the value of each expression if $a = 5$ and $b = -3$.

a) $6a - b$

b) $a^2 - b^2$

c) $9b + 2a$

2. Solve $9x + 7y = 63$ for y.

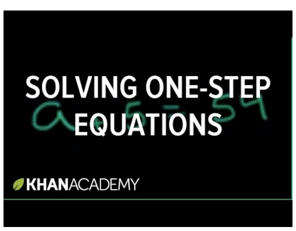
3. Solve $9x + 7y = 63$ for x.

Solving One-Step Linear Equations

Objective

To solve simple linear equations with one step.

Watch This



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Khan Academy: Solving One-Step Equations

Guidance

When solving an equation for a variable, you must get the variable *by itself.* All the equations in this lesson are **linear equations.** That means the equation can be simplified to $ax + b = c$, where a, b, and c are real numbers. Here we will only deal with using one operation; addition, subtraction, multiplication or division.

Example A

Solve $7 + y = 16$ for y.

Solution: This problem is simple and you could probably solve it in your head. However, to start good practices, you should always use algebra to solve any equation. Even if the problem seems easy, equations will get more difficult to solve.

To solve an equation for a variable, you must do the opposite, or undo, whatever is on the same side as the variable. 7 is being added to y, so we must subtract 7 from both sides. Notice that this is very similar to the last concept in the previous concept (*Solving Algebraic Equations for a Variable*).

$$\begin{aligned} \cancel{7} + y &= 16 \\ \underline{-\cancel{7} \qquad} & \underline{\; -7} \\ y &= 9 \end{aligned}$$

You can check that $y = 9$ is correct by plugging 9 back into the original equation. 7 + 9 does equal 16, so we know that we found the correct answer.

Example B

Solve $-7h = 84$.

Solution: Recall that $-7h = -7 \times h$, so the opposite, or **inverse**, operation of multiplication is division. Therefore, we must divide both sides by -7 to solve for h.

$$\begin{aligned} \frac{-\cancel{7}h}{-\cancel{7}} &= \frac{84}{-7} \\ h &= -12 \end{aligned}$$

Again, check your work. $-7 \cdot -12$ is equal to 84, so we know our answer is correct.

Example C

Solve $\frac{3}{8}x = \frac{3}{2}$.

Solution: The variable is being multiplied by a fraction. Instead of dividing by a fraction, we multiply by the **reciprocal** of $\frac{3}{8}$, which is $\frac{8}{3}$.

$$\begin{aligned} \cancel{\frac{8}{3}} \cdot \cancel{\frac{3}{8}} x &= \frac{\cancel{3}}{2} \cdot \frac{8}{\cancel{3}} \\ x &= \frac{8}{2} = 4 \end{aligned}$$

Check the answer: $\frac{3}{_2\cancel{8}} \cdot \cancel{4} = \frac{3}{2}$. This is correct, so we know that $x = 4$ is the answer.

Guided Practice

Solve the following equations for the given variable. Check your answer.

1. $5 + j = 17$

2. $\frac{h}{6} = -11$

Answers

1. Subtract 5 from both sides to solve for j.

$$\begin{aligned} \cancel{5} + j &= 17 \\ \underline{-\cancel{5} \qquad} & \underline{\; -5} \\ j &= 12 \end{aligned}$$

Check the answer: $5 + 12 = 17$ $\boxed{\checkmark}$

2. h is being divided by 6. To undo division, we must multiply both sides by 6.

$$\cancel{6} \cdot \frac{h}{\cancel{6}} = -11.6$$
$$h = -66$$

Check the answer: $\frac{-66}{6} = -11$ $\boxed{\checkmark}$

Vocabulary

Linear Equation
An equation in one variable without exponents. Linear equations are in the form $ax \pm b = c$, where a, b, and c are real numbers.

Inverse
The opposite operation of a given operation in an equation. For example, subtraction is the inverse of addition and multiplication is the inverse of division.

Reciprocal
The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Problem Set

Solve each equation below and check your answer. Reduce any fractions.

1. $-3 + x = -1$
2. $r + 6 = 2$
3. $5s = 30$
4. $-8k = -64$
5. $\frac{m}{-4} = 14$
6. $90 = 10n$
7. $-16 = y - 5$
8. $\frac{6}{7}d = 36$
9. $6 = -\frac{1}{3}p$
10. $u - \frac{3}{4} = \frac{5}{6}$
11. $\frac{8}{5}a = -\frac{72}{13}$
12. $\frac{7}{8} = b + \frac{1}{2}$
13. $w - (-5) = 16$
14. $\frac{1}{4} = b - \left(-\frac{2}{5}\right)$
15. $\frac{3}{5}q = -\frac{12}{11}$
16. $\frac{t}{12} = -4$
17. $45 = 15x$
18. $7 = \frac{g}{-8}$

Challenge Solve the two equations below. Be careful!

19. $14 - z = -3$
20. $\frac{-k}{9} = 5$

Solving Two-Step Equations

Objective

To solve linear equations involving two steps.

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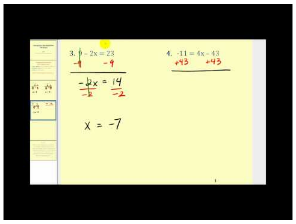

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James Sousa: Solving Two Step Equations

Guidance

Now, we will compound steps to solve slightly more complicated equations that the previous concept. We will continue to undo the operations that are in an equation. Remember that order matters. To undo the operations, work in the opposite order of the Order of Operations.

Example A

Solve $4x-9=11$

Solution: Now we need to undo two operations. Therefore, we need to get everything on the other side of the equals sign to solve for x. First, add 9 to both sides.

$$\begin{aligned} 4x-\cancel{9}&=11\\ \underline{+\cancel{9}\quad}&\underline{\;+9}\\ 4x&=20 \end{aligned}$$

Now, undo the multiplication. Divide both sides by 4 to solve for x.

$$\frac{\cancel{4}x}{\cancel{4}}=\frac{20}{4}$$
$$x=5$$

Check your answer: $4(5)-9=20-9=11\ \boxed{\checkmark}$

Example B

Solve $-\frac{1}{2}g+5=14$.

Solution: First, subtract 5 from both sides.

$$\begin{aligned} -\frac{1}{2}g+\cancel{5}&=14\\ \underline{-\cancel{5}\quad}&\underline{\;-5}\\ -\frac{1}{2}g&=9 \end{aligned}$$

Multiplying by one-half is the same as dividing by 2. To undo this fraction, multiply both sides by –2.

$$\begin{aligned}-\frac{1}{2}g &= 9\\ -\cancel{2}\cdot -\frac{1}{\cancel{2}}g &= 9\cdot -2\\ g &= -18\end{aligned}$$

Check your answer: $-\frac{1}{2}(-18)+5=9+5=14$ $\boxed{\checkmark}$

Example C

Solve $\frac{3}{4}x-\frac{2}{3}=\frac{7}{3}$.

Solution: Even though this problem involves fractions, it is still solved the same way. First, add $\frac{2}{3}$ to both sides and then multiply both sides by the reciprocal of $\frac{3}{4}$.

$$\begin{aligned}\frac{3}{4}x-\cancel{\frac{2}{3}} &= \frac{7}{3}\\ +\cancel{\frac{2}{3}} &\ +\frac{2}{3}\\ \hline \frac{3}{4}x &= \frac{9}{3}=3\\ \cancel{\frac{4}{3}}\cdot\cancel{\frac{3}{4}}x &= \cancel{3}\cdot\frac{4}{\cancel{3}}\\ x &= 4\end{aligned}$$

Check your answer: $\frac{3}{4}(4)-\frac{2}{3}=3-\frac{2}{3}=\frac{9}{3}-\frac{2}{3}=\frac{7}{3}$ $\boxed{\checkmark}$.

For two-step equations:

1. First, undo any addition or subtraction.
2. Second, undo any multiplication or division.

Guided Practice

Solve the following equations and check your answer.

1. $\frac{x}{-6}-7=-11$

2. $18=6-\frac{2}{5}y$

Answers

1. Follow the steps from above. First add 7 to both sides and then multiply both sides by –6.

$$\begin{aligned}\frac{x}{-6}-\cancel{7} &= -11\\ +\cancel{7} &\ \ +7\\ \hline \frac{x}{-6} &= -4\\ -\cancel{6}\cdot\frac{x}{-\cancel{6}} &= -4\cdot -6\\ x &= 24\end{aligned}$$

Check your answer: $\frac{24}{-6}-7=-4-7=-11$ $\boxed{\checkmark}$

2. Again, follow the steps. Here, we will subtract 6 and then multiply by $-\frac{5}{2}$.

$$\begin{aligned}
18 &= \cancel{6}-\frac{2}{5}y \\
\underline{-6} &\underline{\;-\cancel{6}\qquad} \\
12 &= -\frac{2}{5}y \\
-\frac{5}{2}\cdot 12 &= -\frac{\cancel{2}}{\cancel{5}}y\cdot-\frac{\cancel{5}}{\cancel{2}} \\
-30 &= y
\end{aligned}$$

Check your answer: $6-\frac{2}{5}(-30)=6+12=18$ $\boxed{\checkmark}$

Problem Set

Solve the following equations and check your answer. Reduce any fractions.

1. $2x-5=-17$
2. $-4x+3=-5$
3. $-1=\frac{x}{2}-6$
4. $\frac{1}{3}x+11=-2$
5. $\frac{3}{4}=\frac{1}{2}-\frac{1}{8}x$
6. $-18=\frac{x}{-5}-3$
7. $\frac{5}{6}x+4=29$
8. $-11x+4=125$
9. $6-x=-22$
10. $\frac{2}{7}x+8=20$
11. $\frac{4}{5}=-\frac{2}{5}+\frac{3}{2}x$
12. $15-\frac{x}{-9}=21$
13. $1.4x-5.6=2.8$
14. $14.4=-2.7x-1.8$
15. $-\frac{5}{4}=\frac{1}{6}x+\frac{3}{12}$

When dealing with fractions, one way to "get rid" of them is to multiply everything in the equation by the Least Common Denominator (LCD) for the entire equation. Try this technique with the following equations. We will get you started in #16.

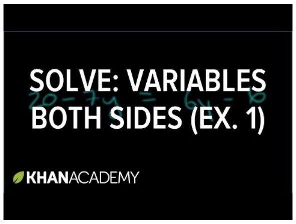
16. $\frac{3}{8}x-\frac{2}{5}=\frac{7}{4}$ The LCD of 8, 5, and 4 is 40. $40\left(\frac{3}{8}x-\frac{2}{5}=\frac{7}{4}\right)$. Distribute the 40 and solve.
17. $\frac{10}{3}x+\frac{3}{4}=\frac{5}{2}$
18. $\frac{9}{10}x-\frac{1}{2}=\frac{2}{3}$
19. Solve #15 using the LCD method.
20. Which method is easier for you to solve equations with fractions?

Solving Multi-Step Equations

Objective

To solve multi-step linear equations involving the Distributive Property, combining like terms or having the variable on both sides of the equals sign.

Watch This



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Khan Academy: Multi-step equations 1

Guidance

The types of equations in this concept involve at least three steps. Keep in mind that the last two steps when solving a linear equation will always be the same: add or subtract the number that is on the same side of the equals sign as the variable, then multiply or divide by the number with the variable.

Example A

Solve $3(x-5)+4=10$.

Solution: When solving more complicated equations, start with one side and simplify as much as you can. The left side of this equation looks more complicated, so let's simplify it by using the Distributive Property and combining like terms.

$$\begin{aligned}3(x-5)+4&=10\\3x-15+4&=10\\3x-11&=10\end{aligned}$$

Now, this looks like an equation from the previous concept. Continue to solve.

$$\begin{array}{r}3x-\cancel{11}=10\\ \underline{+\cancel{11}\ +11}\\ \dfrac{\cancel{3}x}{\cancel{3}}=\dfrac{21}{3}\\ x=7\end{array}$$

Check your answer: $3(7-5)+4=3\cdot2+4=6+4=10\ \boxed{\checkmark}$

Example B

Solve $8x-17=4x+23$.

Solution: This equation has x on both sides of the equals sign. Therefore, we need to move one of the x terms to the other side of the equation. It does not matter which x term you move. We will move the $4x$ to the other side so that, when combined, the x term is positive.

$$\begin{aligned}
8x - 17 &= \cancel{4x} + 23 \\
\underline{-4x \quad} & \underline{\quad -\cancel{4x} \quad} \\
4x - \cancel{17} &= 23 \\
\underline{\quad +\cancel{17}} & \underline{\; +17 \quad} \\
\frac{\cancel{4}x}{\cancel{4}} &= \frac{40}{4} \\
x &= 10
\end{aligned}$$

Check your answer:

$$\begin{aligned}
8(10) - 17 &= 4(10) + 23 \\
80 - 17 &= 40 + 23 \; \boxed{\checkmark} \\
63 &= 63
\end{aligned}$$

Example C

Solve $2(3x-1)+2x = 5-(2x-3)$.

Solution: This equation combines the two previous examples. First, use the Distributive Property.

$$\begin{aligned}
2(3x-1)+2x &= 5-(2x-3) \\
6x-2+2x &= 5-2x+3
\end{aligned}$$

Don't forget to distribute the negative sign in front of the second set of parenthesis. Treat it like distributing a –1. Now, combine like terms and solve the equation.

$$\begin{aligned}
8x - \cancel{2} &= 8 - \cancel{2x} \\
\underline{+2x + \cancel{2}} & \underline{+ 2 + \cancel{2x}} \\
\frac{\cancel{10}x}{\cancel{10}} &= \frac{10}{10} \\
x &= 1
\end{aligned}$$

Check your answer:

$$\begin{aligned}
2(3(1)-1)+2(1) &= 5-(2(1)-3) \\
2 \cdot 2 + 2 &= 5 - (-1) \; \boxed{\checkmark} \\
4 + 2 &= 6
\end{aligned}$$

Guided Practice

Solve each equation below and check your answer.

1. $\frac{3}{4} + \frac{2}{3}x = 2x + \frac{5}{6}$

2. $0.6(2x-7) = 5x - 5.1$

Answers

1. Use the LCD method introduced in the problem set from the previous concept. Multiply every term by the LCD of 4, 3, and 6.

$$\begin{aligned} 12\left(\frac{3}{4}+\frac{2}{3}x = 2x+\frac{5}{6}\right) \\ 9+8x = 24x+10 \end{aligned}$$

Now, combine like terms. Follow the steps from Example B.

$$\begin{aligned} 9+\cancel{8x} &= 24x+\cancel{10} \\ \underline{-10-\cancel{8x}} &\quad \underline{-8x-\cancel{10}} \\ \frac{-1}{16} &= \frac{\cancel{16}x}{\cancel{16}} \\ -\frac{1}{16} &= x \end{aligned}$$

Check your answer:

$$\begin{aligned} \frac{3}{4}+\frac{2}{3}\left(-\frac{1}{16}\right) &= 2\left(-\frac{1}{16}\right)+\frac{5}{6} \\ \frac{3}{4}-\frac{1}{24} &= -\frac{1}{8}+\frac{5}{6} \\ \frac{18}{24}-\frac{1}{24} &= -\frac{3}{24}+\frac{20}{24}\ \boxed{\checkmark} \\ \frac{17}{24} &= \frac{17}{24} \end{aligned}$$

2. Even though there are decimals in this problem, we can approach it like any other problem. Use the Distributive Property and combine like terms.

$$\begin{aligned} 0.6(2x+7) &= 4.3x-5.1 \\ 1.2x+\cancel{4.2} &= \cancel{4.3x}-5.1 \\ \underline{-4.3x-\cancel{4.2}} &\quad \underline{-\cancel{4.3x}-4.2} \\ \frac{-\cancel{3.1}x}{-\cancel{3.1}} &= \frac{-9.3}{-3.1} \\ x &= 3 \end{aligned}$$

Check your answer:

$$\begin{aligned} 0.6(2(3)+7) &= 4.3(3)-5.1 \\ 0.6\cdot 13 &= 12.9-5.1\ \boxed{\checkmark} \\ 7.8 &= 7.8 \end{aligned}$$

Problem Set

Solve each equation and check your solution.

1. $-6(2x-5) = 18$
2. $2-(3x+7) = -x+19$
3. $3(x-4) = 5(x+6)$
4. $x-\frac{4}{5} = \frac{2}{3}x+\frac{8}{15}$
5. $8-9x-5 = x+23$
6. $x-12+3x = -2(x+18)$
7. $\frac{5}{2}x+\frac{1}{4} = \frac{3}{4}x+2$
8. $5\left(\frac{x}{3}+2\right) = \frac{32}{3}$
9. $7(x-20) = x+4$
10. $\frac{2}{7}\left(x+\frac{2}{3}\right) = \frac{1}{5}\left(2x-\frac{2}{3}\right)$
11. $\frac{1}{6}(x+2) = 2\left(\frac{3x}{2}-\frac{5}{4}\right)$
12. $-3(-2x+7)-5x = 8(x-3)+17$

Challenge Solve the equations below. Check your solution.

13. $\frac{3x}{16}+\frac{x}{8} = \frac{3x+1}{4}+\frac{3}{2}$
14. $\frac{x-1}{11} = \frac{2}{5}\cdot\frac{x+1}{3}$
15. $\frac{3}{x} = \frac{2}{x+1}$

2.13 Solving Linear Inequalities

Objective

To solve and graph any linear inequality in one variable.

Review Queue

Solve the following equations.

1. $6x - 11 = x + 14$
2. $\frac{4}{3}x + 1 = -15$
3. Determine if $x = -2$ is a solution to $9x - 7 = x + 12$.
4. Plot $-3, -1\frac{1}{4}$, $\sqrt{2}$ and 2.85 on a number line.

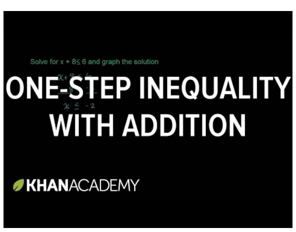
Solving Basic Inequalities

Objective

To determine if a solution works for a given inequality, graph solutions on a number line, and solve basic linear inequalities.

Watch This

First watch this video.



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Khan Academy: One-Step Inequalities

Then watch this video.

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Khan Academy: One-Step Inequalities 2

Guidance

Solving a linear inequality is very similar to solving a linear equality, or equation. There are a few very important differences. We no longer use an equal sign. There are four different inequality signs, shown below.

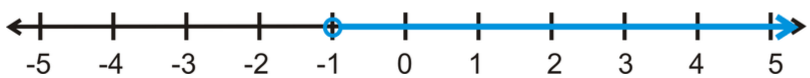
$<$ Less than

$>$ Greater than

$\leq$ Less than or equal to

$\geq$ Greater than or equal to

Notice that the line underneath the $\leq$ and $\geq$ signs indicates "equal to." The inequality $x > -1$ would be read, "x is greater than -1." We can also graph these solutions on a number line. To graph an inequality on a number line, shading is used. This is because an inequality is a range of solutions, not just one specific number. To graph $x > -1$, it would look like this:



Notice that the circle at -1 is *open*. This is to indicate that -1 is *not* included in the solution. A <sign would also have an open circle. If the inequality was a $\geq$ or $\leq$ sign, then the circle would be closed, or filled in. Shading to the right of the circle shows that any number greater than -1 will be a solution to this inequality.

Example A

Is $x = -8$ a solution to $\frac{1}{2}x + 6 > 3$?

Solution: Plug in -8 for x and test this solution.

$$\begin{aligned}
\frac{1}{2}(-8) + 6 &> 3 \\
-4 + 6 &> 3 \\
2 &> 3
\end{aligned}$$

Of course, 2 cannot be greater than 3. Therefore, this is not a valid solution.

Example B

Solve and graph the solution to $2x - 5 \leq 17$.

Solution: For the most part, solving an inequality is the same as solving an equation. The major difference will be addressed in Example C. This inequality can be solved just like an equation.

$$\begin{aligned}
2x - \cancel{5} &\leq 17 \\
\underline{+\cancel{5}} &\underline{\ \ + 5} \\
\frac{\cancel{2}x}{\cancel{2}} &\leq \frac{22}{2} \\
x &\leq 2
\end{aligned}$$

Test a solution, $x = 0 : 2(0) - 5 \leq 17 \rightarrow -5 \leq 17$ $\boxed{\checkmark}$

Plotting the solution, we get:

Always test a solution that is in the solution range. It will help you determine if you solved the problem correctly.

Example C

Solve and graph $-6x+7 \leq -29$.

Solution: When solving inequalities, be careful with negatives. Let's solve this problem the way we normally would an equation.

$$\begin{aligned}
-6x+\cancel{7} &\leq -29 \\
-\cancel{7} \quad & \quad -7 \\
\hline
\frac{-\cancel{6}x}{-\cancel{6}x} &\leq \frac{-36}{-6} \\
x &\leq 6
\end{aligned}$$

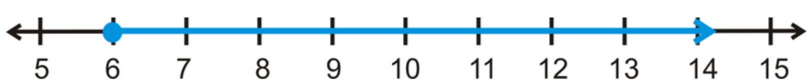
Let's check a solution. If x is less than or equal to 6, let's test 1.

$$\begin{aligned}
-6(1)+7 &\leq -29 \\
-6+7 &\leq -29 \\
1 &\cancel{\leq} -29
\end{aligned}$$

This is not a true inequality. To make this true, we must ***flip*** the inequality. Therefore, **whenever we multiply or divide by a negative number, we must flip the inequality sign.** The answer to this inequality is actually $x \geq 6$. Now, let's test a number in this range.

$$\begin{aligned}
-6(10)+7 &\leq -29 \\
-60+7 &\leq -29 \\
-60 &\leq -29
\end{aligned}$$

This is true. The graph of the solution is:



Guided Practice

1. Is $x=-5$ a solution to $-3x+7>12$?

Solve the following inequalities and graph.

2. $\frac{3}{8}x+5<26$

3. $11<4-x$

Answers

1. Plug -5 into the inequality.

$$\begin{aligned}
-3(-5)+7 &> 12 \\
15+7 &> 12
\end{aligned}$$

This is true because 22 is larger than 12. -5 is a solution.

2. No negatives with the $x-$term, so we can solve this inequality like an equation.

$$\begin{aligned}\frac{3}{8}x+\cancel{5} &< 26\\ \underline{-\cancel{5}\quad -5}&\\ \cancel{\frac{8}{3}}\cdot\cancel{\frac{3}{8}}x &< 21\cdot\frac{8}{3}\\ x &< 56\end{aligned}$$

Test a solution, $x=16: \frac{3}{8}(16)+5<26 \boxed{\checkmark}$

$6+5<26$

The graph looks like:

3. In this inequality, we have a negative $x-$term. Therefore, we will need to flip the inequality.

$$\begin{aligned}11 &< \cancel{4}-x\\ \underline{-4\quad -\cancel{4}}&\\ \frac{7}{-1} &< \frac{\cancel{-}x}{\cancel{-1}}\\ -7 &> x\end{aligned}$$

Test a solution, $x=-10: 11<4-(-10) \boxed{\checkmark}$

$11<14$

Notice that we FLIPPED the inequality sign when we DIVIDED by -1. Also, this equation can also be written $x<-7$. Here is the graph:

Vocabulary

Inequality
A mathematical statement that relates two expressions that are not equal.

Problem Set

Solve and graph each inequality.

1. $x+5>-6$
2. $2x\geq 14$
3. $4<-x$
4. $3x-4\leq 8$

5. $21 - 8x < 45$
6. $9 > x - 2$
7. $\frac{1}{2}x + 5 \geq 12$
8. $54 \leq -9x$
9. $-7 < 8 + \frac{5}{6}x$
10. $10 - \frac{3}{4}x < -8$
11. $4x + 15 \geq 47$
12. $0.6x - 2.4 < 4.8$
13. $1.5 > -2.7 - 0.3x$
14. $-11 < 12x + 121$
15. $\frac{1}{2} - \frac{3}{4}x \leq -\frac{5}{8}$

For questions 16 and 17, write the inequality statement given by the graph below.

16. 

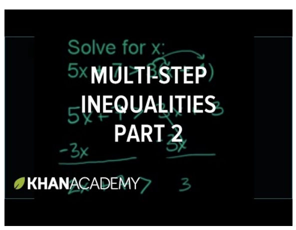
17. 

Multi-Step Inequalities

Objective

To solve more complicated inequalities involving variables on both sides and the Distributive Property.

Watch This



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Khan Academy: Multi-Step Inequalities 2

Guidance

Like multi-step equations, multi-step inequalities can involve having variables on both sides, the Distributive Property, and combining like terms. Again, the only difference when solving inequalities is the sign must be flipped when multiplying or dividing by a negative number.

Example A

Is $x = -3$ a solution to $2(3x - 5) \leq x + 10$?

Solution: Plug in -3 for x and see if the inequality is true.

$$\begin{aligned} 2(3(-3) - 5) &\leq (-3) + 10 \\ 2(-9 - 5) &\leq 7 \\ 2 \cdot -14 &\leq 7 \\ -28 &\leq 7 \end{aligned}$$

This is a true inequality statement. -3 is a solution.

Example B

Solve and graph the inequality from Example A.

Solution: First, distribute the 2 on the left side of the inequality.

$$\begin{aligned} 2(3x-5) &\leq x+10 \\ 6x-10 &\leq x+10 \end{aligned}$$

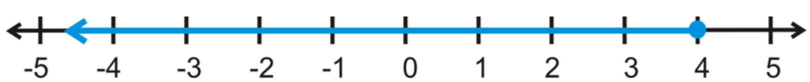
Now, subtract the x on the right side to move it to the left side of the inequality. You can also add the 10's together and solve.

$$\begin{aligned} 6x - \cancel{10} &\geq \cancel{x} + 10 \\ \underline{-x + \cancel{10}} & \underline{\; - \cancel{x} + 10} \\ \frac{\cancel{5}x}{\cancel{5}} &\leq \frac{20}{5} \\ x &\leq 4 \end{aligned}$$

Test a solution, $x = 0 : 2(3(0)-5) \leq 0+10 \; \boxed{\checkmark}$

$-10 \leq 10$

The graph looks like:



Example C

Solve $8x-5-4x \geq 37-2x$

Solution: First, combine like terms on the left side. Then, solve for x.

$$\begin{aligned} 8x-5-4x &\geq 37-2x \\ 4x - \cancel{5} &\geq 37 - \cancel{2x} \\ \underline{+2x + \cancel{5}} & \underline{\; +5 + \cancel{2x}} \\ \frac{\cancel{6}x}{\cancel{6}} &\geq \frac{42}{6} \\ x &\geq 7 \end{aligned}$$

Test a solution, $x = 10 : 8(10)-5-4(10) \geq 37-2(10)$

$80-5-40 \geq 37-20 \; \boxed{\checkmark}$

$35 \geq 17$

Guided Practice

1. Is $x = 12$ a solution to $-3(x-10)+18 \geq x-25$?

Solve and graph the following inequalities.

2. $-(x+16)+3x > 8$

3. $24-9x < 6x-21$

Answers

1. Plug in 12 for x and simplify.

$$\begin{aligned} -3(12-10)+18 &\geq 12-35 \\ -3\cdot 2+18 &\geq -13 \\ -6+18 &\geq -13 \end{aligned}$$

This is true because $12 \geq -13$, so 12 is a solution.

2. Distribute the negative sign on the left side and combine like terms.

$$\begin{aligned} -(x+16)+3x &> 8 \\ -x-16+3x &> 8 \\ 2x-\cancel{16} &> 8 \\ +\cancel{16} &+16 \\ \hline \frac{\cancel{2}x}{\cancel{2}} &> \frac{24}{2} \\ x &> 12 \end{aligned}$$

Test a solution, $x=15$:

$$\begin{aligned} -(15+16)+3(15) &> 8 \\ -31+45 &> 8 \ \boxed{\checkmark} \\ 14 &> 8 \end{aligned}$$

3. First, add $9x$ to both sides and add 21 to both sides.

$$\begin{aligned} 24-\cancel{9x} &< 6x-21 \\ +\cancel{9x} &\ \ +9x \\ \hline 24 &< 15x-\cancel{21} \\ +21 &\qquad +\cancel{21} \\ \hline \frac{45}{15} &< \frac{\cancel{15}x}{\cancel{15}} \\ 3 &< x \end{aligned}$$

Test a solution, $x=10$:

$$\begin{aligned} 24-9(10) &< 6(10)-21 \\ 24-90 &< 60-21 \ \boxed{\checkmark} \\ -66 &< 39 \end{aligned}$$

Problem Set

Determine if the following numbers are solutions to $-7(2x-5)+12 > -4x-13$.

1. $x=4$
2. $x=10$
3. $x=6$

Solve and graph the following inequalities.

4. $2(x-5) \geq 16$
5. $-4(3x+7) < 20$
6. $15x-23 > 6x-17$
7. $5x+16+2x \leq -19$
8. $4(2x-1) \geq 3(2x+1)$
9. $11x-17-2x \leq -(x-23)$

Solve the following inequalities.

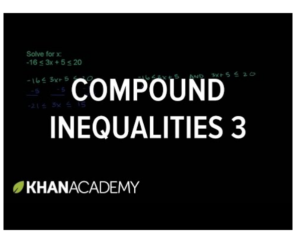
10. $5-5x > 4(3-x)$
11. $-(x-1)+10 < -3(x-3)$
12. Solve $5x+4 \leq -2(x+3)$ by adding the $2x$ term on the right to the left-hand side.
13. Solve $5x+4 \leq -2(x+3)$ by *subtracting* the $5x$ term on the left to the right-hand side.
14. Compare your answers from 12 and 13. What do you notice?
15. **Challenge** Solve $3x-7 > 3(x+3)$. What happens?

Compound Inequalities

Objective

To solve two inequalities that have been joined together by the words "and" and "or."

Watch This



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Khan Academy: Compound Inequalities 3

Guidance

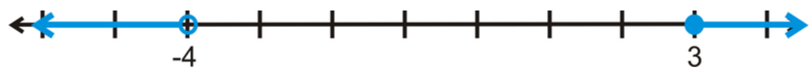
Compound inequalities are inequalities that have been joined by the words "and" or "or." For example:

$-2 < x \leq 5$ Read, "x is greater than -2 *and* less than or equal to 5."

$x \geq 3$ or $x < -4$ Read, "x is greater than or equal to 3 *or* less than -4."

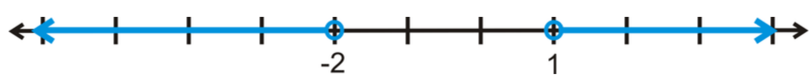
Notice that both of these inequalities have two inequality signs. So, it is like solving or graphing two inequalities at the same time. When graphing, look at the inequality to help you. The first compound inequality above, $-2 < x \leq 5$, has the x in between -2 and 5, so the shading will also be between the two numbers.

And, with the "or" statement, the shading will go in opposite directions.



Example A

Write the inequality statement given by the graph below.



Solution: Because the shading goes in opposite directions, we know this is an "or" statement. Therefore, the statement is $x < -2$ or $x > 1$.

Example B

Solve and graph $-3 < 2x+5 \leq 11$.

Solution: This is like solving two inequalities at the same time. You can split the statement apart to have two inequalities, $-3 < 2x+5$ and $2x+5 \leq 11$ and solve. You can also leave the compound inequality whole to solve.

$$\begin{aligned}
&-3 < 2x+\cancel{5} \leq 11 \\
&\underline{-5 \qquad -\cancel{5} \quad -5} \\
&\frac{-8}{2} < \frac{\cancel{2}x}{\cancel{2}} \leq \frac{6}{2} \\
&-4 < x \leq 3
\end{aligned}$$

Test a solution, $x = 0$:

$$\begin{aligned}
-3 < 2(0)+5 \leq 11 \\
-3 < 5 \leq 11 \boxed{\checkmark}
\end{aligned}$$

Here is the graph:

Example C

Solve and graph $-32 > -5x+3$ or $x-4 \leq 2$.

Solution: When solving an "or" inequality, solve the two inequalities separately, but show the solution on the same number line.

$$\begin{array}{lll} -32 > -5x + \cancel{3} & or & x - \cancel{4} \leq 2 \\ \underline{-3 \qquad\quad -\cancel{3}} & & \underline{+\cancel{4} \ +4} \\ \frac{-35}{-5} > \frac{-\cancel{5}x}{-\cancel{5}} & & x \leq 6 \\ 7 < x & & \end{array}$$

Notice that in the first inequality, we had to flip the inequality sign because we divided by -5. Also, it is a little more complicated to test a solution for these types of inequalities. You still test one point, but it will only work for one of the inequalities. Let's test $x = 10$. First inequality: $-32 > -5(10) + 3 \rightarrow -32 > -47$. Second inequality: $10 - 4 \leq 2 \rightarrow 5 \nleq 2$. Because $x = 10$ works for the first inequality, is a solution. Here is the graph.

Guided Practice

1. Graph $-7 \leq x \leq -1$ on a number line.

Solve the following compound inequalities and graph.

2. $5 \leq -\frac{2}{3}x + 1 \leq 15$

3. $\frac{x}{4} - 7 > 5$ or $\frac{8}{5}x + 2 \leq 18$

Answers

1. This is an "and" inequality, so the shading will be between the two numbers.

2. Solve this just like Example B.

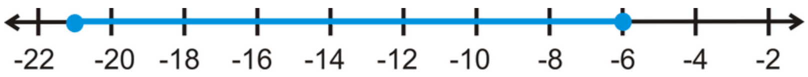
$$\begin{array}{c} 5 \leq -\frac{2}{3}x + \cancel{1} \leq 15 \\ \underline{-1 \qquad\quad -\cancel{1} \quad -1} \\ 4 \leq -\frac{2}{3}x \leq 14 \\ -\frac{3}{2}\left(4 \leq -\frac{2}{3}x \leq 14\right) \\ -6 \geq x \geq -21 \end{array}$$

Test a solution, $x = -10:$

$$\begin{array}{c} 5 \leq -\frac{2}{3}(-12) + 1 \leq 15 \\ 5 \leq 9 \leq 15 \ \boxed{\checkmark} \end{array}$$

This solution can also be written $-21 \leq x \leq -6$.

The graph is:



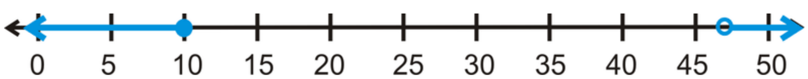
3. This is an "or" compound inequality. Solve the two inequalities separately.

$$\begin{array}{ccc}
\frac{x}{4} - \cancel{7} > 5 & or & \frac{8}{5}x + \cancel{2} \le 18 \\
\underline{+\cancel{7}\ +7} & & \underline{-\cancel{2}\ \ -2} \\
\cancel{4} \cdot \frac{x}{4} > 12 \cdot 4 & or & \cancel{\frac{5}{8}} \cdot \cancel{\frac{8}{5}}x \le 16 \cdot \frac{5}{8} \\
x > 48 & or & x \le 10
\end{array}$$

Test a solution, $x = 0$:

$$\begin{array}{ccc}
\frac{0}{4} - 7 > 5 & or & \frac{8}{5}(0) + 2 \le 18 \\
-7 \not> 5 & or & 2 \le 18 \ \boxed{\checkmark}
\end{array}$$

Notice that $x = 0$ is a solution for the second inequality, which makes it a solution for the entire compound inequality. Here is the graph:



On problems 2 and 3 we changed the scale of the number line to accommodate the solution.

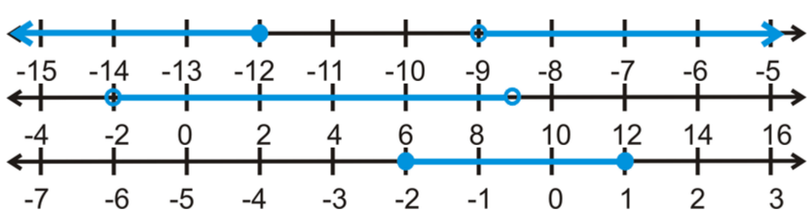
Problem Set

Graph the following compound inequalities. Use an appropriate scale.

1. $-1 < x < 8$
2. $x > 5$ or $x \le 3$
3. $-4 \le x \le 0$

Write the compound inequality that best fits each graph below.

4.
5.
6.



Solve each compound inequality and graph the solution.

7. $-11 < x - 9 \le 2$
8. $8 \le 3 - 5x < 28$
9. $2x - 7 > -13$ or $\frac{1}{3}x + 5 \le 1$
10. $0 < \frac{x}{5} < 4$
11. $-4x + 9 < 35$ or $3x - 7 \le -16$

12. $\frac{3}{4}x+7 \geq -29$ or $16-x > 2$
13. $3 \leq 6x-15 < 51$
14. $-20 < -\frac{3}{2}x+1 < 16$
15. **Challenge** Write a compound inequality whose solutions are all real numbers. Show why this is true.

2.14 Solving Absolute Value Equations and Inequalities

Objective

To solve equations and inequalities that involved absolute value.

Review Queue

Solve the following equations and inequalities

1. $5x-14=21$

2. $1<2x-5<9$

3. $-\frac{3}{4}x+7\geq 19$

4. $4(2x+7)-15=53$

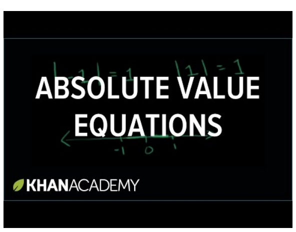
Solving Absolute Value Equations

Objective

To solve absolute value equations.

Watch This

Watch the first part of this video.



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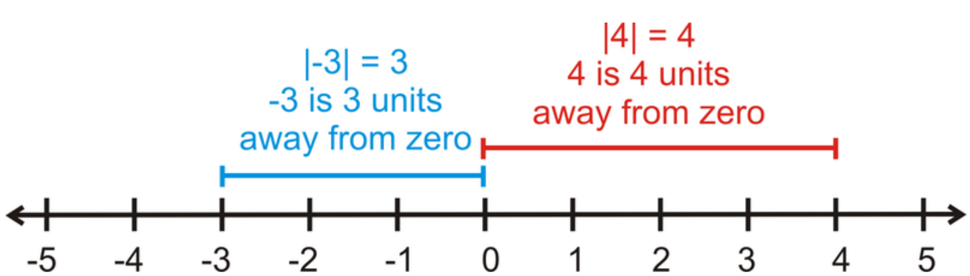
Khan Academy: Absolute Value Equations

Guidance

Absolute value is the distance a number is from zero. Because distance is always positive, the absolute value will always be positive. Absolute value is denoted with two vertical lines around a number, $|x|$.



$|5|=5$ $\quad|-9|=9$ $\quad|0|=0$ $\quad|-1|=1$

When solving an absolute value equation, the value of x could be two different possibilities; whatever makes the absolute value positive OR whatever makes it negative. Therefore, there will always be TWO answers for an absolute value equation.

If $|x| = 1$, then x can be 1 or -1 because $|1| = 1$ and $|-1| = 1$.

If $|x| = 15$, then x can be 15 or -15 because $|15| = 15$ and $|-15| = 15$.

From these statements we can conclude:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Example A

Determine if $x = -12$ is a solution to $|2x - 5| = 29$.

Solution: Plug in -12 for x to see if it works.

$$\begin{aligned} |2(-12) - 5| &= 29 \\ |-24 - 5| &= 29 \boxed{\checkmark} \\ |-29| &= 29 \end{aligned}$$

-12 is a solution to this absolute value equation.

Example B

Solve $|x + 4| = 11$.

Solution: There are going to be two answers for this equation. $x + 4$ can equal 11 or -11.

$$\begin{array}{ccc} & |x+4| = 11 & \\ & \swarrow \searrow & \\ x+4 = 11 & & x+4 = -11 \\ & or & \\ x = 7 & & x = -15 \end{array}$$

Test the solutions:

$$\begin{array}{ll} |7+4| = 11 & |-15+4| = 11 \\ |11| = 11 \boxed{\checkmark} & |-11| = 11 \boxed{\checkmark} \end{array}$$

Example C

Solve $\left|\frac{2}{3}x - 5\right| = 17$.

Solution: Here, what is inside the absolute value can be equal to 17 or -17.

$$\left|\frac{2}{3}x-5\right|=17$$

$$\swarrow \searrow$$

$$\begin{aligned} \frac{2}{3}x-5&=17 & \frac{2}{3}x-5&=-17 \\ \frac{2}{3}x&=22 \qquad or & \frac{2}{3}x&=-12 \\ x&=22\cdot\frac{3}{2} & x&=-12\cdot\frac{3}{2} \\ x&=33 & x&=-18 \end{aligned}$$

Test the solutions:

$$\begin{aligned} \left|\frac{2}{3}(33)-5\right|&=17 & \left|\frac{2}{3}(-18)-5\right|&=17 \\ |22-5|&=17\ \boxed{\checkmark} & |-12-5|&=17\ \boxed{\checkmark} \\ |17|&=17 & |-17|&=17 \end{aligned}$$

Guided Practice

1. Is $x=-5$ a solution to $|3x+22|=6$?

Solve the following absolute value equations.

2. $|6x-11|+2=41$

3. $\left|\frac{1}{2}x+3\right|=9$

Answers

1. Plug in -5 for x to see if it works.

$$\begin{aligned} |3(-5)+22|&=6 \\ |-15+22|&=6 \\ |-7|&\neq 6 \end{aligned}$$

-5 is not a solution because $|-7|=7$, not 6.

2. Find the two solutions. Because there is a 2 being added to the left-side of the equation, we first need to subtract it from both sides so the absolute value is by itself.

$$\begin{array}{c} |6x-11|+2=41 \\ |6x-11|=39 \\ \swarrow \searrow \\ 6x-11=39 \quad 6x-11=-39 \\ 6x=50 \qquad 6x=-28 \\ x=\frac{50}{6} \quad or \quad x=-\frac{28}{6} \\ =\frac{25}{3} \text{ or } 8\frac{1}{3} \quad =-\frac{14}{3} \text{ or } -4\frac{2}{3} \end{array}$$

Check both solutions. It is easier to check solutions when they are improper fractions.

$$\begin{array}{ccc} \left|6\left(\frac{25}{3}\right)-11\right|=39 & & \left|6\left(-\frac{14}{3}\right)-11\right|=39 \\ |50-11|=39 \boxed{\checkmark} & and & |-28-11|=39 \boxed{\checkmark} \\ |39|=39 & & |-39|=39 \end{array}$$

3. What is inside the absolute value is equal to 9 or -9.

$$\begin{array}{c} \left|\frac{1}{2}x+3\right|=9 \\ \swarrow \searrow \\ \frac{1}{2}x+3=9 \quad \frac{1}{2}x+3=-9 \\ \frac{1}{2}x=6 \quad or \quad \frac{1}{2}x=-12 \\ x=12 \qquad x=-24 \end{array}$$

Test solutions:

$$\begin{array}{cc} \left|\frac{1}{2}(12)+3\right|=9 & \left|\frac{1}{2}(-24)+3\right|=9 \\ |6+3|=9 \boxed{\checkmark} & |-12+3|=9 \boxed{\checkmark} \\ |9|=9 & |-9|=9 \end{array}$$

Vocabulary

Absolute Value
: The positive distance from zero a given number is.

Problem Set

Determine if the following numbers are solutions to the given absolute value equations.

1. $|x-7|=16; 9$

2. $\left|\frac{1}{4}x+1\right| = 4; -8$
3. $|5x-2| = 7; -1$

Solve the following absolute value equations.

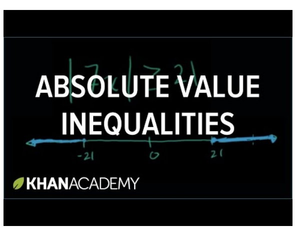
4. $|x+3| = 8$
5. $|2x| = 9$
6. $|2x+15| = 3$
7. $\left|\frac{1}{3}x-5\right| = 2$
8. $\left|\frac{x}{6}+4\right| = 5$
9. $|7x-12| = 23$
10. $\left|\frac{3}{5}x+2\right| = 11$
11. $|4x-15|+1 = 18$
12. $|-3x+20| = 35$
13. Solve $|12x-18| = 0$. What happens?
14. **Challenge** When would an absolute value equation have no solution? Give an example.

Absolute Value Inequalities

Objective

To solve absolute value inequalities.

Watch This



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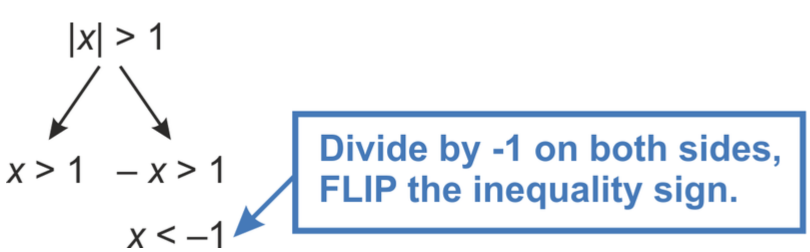
Khan Academy: Absolute Value Inequalities

Guidance

Like absolute value equations, absolute value inequalities also will have two answers. However, they will have a range of answers, just like compound inequalities.

$|x| > 1$ This inequality will have two answers, when x is 1 and when $-x$ is 1. But, what about the inequality sign? The two possibilities would be:



Notice in the second inequality, we did not write $x > -1$. This is because what is inside the absolute value sign can be positive or negative. Therefore, if x is negative, then $-x > 1$. It is a very important difference between the two inequalities. Therefore, for the first solution, we leave the inequality sign the same and for the second solution we need to change the sign of the answer AND flip the inequality sign.

Example A

Solve $|x+2| \leq 10$.

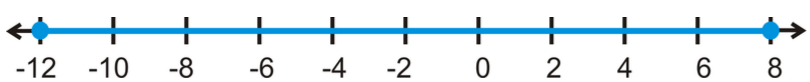
Solution: There will be two solutions, one with the answer and sign unchanged and the other with the inequality sign flipped and the answer with the opposite sign.

$$\begin{array}{cc} \multicolumn{2}{c}{|x+2| \leq 10} \\ \multicolumn{2}{c}{\swarrow \searrow} \\ x+2 \leq 10 & x+2 \geq -10 \\ x \leq 8 & x \geq -12 \end{array}$$

Test a solution, $x = 0$:

$$\begin{aligned} |0+2| &\leq 10 \\ |2| &\leq 10 \boxed{\checkmark} \end{aligned}$$

When graphing this inequality, we have



Notice that this particular absolute value inequality has a solution that is an "and" inequality because the solution is between two numbers.

If $|ax+b| < c$, then $-c < ax+b < c$.

If $|ax+b| \leq c$, then $-c \leq ax+b \leq c$.

If $|ax+b| > c$, then $ax+b < -c$ or $ax+b > c$.

If $|ax+b| \geq c$, then $ax+b \leq -c$ or $ax+b \geq c$.

If you are ever confused by the rules above, always test one or two solutions and graph it.

Example B

Solve and graph $|4x-3| > 9$.

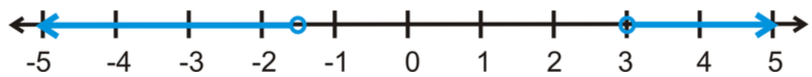
Solution: Break apart the absolute value inequality to find the two solutions.

$$\begin{array}{cc} \multicolumn{2}{c}{|4x-3| > 9} \\ \multicolumn{2}{c}{\swarrow \searrow} \\ 4x-3 > 9 & 4x-3 < -9 \\ 4x > 12 & 4x < -6 \\ x > 3 & x < -\frac{3}{2} \end{array}$$

Test a solution, $x = 5$:

$$\begin{aligned} |4(5)-3| &> 9 \\ |20-3| &> 9 \boxed{\checkmark} \\ 17 &> 9 \end{aligned}$$

The graph is:



Example C

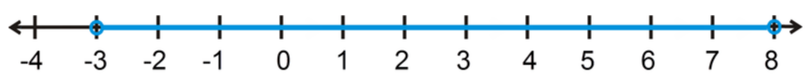
Solve $|-2x+5| < 11$.

Solution: Given the rules above, this will become an "and" inequality.

$$\begin{array}{cc} \multicolumn{2}{c}{|-2x+5| < 11} \\ \swarrow & \searrow \\ -2x+5 < 11 & -2x+5 > -11 \\ -2x < 6 & -2x > -16 \\ x > -3 & x < 8 \end{array}$$

The solution is x is greater than -3 and less than 8. In other words, the solution is $-3 < x < 8$.

The graph is:



Guided Practice

1. Is $x = -4$ a solution to $|15-2x| > 9$?

2. Solve and graph $\left|\frac{2}{3}x+5\right| \le 17$.

Answers

1. Plug in -4 for x to see if it works.

$$\begin{aligned} |15-2(-4)| &> 9 \\ |15+8| &> 9 \\ |23| &> 9 \\ 23 &> 9 \end{aligned}$$

Yes, -4 works, so it is a solution to this absolute value inequality.

2. Split apart the inequality to find the two answers.

$$\left|\frac{2}{3}x+5\right| \leq 17$$

$$\swarrow \searrow$$

$$\begin{array}{ll}
\left|\frac{2}{3}x+5\right| \leq 17 & \frac{2}{3}x+5 \geq -17 \\
\frac{2}{3}x \leq 12 & \frac{2}{3}x \geq -22 \\
x \leq 12 \cdot \frac{3}{2} & x \geq -22 \cdot \frac{3}{2} \\
x \leq 18 & x \geq -33
\end{array}$$

Test a solution, $x = 0$:

$$\left|\frac{2}{3}(0)+5\right| \leq 17$$
$$|5| \leq 17 \boxed{\checkmark}$$
$$5 \leq 17$$

Problem Set

Determine if the following numbers are solutions to the given absolute value inequalities.

1. $|x-9| > 4; 10$
2. $\left|\frac{1}{2}x-5\right| \leq 1; 8$
3. $|5x+14| \geq 29; -8$

Solve and graph the following absolute value inequalities.

4. $|x+6| > 12$
5. $|9-x| \leq 16$
6. $|2x-7| \geq 3$
7. $|8x-5| < 27$
8. $\left|\frac{5}{6}x+1\right| > 6$
9. $|18-4x| \leq 2$
10. $\left|\frac{3}{4}x-8\right| > 13$
11. $|6-7x| \leq 34$
12. $|19+3x| \geq 46$

Solve the following absolute value inequalities. a is greater than zero.

13. $|x-a| > a$
14. $|x+a| \leq a$
15. $|a-x| \leq a$

2.15 Interpreting Word Problems

Objective

To learn how to write a word problem as a mathematical equation.

Review Queue

1. Solve $3(4x-7)+18=5x+53$.

2. How many feet are in a yard? A mile?

3. List 3 words that mean "add."

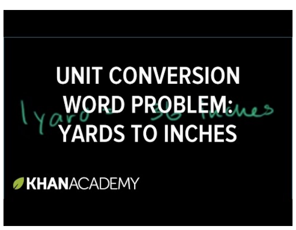
4. List 3 words that mean "subtract."

Unit Conversion

Objective

To convert units of measure into a different unit.

Watch This



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Khan Academy: Unit Conversion with Fractions

Guidance

One part of word problems is the unit of measure. It can be confusing when we don't know what the problem is asking for. For example, how many feet are in a mile? How many cups are in a gallon? Here are a few conversions between different units of measure.

TABLE 2.6:

2 cups (c)	1 pint
2 pints (pt)	1 quart
4 quarts (qt)	1 gallon (gal)
12 inches (in)	1 foot
3 feet (ft)	1 yard
1760 yards (yd)	1 mile (mi)
16 ounces (oz)	1 pound (lb)
2000 lbs	1 ton
100 centimeters (cm)	1 meter (m)
1000 meters	1 kilometer (km)
2.2 cm	1 inch

Example A

How many cups are in a gallon?

Solution: There are 2 cups in a pint, 2 pints in a quart and 4 quarts in a gallon.

$\frac{2c}{1pt} \cdot \frac{2pt}{1qt} \cdot \frac{4pt}{1gal}$ Cancel out the like terms and multiply across.

$\frac{2c}{\cancel{1pt}} \cdot \frac{2\cancel{pt}}{1\cancel{qt}} \cdot \frac{4\cancel{pt}}{1gal} = \frac{16c}{1gal}$ There are 16 cups in one gallon.

Make sure to always cancel any units that are in the numerator and denominator of these fractions. Fractions like these are called **unit rates** because the base is one unit. We write out the unit conversion problems in this way so that we ensure that all of the correct units are canceled out.

Example B

How many feet are in 16 yards?

Solution: This problem is not a conversion problem, but asking to extend your knowledge of how many feet are in a yard. We know that there are 3 feet in a yard; therefore there will be $3 \cdot 16 = 48$ *feet* in 16 yards.

Another way to solve this problem is in a ratio, below:

$$\frac{3ft}{1yd} = \frac{xft}{16yd} \text{ To solve a ratio, we cross-multiply.}$$
$$3ft \cdot 16yd = 1yd \cdot xft$$
$$\frac{48ft \cdot \cancel{yd}}{1\cancel{yd}} = xft \text{ Here, } x = 48 \text{ } feet \text{ and we show that the appropriate units cancel.}$$

Guided Practice

1. How many centimeters are in a foot?

2. How many ounces are in 3.5 pounds?

Answers

1. This problem is just like Example A. Set up the conversion.

$$\frac{2.2cm}{1in} \cdot \frac{12in}{1ft} \quad \text{Cancel out the inches and multiply.}$$
$$\frac{2.2cm}{1\cancel{in}} \cdot \frac{12\cancel{in}}{1ft} = \frac{26.4cm}{1ft}$$

2. This problem is just like Example B. If there are 16 ounces in a pound, then there will be $16 \cdot 3.5 = 56$ *ounces* in 3.5 pounds.

Problem Set

For questions 1-6, set up a unit conversion to find:

1. Feet in a mile?
2. Cups in a quart?
3. Centimeters in a kilometer?
4. Pints in a gallon?
5. Centimeters in a mile?
6. Gallons in a quart?

7. How many inches are in 5.25 yards?
8. How many pints are in 7.5 gallons?
9. How many pounds are in 2.6 tons?
10. How many centimeters are in 4.75 meters?
11. Claire is making chocolate chip cookies. If the recipe calls for 3.5 cups of flour, how many cups will Claire need to use if she triples the recipe?
12. The recipe above calls for 8 oz. of chocolate chips. Claire wants to make the cookies with three-quarters bittersweet chips and one-quarter semi-sweet chips. Again, tripling the recipe, how many ounces of each type of chocolate chip will she need?

Using Algebraic Models

Objective

To write and solve an algebraic equation from a word problem.

Guidance

Word problems are some of the hardest types of problems for students to grasp. There are a few steps to solving any word problem:

1. Read the problem at least twice.
2. Cross out any unnecessary words, circle any numbers or words that represent mathematical operators, or translate words into mathematical expressions.
3. Write an equation and solve.

To help you with steps 2 and 3, generate a list of words that represent: add, subtract, multiply, divide, equal, etc. Here are a few to get you started.

TABLE 2.7:

Operation	Alternate Terms
add	sum ; plus ; and ; increase ; more (than)
multiply	times ; of ; product ; double (x2) ; triple (x3)
subtract	difference ; minus ; decrease ; less (than)
equal	is ; total ; to ; made/make ; spend/spent
divide	quotient ; half ($\div 2$) ; third ($\div 3$)
variable	how many __ ; how much __ ; what amount (of) __

See if you can add anything to these lists. Then, use this chart to help you with decoding word problems.

Example A

Two consecutive numbers add up to 55. What are the two numbers?

Solution: First, translate the statement. "Consecutive" means numbers that are one after the other. So if the first number is x, then the second number will be $x+1$. And, they add up to 55. The equation is: $x+(x+1)=55$

We put $x+1$ in parenthesis to show that it is a separate number. Solve the equation.

$$\begin{aligned} x+x+1 &= 55 \\ 2x+1 &= 55 \\ 2x &= 54 \\ x &= 27 \end{aligned}$$

The smaller number is 27, and the larger number will be 28. $27+28=55$ $\boxed{\checkmark}$

Sometime you may encounter problems with "consecutive even numbers" or "consecutive odd numbers." All even numbers are divisible by 2, so the smallest should be $2x$, then the next even number would be $2x+2$. For consecutive odd numbers, they will always be an even number plus 1, 3, 5, etc. So, the smaller odd number would be $2x+1$ and the larger odd number would be $2x+3$.

Example B

Over the Winter Break, you worked at a clothing store and made $9.00 an hour. For the two weeks you worked 65 hours of regular pay and 10 hours of overtime (time and a half). How much money did you make?

Solution: First, we need to figure out how much you make for overtime. Time and a half would be $\$9.00+\$4.50=\$13.50$ an hour. So, you made:

$$\$9.00(65)+\$13.50(10)=\$585.00+\$135.00=\$720.00$$

Guided Practice

1. Elise is taking piano lessons. The first lesson is twice as expensive as each additional lesson. Her mom spends $270 for 8 lessons. How much was the first lesson?

2. Javier needs to get a tank of gas. Gas costs $3.79 per gallon. How much money does Javier need to fill up his 16 gallon tank?

Answers

1. Translate each statement.

first lesson is twice as expensive as each additional lesson: call the regularly priced lessons l. Then, the first lesson will be $2l$.

mom spends $270 for 8 lessons: first lesson, $2l+7l=\$270$

Solve:

$$\begin{aligned} 2l+7l &= 270 \\ 9l &= 270 \\ l &= 30 \end{aligned}$$

The regularly priced lessons are $30. The first lesson will be $60.

2. This problem wants to know how much money Javier needs to fill up his gas tank. Gas costs $3.79 per gallon and he needs 16 gallons of gas. It will cost $\$3.79 \cdot 16=\$60.64$ to fill up his tank.

Problem Set

Answer each question to the best of your ability.

1. The average speed on highway 101 is 65 miles per hour (mph). Assuming you drive the speed limit, how long will it take you to drive 350 miles? Use the formula $distance = rate \cdot time$. Round your answer to two decimal places.
2. Using the information in #1, how many miles did you drive on highway 101 if you drove for 2.5 hours?
3. The sum of two consecutive numbers is 79. Find the two numbers.
4. The sum of two consecutive odd numbers is 44. Find the two odd numbers.
5. You borrowed $350 from your parents for a new Wii and games. They are not going to charge you interest, but you need to pay them back as quickly as possible. If you pay them $15 per week, how long will it take you to pay them back?
6. George is building a rectangular, fenced-in dog run. He has 120 feet of fencing and wants the length to be 20 feet greater than the width. If you use all the fencing, find the length and width of the dog run.
7. Cynthia is selling chocolate bars for a fundraiser for school. Each bar costs $1.50. If she needs to raise $225, how many chocolate bars does she need to sell?
8. Harriet bakes and sells cookies to local stores. Her cost for one dozen cookies is $2.75 and she sells them to stores for $7.00 (per dozen). How many dozen cookies does she need to make to earn $500? Round to the nearest dozen.
9. A football field is a rectangle where the length is 100 yards. If the total perimeter is 1040 feet, what is the width of a football field? Leave your answer in feet.
10. **Challenge** The sum of *three* consecutive even numbers is 138. What are the three numbers?

2.16 Systems of Two Equations and Two Unknowns

Learning Objectives

Here you will review how to solve a system of two equations and two unknowns using the elimination method.

The cost of two cell phone plans can be written as a **system of equations** based on the number of minutes used and the base monthly rate. As a consumer, it would be useful to know when the two plans cost the same and when is one plan cheaper.

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

Plan B has a lower starting cost, but since it costs more per minute, it may not be the right plan for someone who likes to spend a lot of time on the phone. When do the two plans cost the same amount?

Solving Systems of Equations with Two Unknowns

There are many ways to solve a system that you have learned in the past including substitution and graphical intersection. Here you will focus on solving using elimination because the knowledge and skills used will transfer directly into using matrices.

When solving a system, the first thing to do is to count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. Two equations and two variables can be solved, but one equation with two variables cannot.

Get into the habit of always writing systems in **standard form**: $Ax+By=C$. This will help variables line up, avoid +/- errors and lay the groundwork for using matrices. Once two equations with two variables are in standard form, decide which variable you want to eliminate, scale each equation as necessary by multiplying through by constants and then add the equations together. This procedure should reduce both the number of equations and the number of variables leaving one equation and one variable. Solve and substitute to determine the value of the second variable.

Here is a system of two equations and two variables in standard form: $5x+12y=72$ and $3x-2y=18$. Notice that there is an x column and a y column on the left hand side and a constant column on the right hand side when you rewrite the equations as shown. Also notice that if you add the system as written no variable will be eliminated.

Equation 1: $5x+12y=72$

Equation 2: $3x-2y=18$

Strategically choose to eliminate y by scaling the second equation by 6 so that the coefficient of y will match at 12 and -12.

$$\begin{aligned} 5x+12y&=72\\ 18x-12y&=108 \end{aligned}$$

Add the two equations:

$$23x = 180$$
$$x = \frac{180}{23}$$

The value for x could be substituted into either of the original equations and the result could be solved for y; however, since the value is a fraction it will be easier to repeat the elimination process in order to solve for x. This time you will take the first two equations and eliminate x by making the coefficients of x to be 15 and -15. Scale the first equation by a factor of 3 and scale the second equation by a factor of -5.

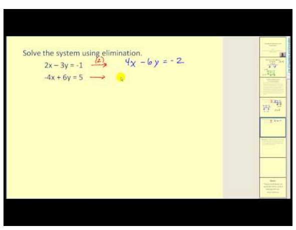
Equation 1: $15x + 36y = 216$

Equation 2: $-15x + 10y = -90$

Adding the two equations:

$$0x + 46y = 126$$
$$y = \frac{126}{46} = \frac{63}{23}$$

The point $\left(\frac{180}{23}, \frac{63}{23}\right)$ is where these two lines intersect.



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Examples

Example 1

Earlier, you were asked about two phone plans.

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

If you want to find out when the two plans cost the same, you can represent each plan with an equation and solve the system of equations. Let y represent cost and x represent number of minutes.

$$y = 0.10x + 40$$
$$y = 0.50x + 25$$

First you put these equations in standard form.

$$x - 10y = -400$$
$$x - 2y = -50$$

Then you scale the second equation by -1 and add the equations together and solve for y.

$$-8y = -350$$
$$y = 43.75$$

To solve for x, you can scale the second equation by -5, add the equations together and solve for x.

$$-4x = -150$$
$$x = 37.5$$

The equivalent costs of plan A and plan B will occur at 37.5 minutes of talk time with a cost of \$43.75.

Example 2

Solve the following system of equations:

$$6x - 7y = 8$$
$$15x - 14y = 21$$

Scaling the first equation by -2 will allow the y term to be eliminated when the equations are summed.

$$-12x + 14y = -16$$
$$15x - 14y = 21$$

The sum is:

$$3x = 5$$
$$x = \frac{5}{3}$$

You can substitute x into the first equation to solve for y.

$$6 \cdot \frac{5}{3} - 7y = 8$$
$$10 - 7y = 8$$
$$-7y = -2$$
$$y = \frac{2}{7}$$

The point $\left(\frac{5}{3}, \frac{2}{7}\right)$ is where these two lines intersect.

Example 3

Solve the following system using elimination:

$$\begin{aligned} 5x - y &= 22 \\ -2x + 7y &= 19 \end{aligned}$$

Start by scaling the first equation by 7 and notice that the y coefficient will immediately be eliminated when the equations are summed.

$$\begin{aligned} 35x - 7y &= 154 \\ -2x + 7y &= 19 \end{aligned}$$

Add, solve for $x = \frac{173}{33}$. Instead of substituting, practice eliminating x by scaling the first equation by 2 and the second equation by 5.

$$\begin{aligned} 10x - 2y &= 44 \\ -10x + 35y &= 95 \end{aligned}$$

Add, solve for y.

Final Answer: $\left(\frac{173}{33}, \frac{139}{33}\right)$

Example 4

Solve the following system of equations:

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ \frac{1}{x} + \frac{1}{y} &= 4 \end{aligned}$$

The strategy of elimination still applies. You can eliminate the $\frac{1}{y}$ term if the second equation is scaled by a factor of -2.

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ -2 \cdot \frac{1}{x} - 2 \cdot \frac{1}{y} &= -8 \end{aligned}$$

Add the equations together and solve for x.

$$\begin{aligned} -3 \cdot \frac{1}{x} + 0 \cdot \frac{1}{y} &= 3 \\ -3 \cdot \frac{1}{x} &= 3 \\ \frac{1}{x} &= -1 \\ x &= -1 \end{aligned}$$

Substitute into the second equation and solve for y.

$$\begin{aligned} \frac{1}{-1} + \frac{1}{y} &= 4 \\ -1 + \frac{1}{y} &= 4 \\ \frac{1}{y} &= 5 \\ y &= \frac{1}{5} \end{aligned}$$

The point $\left(-1, \frac{1}{5}\right)$ is the point of intersection between these two curves.

Example 5

Solve the following system using elimination:

$$\begin{aligned} 11 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} &= -38 \\ 9 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= -25 \end{aligned}$$

To eliminate $\frac{1}{y}$, scale the first equation by 2 and the second equation by 5.

To eliminate $\frac{1}{x}$, scale the first equation by -9 and the second equation by 11.

Final Answer: $\left(-\frac{1}{3}, 1\right)$

Review

Solve each system of equations using the elimination method.

1. $x + y = -4; -x + 2y = 13$
2. $\frac{3}{2}x - \frac{1}{2}y = \frac{1}{2}; -4x + 2y = 4$
3. $6x + 15y = 1; 2x - y = 19$
4. $x - \frac{2y}{3} = \frac{-2}{3}; 5x - 2y = 10$
5. $-9x - 24y = -243; \frac{1}{2}x + y = \frac{21}{2}$

6. $5x+\frac{28}{3}y=\frac{176}{3}; y+x=10$

7. $2x-3y=50; 7x+8y=-10$

8. $2x+3y=1; 2y=-3x+14$

9. $2x+\frac{3}{5}y=3; \frac{3}{2}x-y=-5$

10. $5x=9-2y; 3y=2x-3$

11. How do you know if a system of equations has no solution?

12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?

13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.

14. Solve:

$$\begin{aligned} 12\cdot\frac{1}{x}-18\cdot\frac{1}{y}&=4\\ 8\cdot\frac{1}{x}+9\cdot\frac{1}{y}&=5 \end{aligned}$$

15. Solve:

$$\begin{aligned} 14\cdot\frac{1}{x}-5\cdot\frac{1}{y}&=-3\\ 7\cdot\frac{1}{x}+2\cdot\frac{1}{y}&=3 \end{aligned}$$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.1.

2.17 Systems of Three Equations and Three Unknowns

Learning Objectives

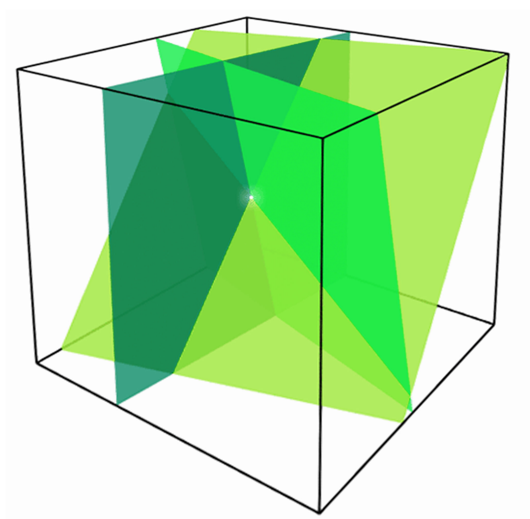
Here you will extend your knolwedge of systems of equations to three equations and three unknowns.

Later, you will learn about matrices and how to row reduce which will allow you to solve systems of equations in a new way. In order to set you up so that using matrices is logical and helpful, it is important to first solve a few systems of three equations using a very specific type of variable elimination.

When solving systems, what are you allowed to do to each equation?

Solving Systems of Equations with Three Unknowns

A system of three equations with three unknowns represents three planes in three dimensional space. When solving the system, you are figuring out how the planes intersect. One way that three planes could intersect is in a point:



A system of equations that has at least one solution is called a **consistent system**.

It is also possible for two or more planes to be parallel or each pair of planes to intersect in a line. In either of these cases the three planes do not intersect at a single point and the system is said to have no solution. A system of equations with no solutions is called an **inconsistent system**. If the three planes intersect at a line or a plane, there are an **infinite number of solutions.**

The following system of equations has the solution (1, 3, 7). You can verify this by substituting 1 for x, 3 for y, and 7 for z into each equation.

$$\begin{aligned} x+2y-z&=0\\ 7x-0y+z&=14\\ 0x+y+z&=10 \end{aligned}$$

One thing to be mindful of when given a system of equations is whether or not the equations are linearly independent. Three equations are **linearly independent** if each equation cannot be produced by a **linear combination** of the other two. Remember that a linear combination means that one equation can be written as the sum of multiples of the others.

When solving a system of three equations and three variables, there are a few general guidelines that can be helpful:

- Start by trying to eliminate the first variable in the second row.
- Next eliminate the first and second variables in the third row. This will create zero coefficients in the lower right hand corner.
- Repeat this process for the upper right hand corner and you should end up with a very nice diagonal indicating what x, y and z equal.

Take the system of equations mentioned above:

$$\begin{aligned} x+2y-z &= 0 \\ 7x-0y+z &= 14 \\ 0x+y+z &= 10 \end{aligned}$$

There are a number of ways to solve this system. Common techniques involve swapping rows, dividing and multiplying a row by a constant and adding or subtracting a multiple of one row to another.

Step 1: Swap rows 2 and 3. Change -0 to +0.

$$\begin{aligned} x+2y-z &= 0 \\ 0x+y+z &= 10 \\ 7x+0y+z &= 14 \end{aligned}$$

Step 2: Subtract 7 times row 1 to row 3.

$$\begin{aligned} x+2y-z &= 0 \\ 0x+y+z &= 10 \\ 0x-14y+8z &= 14 \end{aligned}$$

Step 3: Add 14 times row 2 to row 3

$$\begin{aligned} x+2y-z &= 0 \\ 0x+y+z &= 10 \\ 0x+0y+22z &= 154 \end{aligned}$$

Step 4: Divide row 3 by 22.

$$\begin{aligned} x+2y-z&=0\\ 0x+y+z&=10\\ 0x+0y+z&=7 \end{aligned}$$

Step 5: Subtract row 3 from row 2

$$\begin{aligned} x+2y-z&=0\\ 0x+y+0z&=3\\ 0x+0y+z&=7 \end{aligned}$$

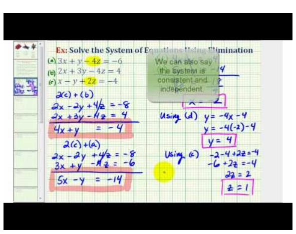
Step 6: Add row 3 to row 1

$$\begin{aligned} x+2y+0z&=7\\ 0x+y+0z&=3\\ 0x+0y+z&=7 \end{aligned}$$

Step 7: Subtract 2 times row 2 to row 1.

$$\begin{aligned} x+0y+0z&=1\\ 0x+y+0z&=3\\ 0x+0y+z&=7 \end{aligned}$$

The solution to the system is (1, 3, 7) exactly as stated earlier.



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Examples

Example 1

Earlier, you were asked what you are allowed to do when solving a system of three equations. When solving a system of three equations with three unknowns, you are allowed to add and subtract rows, swap rows and scale rows. These three operations should allow you to eliminate the coefficients of the variables in a systematic way.

Example 2

Is the following system linearly independent or dependent? How do you know?

$$\begin{aligned}3x+2y+z&=8\\x+y+z&=3\\5x+4y+3z&=14\\6x+6y+6z&=18\end{aligned}$$

With four equations and three unknowns there must be at least one dependent equation. The simplest method of seeing linearly dependence is to notice that one equation is just a multiple of the other. In this case the fourth equation is just six times the second equation and so it is dependent.

Most people will not notice that the third equation is also dependent. It is common to start doing a problem and notice somewhere along the way that all the variables in a row disappear. This means that the original equations were dependent. In this case, the third equation is the first equation plus two times the second equation. This means they are dependent.

Example 3

Reduce the following system to a system of two equations and two unknowns.

$$\begin{aligned}3x+2y+z&=7\\4x+0y+z&=6\\6x-y+0z&=5\end{aligned}$$

Strategically swapping rows so that the zero coefficients do not live on the diagonal is a clever starting move.

Step 1: Swap rows 2 and 3.

$$\begin{aligned}3x+2y+z&=7\\6x-y+0z&=5\\4x+0y+z&=6\end{aligned}$$

Step 2: Scale row 3 by a factor of 3. Subtract 2 times row 1 from for 2.

$$\begin{aligned}3x+2y+z&=7\\0x-5y-2z&=-9\\12x+0y+3z&=18\end{aligned}$$

Step 3: Subtract 4 times row 1 from row 3.

$$3x+2y+z=7$$
$$0x-5y-2z=-9$$
$$0x-8y-z=-10$$

Step 4: Scale the second row by 8 and the third row by 5.

$$3x+2y+z=7$$
$$0x-40y-16z=-72$$
$$0x-40y-5z=-50$$

Step 5: Subtract row 2 from row 3.

$$3x+2y+z=7$$
$$0x-40y-16z=-72$$
$$0x+0y+11z=+22$$

Step 6: Scale row 3 to find z.

$$3x+2y+z=7$$
$$0x-40y-16z=-72$$
$$0x+0y+z=2$$

Now that $z=2$, rewrite the system so it becomes a system of three equations and three unknowns. Any iteration of the first two rows will work. This iteration is from step 6.

$$3x+2y+2=7$$
$$0x-40y-32=-72$$

Example 4

When Kaitlyn went to the store with ten dollars she saw that she had some choices about what to buy. She could get one apple, one onion and one basket of blueberries for 9 dollars. She could get two apples and two onions for 10 dollars. She could also get two onions and one basket of blueberries for 10 dollars. Write and solve a system of equations with variables a,o and b representing each of the three things she can buy.

Here is the system of equations:

$$
\begin{aligned}
a+o+b &= 9 \\
2a+2o &= 10 \\
2o+b &= 10
\end{aligned}
$$

Rewrite the system using x, y and z so that o and 0 do not get mixed up. Include coefficients of 0 so that each column represents one variable.

Step 1: Rewrite

$$
\begin{aligned}
1x+1y+1z &= 9 \\
2x+2y+0z &= 10 \\
0x+2y+1z &= 10
\end{aligned}
$$

Step 2: Subtract 2 times row 1 from row 2.

$$
\begin{aligned}
1x+1y+1z &= 9 \\
0x+0y-2z &= -8 \\
0x+2y+1z &= 10
\end{aligned}
$$

Step 3: Swap row 2 and row 3. Then scale row 3.

$$
\begin{aligned}
1x+1y+1z &= 9 \\
0x+2y+1z &= 10 \\
0x+0y+z &= 4
\end{aligned}
$$

At this point you can see from the third equation that $z = 4$. From the second equation, $2y+4 = 10$, so $y = 3$. Finally you can see from the first equation that $x+3+4 = 9$ so $x = 2$. Apples cost 2 dollars each, onions cost 3 dollars each and blueberries cost 4 dollars each.

Example 5

Show that the following system is dependent.

$$
\begin{aligned}
x+y+z &= 9 \\
x+2y+3z &= 22 \\
2x+3y+4z &= 31
\end{aligned}
$$

You could notice that the third equation is simply the sum of the other two. What happens when you do not notice and try to solve the system as if it were independent?

Step 1: Rewrite the system.

$$\begin{aligned} x+y+z&=9\\ x+2y+3z&=22\\ 2x+3y+4z&=31 \end{aligned}$$

Step 2: Subtract 2 times row 1 from row 3.

$$\begin{aligned} x+y+z&=9\\ x+2y+3z&=22\\ 0x+1y+2z&=13 \end{aligned}$$

Step 3: Subtract row 1 from row 2.

$$\begin{aligned} x+y+z&=9\\ 0x+1y+2z&=13\\ 0x+1y+2z&=13 \end{aligned}$$

At this point when you subtract row 2 from row 3, all the coefficients in row 3 disappear. This means that you will end up with the following system of only two equations and three unknowns. Since the unknowns outnumber the equations, the system does not have a solution of one point.

$$\begin{aligned} x+y+z&=9\\ 0x+1y+2z&=13 \end{aligned}$$

Review

1. An equation with three variables represents a plane in space. Describe all the ways that three planes could interact in space.

2. What does it mean for equations to be linearly dependent?

3. How can you tell that a system is linearly dependent?

4. If you have linearly independent equations with four unknowns, how many of these equations would you need in order to get one solution?

5. Solve the following system of equations:

$$\begin{aligned} 3x-4y+z&=-17\\ 6x+y-3z&=4\\ -x-y+5z&=16 \end{aligned}$$

6. Show that the following system is dependent:

$$\begin{aligned}2x-2y+z&=5\\6x+y-3z&=2\\4x+3y-4z&=-3\end{aligned}$$

7. Solve the following system of equations:

$$\begin{aligned}4x+y+z&=15\\-2x+3y+4z&=38\\-x-y+3z&=16\end{aligned}$$

8. Solve the following system of equations:

$$\begin{aligned}3x-2y+3z&=6\\x+3y-3z&=-14\\-x+y+5z&=22\end{aligned}$$

9. Solve the following system of equations:

$$\begin{aligned}3x-y+z&=-10\\6x-2y+2z&=-20\\-x-y+4z&=12\end{aligned}$$

10. Solve the following system of equations:

$$\begin{aligned}x-3y+6z&=-30\\4x+2y-3z&=18\\-8x-3y+2z&=-22\end{aligned}$$

11. Solve the following system of equations.

$$\begin{aligned}x+2y+2z+w&=5\\2x+y+2z-0w&=5\\3x+3y+3z+2w&=12\\x+0y+z+w&=1\end{aligned}$$

A parabola goes through (3, -9.5), (6, -32), and (-4, 8).

12. Write a system of equations that you could use to solve to find the equation of the parabola. Hint: Use the general equation $Ax^2+Bx+C=y$.

13. Solve the system of equations from #12.

A parabola goes through (-2, 3), (2, 19), and (1, 6).

14. Write a system of equations that you could use to solve to find the equation of the parabola. Hint: Use the general equation $Ax^2 + Bx + C = y$.

15. Solve the system of equations from #14.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.2.

2.18 Finding the Slope and Equation of a Line

Objective

To review how to find the slope and equation of a line.

Review Queue

1. Plot the following points on the same graph.

a) (4, -2)

b) (-2, -7)

c) (6, 1)

d) (0, 8)

Solve the following equations for the indicated variable.

2. $3x-4y=12; x$

3. $3x-4y=12; y$

4. $2b+5c=-10; b$

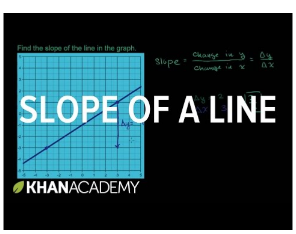
5. $2b+5c=-10; c$

Finding Slope

Objective

To find the slope of a line and between two points.

Watch This



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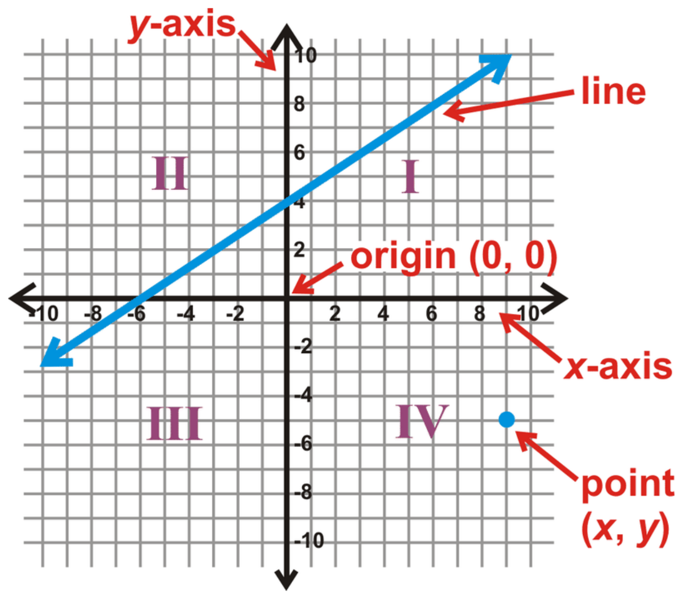
Khan Academy: Slope of a line

Guidance

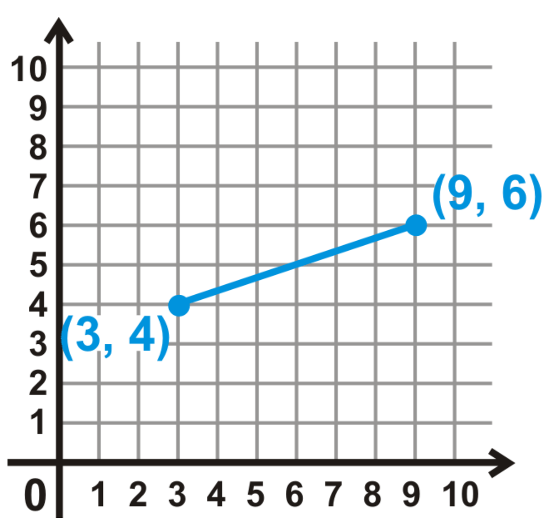
The slope of a line determines how steep or flat it is. When we place a line in the coordinate plane, we can measure the slope, or steepness, of a line. Recall the parts of the coordinate plane, also called an $x-y$ plane and the Cartesian plane, after the mathematician Descartes.



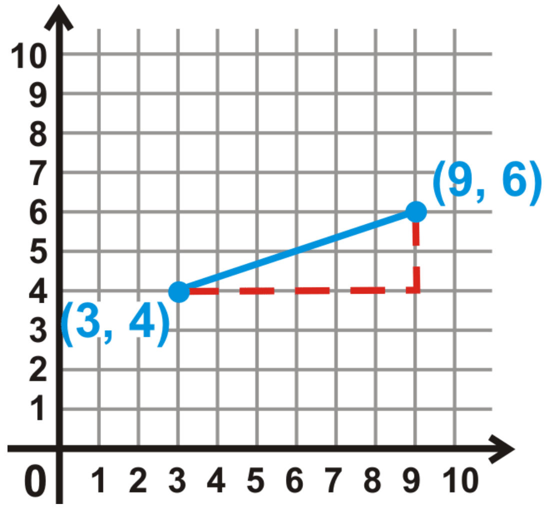
To plot a point, order matters. First, every point is written (x, y), where x is the movement in the $x-$direction and y is the movement in the $y-$direction. If x is negative, the point will be in the 2^{nd} or 3^{rd} quadrants. If y is negative, the point will be in the 3^{rd} or 4^{th} quadrants. The quadrants are always labeled in a counter-clockwise direction and using Roman numerals.

The point in the 4^{th} quadrant would be (9, -5).

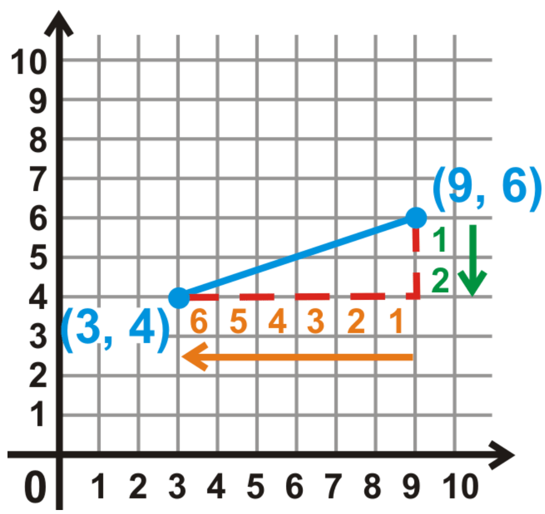
To find the slope of a line or between two points, first, we start with right triangles. Let's take the two points (9, 6) and (3, 4). Plotting them on an $x-y$ plane, we have:



To turn this segment into a right triangle, draw a vertical line down from the higher point, and a horizontal line from the lower point, towards the vertical line. Where the two lines intersect is the third vertex of the slope triangle.



Now, count the vertical and horizontal units along the horizontal and vertical sides (**red** dotted lines).

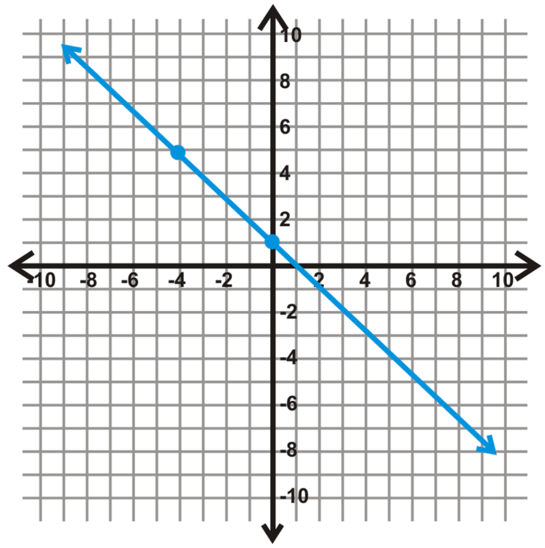


The slope is a fraction with the vertical distance over the horizontal distance, also called the "rise over run." Because the vertical distance goes down, we say that it is -2. The horizontal distance moves towards the negative direction (the left), so we would say that it is -6. So, for slope between these two points, the slope would be $\frac{-2}{-6}$ or $\frac{1}{3}$.

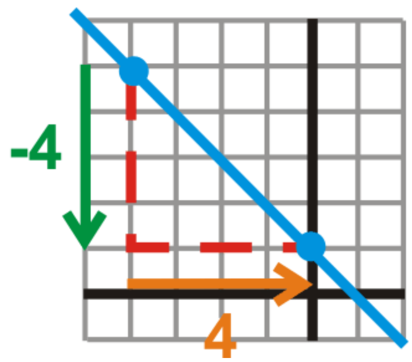
Note: You can also draw the right triangle above the line segment.

Example A

Use a slope triangle to find the slope of the line below.



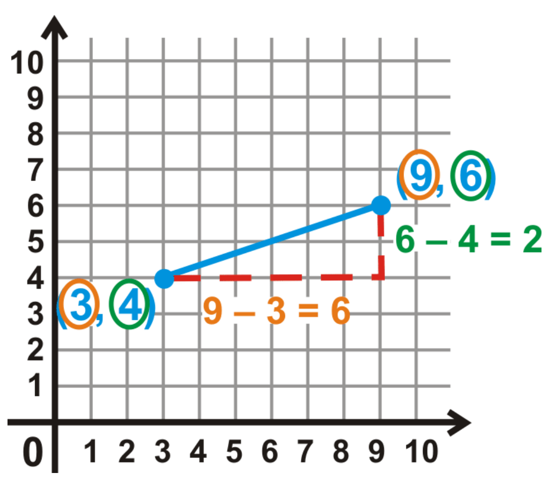
Solution: Notice the two points that are drawn on the line. These are given to help you find the slope. Draw a triangle between these points and find the slope.



From the slope triangle above, we see that the slope is $\frac{-4}{4} = -1$.

Whenever a slope reduces to a whole number, the "run" will always be positive 1. Also, notice that this line points in the opposite direction as the line segment above. We say this line has a *negative* slope because the slope is a negative number and points from the 2^{nd} to 4^{th} quadrants. A line with positive slope will point in the opposite direction and point between the 1^{st} and 3^{rd} quadrants.

If we go back to our previous example with points (9, 6) and (3, 4), we can find the vertical distance and horizontal distance another way.



From the picture, we see that the vertical distance is the same as the difference between the $y-$values and the horizontal distance is the difference between the $x-$values. Therefore, the slope is $\frac{6-4}{9-3}$. We can extend this idea to any two points, (x_1, y_1) and (x_2, y_2).

Slope Formula: For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2-y_1}{x_2-x_1}$. The symbol for slope is m.

It does not matter which point you choose as (x_1, y_1) or (x_2, y_2).

Example B

Find the slope between (-4, 1) and (6, -5).

Solution: Use the Slope Formula above. Set $(x_1, y_1) = (-4, 1)$ and $(x_2, y_2) = (6, -5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-4)}{-5 - 1} = \frac{10}{-6} = -\frac{5}{3}$$

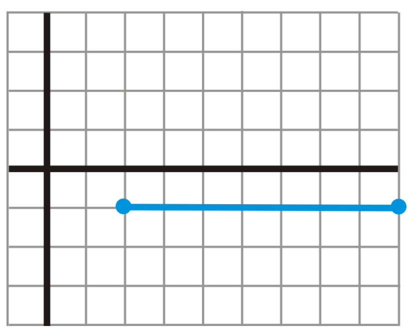
Example C

Find the slope between (9, -1) and (2, -1).

Solution: Use the Slope Formula. Set $(x_1, y_1) = (9, -1)$ and $(x_2, y_2) = (2, -1)$.

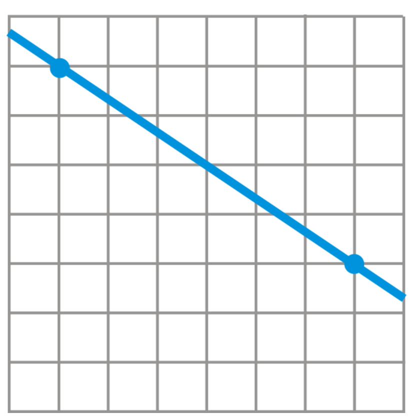
$$m = \frac{-1 - (-1)}{2 - 9} = \frac{0}{-7} = 0$$

Here, we have zero slope. Plotting these two points we have a horizontal line. This is because the $y-$values are the same. Anytime the $y-$values are the same we will have a horizontal line and the slope will be zero.



Guided Practice

1. Use a slope triangle to find the slope of the line below.



2. Find the slope between (2, 7) and (-3, -3).

3. Find the slope between (-4, 5) and (-4, -1).

Answers

1. Counting the squares, the vertical distance is down 6, or -6, and the horizontal distance is to the right 8, or +8. The slope is then $\frac{-6}{8}$ or $-\frac{2}{3}$.

2. Use the Slope Formula. Set $(x_1, y_1) = (2, 7)$ and $(x_2, y_2) = (-3, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-7}{-3-2} = \frac{-10}{-5} = 2$$

3. Again, use the Slope Formula. Set $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (-4, -1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{-4-(-4)} = \frac{-6}{0}$$

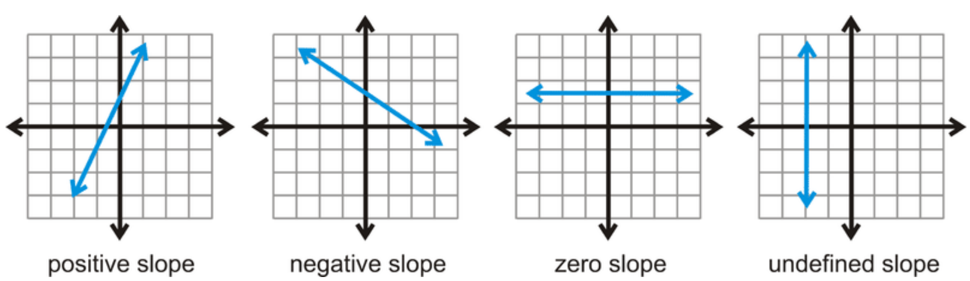
You cannot divide by zero. Therefore, this slope is undefined. If you were to plot these points, you would find they form a vertical line. *All vertical lines have an undefined slope.*

Important Note: Always reduce your slope fractions. Also, if the numerator or denominator of a slope is negative, then the slope is negative. If they are both negative, then we have a negative number divided by a negative number, which is positive, thus a positive slope.

Vocabulary

Slope

The steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. Slope can also be called "rise over run" or "the change in the $y-$values over the change in the $x-$values." The symbol for slope is m.

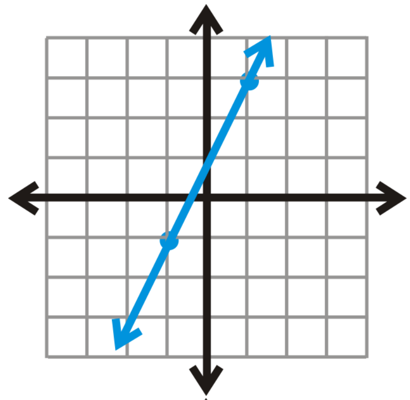


Slope Formula

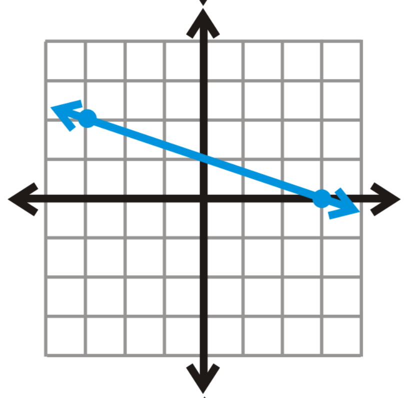
For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2 - y_1}{x_2 - x_1}$.

Problem Set

Find the slope of each line by using slope triangles.



1.



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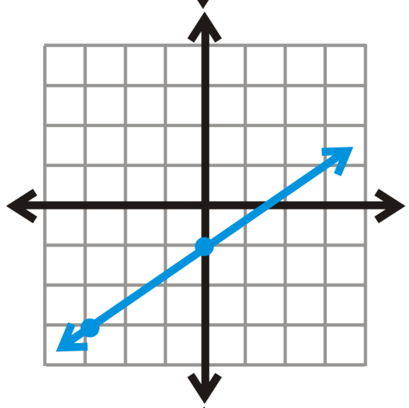

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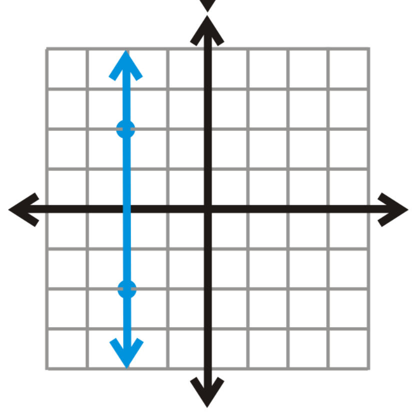

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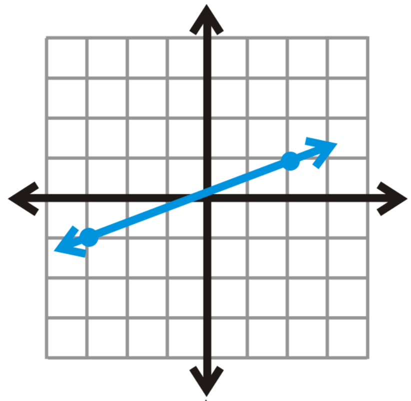
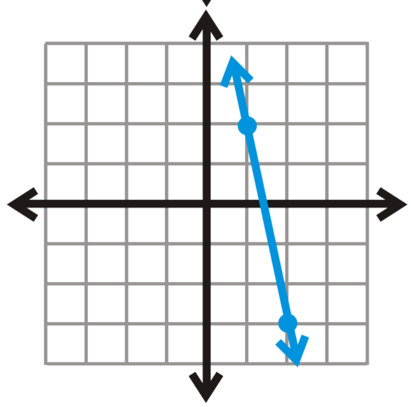
5. 

6. 

Find the slope between each pair of points using the Slope Formula.

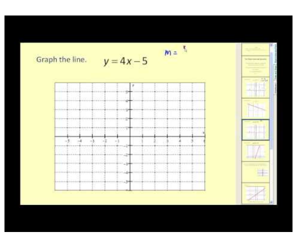
7. (-5, 6) and (-3, 0)
8. (1, -1) and (6, -1)
9. (3, 2) and (-9, -2)
10. (8, -4) and (8, 1)
11. (10, 2) and (4, 3)
12. (-3, -7) and (-6, -3)
13. (4, -5) and (0, -13)
14. (4, -15) and (-6, -11)
15. (12, 7) and (10, -1)
16. **Challenge** The slope between two points (a,b) and (1, -2) is $\frac{1}{2}$. Find a and b.

Finding the Equation of a Line in Slope-Intercept Form

Objective

To find the equation of a line (the slope and $y-$intercept) in slope-intercept form.

Watch This



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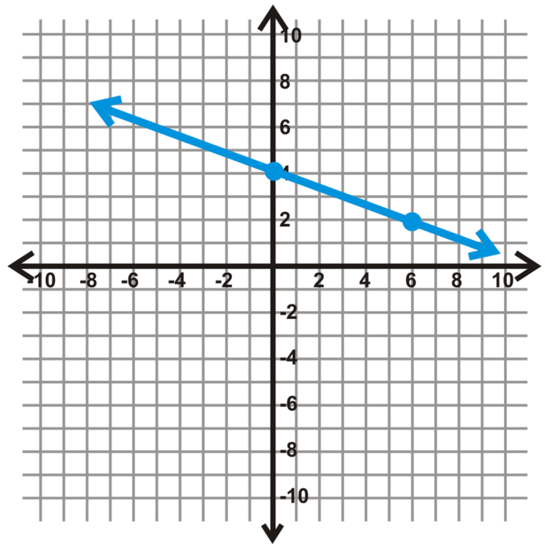
James Sousa: Slope Intercept Form of a Line

Guidance

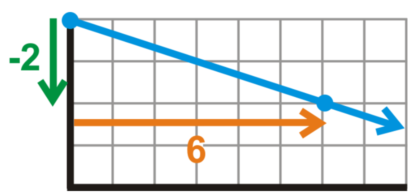
In the previous concept, we found the slope between two points. We will now find the entire equation of a line. Recall from Algebra I that the equation of a line in slope-intercept form is $y = mx + b$, where m is the slope and b is the $y-$intercept. You can find the slope either by using slope triangles or the Slope Formula. To find the $y-$intercept, or b, you can either locate where the line crosses the $y-$axis (if given the graph) or by using algebra.

Example A

Find the equation of the line below.



Solution: Analyze the line. We are given two points on the line, one of which is the $y-$intercept. From the graph, it looks like the line passes through the $y-$axis at (0, 4), making $b = 4$. Now, we need to find the slope. You can use slope triangles or the Slope Formula. Using slope triangles, we have:



From this, we see that the slope is $-\frac{2}{6}$ or $-\frac{1}{3}$.

Plugging our found information into the slope-intercept equation, the equation of this line is $y = -\frac{1}{3}x + 4$.

Alternate Method: If we had used the Slope Formula, we would use (0, 4) and (6, 2), which are the values of the given points.

$$m = \frac{2-4}{6-0} = \frac{-2}{6} = -\frac{1}{3}$$

Example B

The slope of a line is -4 and the $y-$intercept is (0, 3). What is the equation of the line?

Solution: This problem explicitly tells us the slope and $y-$intercept. The slope is -4, meaning $m = -4$. The $y-$intercept is (0, 3), meaning $b = 3$. Therefore, the equation of the line is $y = -4x + 3$.

Example C

The slope of a line is $\frac{1}{2}$ and it passes through the point (4, -7). What is the equation of the line?

Solution: In this problem, we are given m and a point on the line. The point, (4, -7) can be substituted in for x and y in the equation. We need to solve for the $y-$intercept, or b. Plug in what you know to the slope-intercept equation.

$$\begin{aligned} y &= mx + b \\ -7 &= \frac{1}{2}(4) + b \\ -7 &= 2 + b \\ -9 &= b \end{aligned}$$

From this, the equation of the line is $y = \frac{1}{2}x - 9$.

We can test if a point is on a line or not by plugging it into the equation. If the equation holds true, the point is on the line. If not, then the point is not on the line.

Example D

Find the equation of the line that passes through (12, 7) and (10, -1).

Solution: In this example, we are not given the slope or the $y-$intercept. First, we need to find the slope using the Slope Formula.

$$m = \frac{-1-7}{10-12} = \frac{-8}{-2} = 4$$

Now, plug in one of the points for x and y. It does not matter which point you choose because they are both on the line.

$$\begin{aligned} 7 &= 4(12) + b \\ 7 &= 48 + b \\ -41 &= b \end{aligned}$$

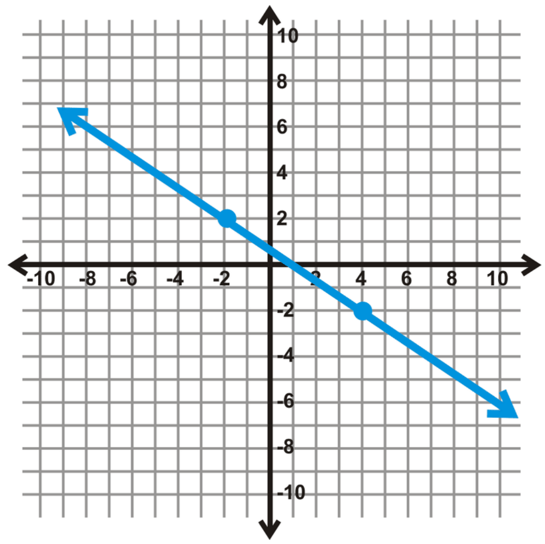
The equation of the line is $y = 4x - 41$.

Guided Practice

1. What is the equation of the line where the slope is 1 and passes through (5, 3)?

2. Find the equation of the line that passes through (9, -4) and (-1, -8).

3. Find the equation of the line below.



Answers

1. We are told that $m = 1, x = 5$, and $y = 3$. Plug this into the slope-intercept equation and solve for b.

$$\begin{aligned} 3 &= 1(5) + b \\ 3 &= 5 + b \\ -2 &= b \end{aligned}$$

The equation of the line is $y = x - 2$

2. First, find the slope.

$$m = \frac{-8-(-4)}{-1-9} = \frac{-4}{-10} = \frac{2}{5}$$

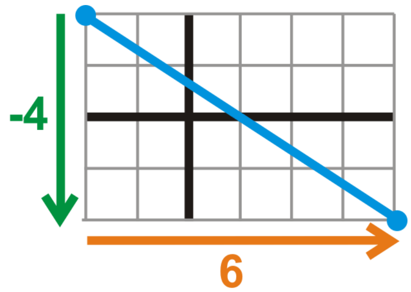
Now, find the $y-$intercept. We will use the second point. Remember, it does not matter which point you use.

$$\begin{aligned} -8 &= \frac{2}{5}(-1) + b \\ -8 &= -\frac{2}{5} + b \\ -7\frac{3}{5} &= b \end{aligned}$$

The equation of the line is $y = \frac{2}{5}x - 7\frac{3}{5}$ or $y = \frac{2}{5}x - \frac{38}{5}$.

When your $y-$intercept is a fraction, make sure it is reduced. Double-check with your teacher on how s/he wants you to leave your answer.

3. We can find the slope one of two ways: using slope triangles or by using the Slope Formula. We are given (by the drawn points in the picture) that (-2, 2) and (4, -2) are on the line. Drawing a slope triangle, we have:



We have that the slope is $-\frac{4}{6}$ or $-\frac{2}{3}$. To find the $y-$intercept, it looks like it is somewhere between 0 and 1. Take one of the points and plug in what you know to the slope-intercept equation.

$$\begin{aligned}2 &= -\frac{2}{3}(-2)+b\\ 2 &= \frac{4}{3}+b\\ \frac{2}{3} &= b\end{aligned}$$

The equation of the line is $y=-\frac{2}{3}x+\frac{2}{3}$.

Vocabulary

Slope-Intercept Form
The equation of a line in the form $y=mx+b$, where m is the slope and b is the $y-$intercept.

$y-$**intercept**
The point where a line crosses the $y-$axis. This point will always have the form $(0,y)$.

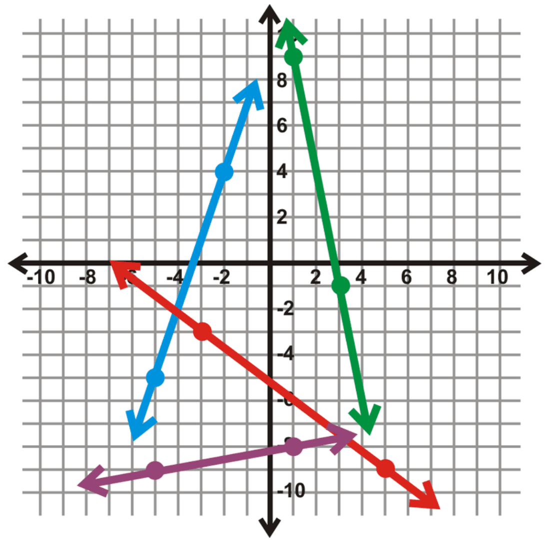
$x-$**intercept**
The point where a line crosses the $x-$axis. This point will always have the form $(x,0)$.

Problem Set

Find the equation of each line with the given information below.

1. slope = 2, $y-$intercept = (0, 3)
2. $m=-\frac{1}{4}$, $b=2.6$
3. slope = -1, $y-$intercept = (0, 2)
4. $x-$intercept = (-2, 0), $y-$intercept = (0, -5)
5. slope $=\frac{2}{3}$ and passes through (6, -4)
6. slope $=-\frac{3}{4}$ and passes through (-2, 5)
7. slope = -3 and passes through (-1, -7)
8. slope = 1 and passes through (2, 4)
9. passes through (-5, 4) and (1, 1)
10. passes through (5, -1) and (-10, -10)
11. passes through (-3, 8) and (6, 5)
12. passes through (-4, -21) and (2, 9)

For problems 13-16, find the equation of the lines in the graph below.



13. Green Line
14. Blue Line
15. Red Line
16. Purple Line
17. Find the equation of the line with zero slope and passes through (8, -3).
18. Find the equation of the line with zero slope and passes through the point (-4, 5).
19. Find the equation of the line with zero slope and passes through the point (a, b).
20. **Challenge** Find the equation of the line with an *undefined* slope that passes through (a, b).

2.19 Standard Form of a Line

Objective

To familiarize students with the standard form of a line, as well as finding the equations of lines that are parallel or perpendicular to a given line.

Review Queue

1. Solve $4x - y = 6$ for y.
2. Solve $4x - 8y = 12$ for y.
3. What are the slope and $y-$intercept of $y = -\frac{2}{3}x + 5$?
4. Define *parallel* and *perpendicular* in your own words.

Standard Form

Objective

To manipulate and use the standard form of a line.

Guidance

Slope-intercept form is one way to write the equation of a line. Another way is called standard form. Standard form looks like $Ax + By = C$, where A, B, and C are all real numbers. In the Review Queue above, the equations from problems 1 and 2 are in standard form. Once they are solved for y, they will be in slope-intercept form.

Example A

Find the equation of a line, in standard form, where the slope is $\frac{3}{4}$ and passes through (4, -1).

Solution: To find the equation in standard form, you need to determine what A, B, and C are. Let's start this example by finding the equation in slope-intercept form.

$$-1 = \frac{3}{4}(4) + b$$
$$-1 = 3 + b$$
$$-4 = b$$

In slope-intercept form, the equation is $y = \frac{3}{4}x - 4$.

To change this to standard form we need to subtract the $x-$term from both sides of the equation.

$$-\frac{3}{4}x + y = -4$$

Example B

The equation of a line is $5x - 2y = 12$. What are the slope and $y-$intercept?

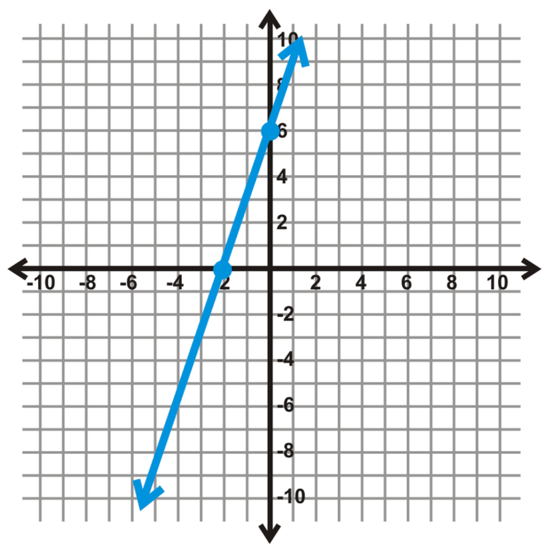
Solution: To find the slope and $y-$intercept of a line in standard form, we need to switch it to slope-intercept form. This means, we need to solve the equation for y.

$$\begin{aligned}
5x-2y&=12\\
-2y&=-5x+12\\
y&=\frac{5}{2}x-6
\end{aligned}$$

From this, the slope is $\frac{5}{2}$ and the $y-$intercept is (0, -6).

Example C

Find the equation of the line below, in standard form.



Solution: Here, we are given the intercepts. The slope triangle is drawn by the axes, $\frac{-6}{-2}=3$. And, the $y-$intercept is (0, 6). The equation of the line, in slope-intercept form, is $y=3x+6$. To change the equation to standard form, subtract the $x-$term to move it over to the other side.

$$-3x+y=6 \text{ } or \text{ } 3x-y=-6$$

Example D

The equation of a line is $6x-5y=45$. What are the intercepts?

Solution: For the $x-$intercept, the $y-$value is zero. Plug in zero for y and solve for x.

$$\begin{aligned}
6x-5y&=45\\
6x-5(0)&=45\\
6x&=45\\
x&=\frac{45}{6} \text{ } or \text{ } \frac{15}{2}
\end{aligned}$$

The $x-$intercept is $\left(\frac{15}{2},0\right)$.

For the $y-$intercept, the $x-$value is zero. Plug in zero for x and solve for y.

$$\begin{aligned}6x - 5y &= 45\\ 6(0) - 5y &= 45\\ 5y &= 45\\ y &= 9\end{aligned}$$

The $y-$intercept is (0, 9).

Guided Practice

1. Find the equation of the line, in standard form that passes through (8, -1) and (-4, 2).

2. Change $2x+3y=9$ to slope-intercept form.

3. What are the intercepts of $3x-4y=-24$?

Answers

1. Like with Example A, we need to first find the equation of this line in $y-$intercept form and then change it to standard form. First, find the slope.

$$\frac{2-(-1)}{-4-8} = \frac{3}{-12} = -\frac{1}{4}$$

Find the $y-$intercept using slope-intercept form.

$$\begin{aligned}2 &= -\frac{1}{4}(-4)+b\\ 2 &= 1+b\\ 1 &= b\end{aligned}$$

The equation of the line is $y=-\frac{1}{4}x+1$.

To change this equation into standard form, add the $x-$term to both sides.

$$\frac{1}{4}x+y=1$$

2. To change $2x+3y=9$ into slope-intercept form, solve for y.

$$\begin{aligned}2x+3y &= 9\\ 3y &= -2x+9\\ y &= -\frac{2}{3}x+3\end{aligned}$$

3. Copy Example D to find the intercepts of $3x-4y=-24$. First, plug in zero for y and solve for x.

$$\begin{aligned}3x-4(0) &= -24\\ 3x &= -24\\ x &= -8\end{aligned}$$

$x-$intercept is (-8, 0)

Now, start over and plug in zero for x and solve for y.

$$\begin{aligned} 3(0) - 4y &= -24 \\ -4y &= -24 \\ y &= 6 \end{aligned}$$

$y-$intercept is (6, 0)

Vocabulary

Standard Form (of a line)
When a line is in the form $Ax+By=C$ and A, B, and C are real numbers.

Problem Set

Change the following equations into standard form.

1. $y=-\frac{2}{3}x+4$
2. $y=x-5$
3. $y=\frac{1}{5}x-1$

Change the following equations into slope-intercept form.

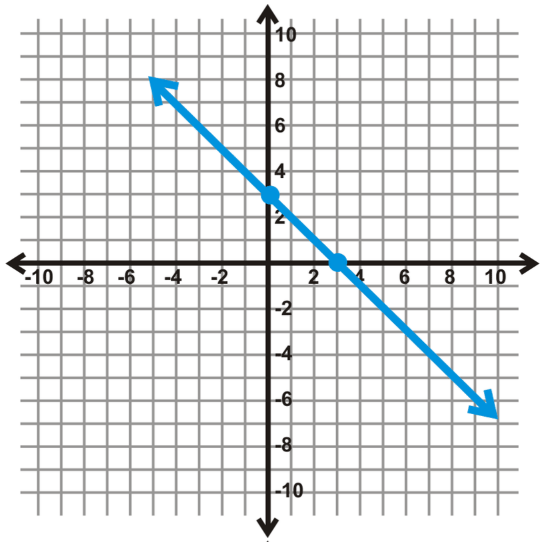
4. $4x+5y=20$
5. $x-2y=9$
6. $2x-3y=15$

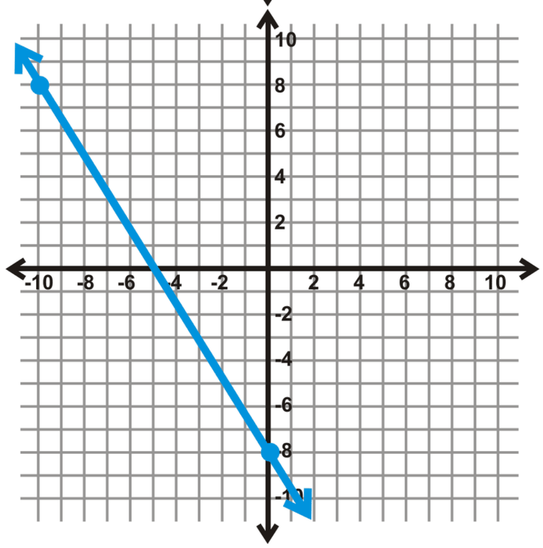
Find the x and $y-$intercepts of the following equations.

7. $3x+4y=12$
8. $6x-y=8$
9. $3x+8y=-16$

Find the equation of the lines below, in standard form.

10. slope = 2 and passes through (3, -5)
11. slope $=-\frac{1}{2}$ and passes through (6, -3).
12. passes through (5, -7) and (-1, 2)
13. passes through (-5, -5) and (5, -3)

14. 

15. 

16. Change $Ax+By=C$ into slope-intercept form.
17. From #16, what are the slope and $y-$intercept equal to (in terms of A,B, and/or C)?
18. Using #16 and #17, find one possible combination of A,B, and C for $y=\frac{1}{2}x-4$. Write your answer in standard form.
19. The measure of a road's slope is called the *grade*. The grade of a road is measured in a percentage, for how many vertical feet the road rises or declines over 100 feet. For example, a road with a grade incline of 5% means that for every 100 horizontal feet the road rises 5 vertical feet. What is the slope of a road with a grade decline of 8%?
20. The population of a small town in northern California gradually increases by about 50 people a year. In 2010, the population was 8500 people. Write an equation for the population of this city and find its estimated population in 2017.

Finding the Equation of Parallel Lines

Objective

To find the equation of a line that is parallel to a given line.

Guidance

When two lines are parallel, they have the same slope and never intersect. So, if a given line has a slope of -2, then any line that is parallel to that line will also have a slope of -2, but it will have a different $y-$intercept.

Example A

Find the equation of the line that is parallel to $y = \frac{2}{3}x - 5$ and passes through (-12, 1).

Solution: We know that the slopes will be the same; however we need to find the $y-$intercept for this new line. Use the point you were given, (-12, 1) and plug it in for x and y to solve for b.

$$\begin{aligned}
y &= \frac{2}{3}x + b \\
1 &= \frac{2}{3}(-12) + b \\
1 &= -8 + b \\
9 &= b
\end{aligned}$$

The equation of the parallel line is $y = \frac{2}{3}x + 9$.

Example B

Write the equation of the line that passes through (4, -7) and is parallel to $y = -2$.

Solution: The line $y = -2$ does not have an $x-$term, meaning it has no slope. This is a horizontal line. Therefore, to find the horizontal line that passes through (4, -7), we only need the $y-$coordinate. The line would be $y = -7$.

The same would be true for vertical lines, but all vertical line equations are in the form $x = a$. The $x-$coordinate of a given point would be what is needed to determine the equation of the parallel vertical line.

Example C

Write the equation of the line that passes through (6, -10) and is parallel to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be parallel to. Use the points (4, -6) and (3, -4) to find the slope.

$$m = \frac{-4-(-6)}{3-4} = \frac{2}{-1} = -2$$

This is the slope of our given line as well as the parallel line. Use the point (6, -10) to find the $y-$intercept of the line that we are trying to find the equation for.

$$\begin{aligned}
-10 &= -2(6) + b \\
-10 &= -12 + b \\
2 &= b
\end{aligned}$$

The equation of the line is $y = -2x + 2$.

Guided Practice

1. Find the equation of the line that is parallel to $x - 2y = 8$ and passes through (4, -3).

2. Find the equation of the line that is parallel to $x = 9$ and passes through (-1, 3).

3. Find the equation of the line that passes through (-5, 2) and is parallel to the line that passes through (6, -1) and (1, 3).

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned} x-2y &= 8 \\ -2y &= -x+8 \quad \text{Now, we know the slope is } \frac{1}{2}\text{. Let's find the new } y- \text{ intercept.} \\ y &= \frac{1}{2}x-4 \end{aligned}$$

$$\begin{aligned} -3 &= \frac{1}{2}(4)+b \\ -3 &= 2+b \\ -5 &= b \end{aligned}$$

The equation of the parallel line is $y=\frac{1}{2}x-5$ or $x-2y=10$.

2. $x=9$ is a vertical line that passes through the $x-$axis at 9. Therefore, we only need to $x-$coordinate of the point to determine the equation of the parallel vertical line. The parallel line through (-1, 3) would be $x=-1$.

3. First, find the slope between (6, -1) and (1, 3).

$$m=\frac{-1-3}{6-1}=\frac{-4}{5}=-\frac{4}{5}$$

This will also be the slope of the parallel line. Use this slope with the given point, (-5, 2).

$$\begin{aligned} 2 &= -\frac{4}{5}(-5)+b \\ 2 &= 1+b \\ 1 &= b \end{aligned}$$

The equation of the parallel line is $y=-\frac{4}{5}x+1$.

Vocabulary

Parallel
When two or more lines are in the same plane and never intersect. These lines will always have the same slope.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is parallel to $x-y=-5$.
2. Passes through (-6, -2) and is parallel to $y=4$.
3. Passes through (-3, 5) and is parallel to $y=-\frac{1}{3}x-1$.
4. Passes through (1, -9) and is parallel to $x=8$.
5. Passes through the $y-$intercept of $2x-3y=6$ and parallel to $x-4y=10$.

6. Passes through (-12, 4) and is parallel to $y = -3x + 5$.
7. Passes through the $x-$intercept of $2x - 3y = 6$ and parallel to $x + 4y = -3$.
8. Passes through (7, -8) and is parallel to $2x + 5y = 14$.
9. Passes through (1, 3) and is parallel to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (-18, -10) and is parallel to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is parallel to the line that passes through (15, 7) and (-1, -1).

Are the pairs of lines parallel? Briefly explain how you know.

12. $x - 2y = 4$ and $-5x + 10y = 16$
13. $3x + 4y = -8$ and $6x + 12y = -1$
14. $5x - 5y = 20$ and $x + y = 7$
15. $8x - 12y = 36$ and $10x - 15y = -15$

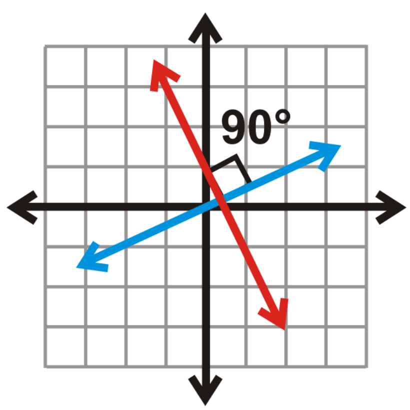
Finding the Equation of Perpendicular Lines

Objective

To find the equation of a line that is perpendicular to a given line and determine if pairs of lines are parallel, perpendicular, or neither.

Guidance

When two lines are perpendicular, they intersect at a 90°, or right, angle. The slopes of two perpendicular lines, are therefore, not the same. Let's investigate the relationship of perpendicular lines.



Investigation: Slopes of Perpendicular Lines

Tools Needed: Pencil, ruler, protractor, and graph paper

1. Draw an $x - y$ plane that goes from -5 to 5 in both the x and y directions.
2. Plot (0, 0) and (1, 3). Connect these to form a line.
3. Plot (0, 0) and (-3, 1). Connect these to form a second line.
4. Using a protractor, measure the angle formed by the two lines. What is it?
5. Use slope triangles to find the slope of both lines. What are they?
6. Multiply the slope of the first line times the slope of the second line. What do you get?

From this investigation, the lines from #2 and #3 are perpendicular because they form a 90° angle. The slopes are 3 and $-\frac{1}{3}$, respectively. When multiplied together, the product is -1. This is true of all perpendicular lines.

The product of the slopes of two perpendicular lines is -1.

If a line has a slope of m*, then the perpendicular slope is* $-\frac{1}{m}$.

Example A

Find the equation of the line that is perpendicular to $2x-3y=15$ and passes through (6, 5).

Solution: First, we need to change the line from standard to slope-intercept form.

$$\begin{aligned}2x-3y&=15\\-3y&=-2x+15\\y&=\frac{2}{3}x-5\end{aligned}$$

Now, let's find the perpendicular slope. From the investigation above, we know that the slopes must multiply together to equal -1.

$$\begin{aligned}\frac{2}{3}\cdot m&=-1\\\cancel{\frac{3}{2}}\cancel{\frac{2}{3}}\cdot m&=-1\cdot\frac{3}{2}\\m&=-\frac{3}{2}\end{aligned}$$

Notice that the perpendicular slope is the *opposite sign and reciprocals* with the original slope. Now, we need to use the given point to find the $y-$intercept.

$$\begin{aligned}5&=-\frac{3}{2}(6)+b\\5&=-9+b\\14&=b\end{aligned}$$

The equation of the line that is perpendicular to $y=\frac{2}{3}x-5$ is $y=-\frac{3}{2}x+14$.

If we write these lines in standard form, the equations would be $2x-3y=15$ and $3x+2y=28$, respectively.

Example B

Write the equation of the line that passes through (4, -7) and is perpendicular to $y=2$.

Solution: The line $y=2$ does not have an $x-$term, meaning it has no slope and a horizontal line. Therefore, to find the perpendicular line that passes through (4, -7), it would have to be a vertical line. Only need the $x-$coordinate. The perpendicular line would be $x=4$.

Example C

Write the equation of the line that passes through (6, -10) and is perpendicular to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be perpendicular to. Use the points (4, -6) and (3, -4) to find the slope.

$$m=\frac{-4-(-6)}{3-4}=\frac{2}{-1}=-2$$

Therefore, the perpendicular slope is the opposite sign and the reciprocal of -2. That makes the slope $\frac{1}{2}$. Use the point (6, -10) to find the $y-$intercept.

$$\begin{aligned}
-10 &= \frac{1}{2}(6)+b \\
-10 &= 3+b \\
-7 &= b
\end{aligned}$$

The equation of the perpendicular line is $y = \frac{1}{2}x - 7$.

Guided Practice

1. Find the equation of the line that is perpendicular to $x - 2y = 8$ and passes through (4, -3).

2. Find the equation of the line that passes through (-8, 7) and is perpendicular to the line that passes through (6, -1) and (1, 3).

3. Are $x - 4y = 8$ and $2x + 8y = -32$ parallel, perpendicular or neither?

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned}
x - 2y &= 8 \\
-2y &= -x + 8 \\
y &= \frac{1}{2}x - 4
\end{aligned}$$

The perpendicular slope will be -2. Let's find the new $y-$intercept.

$$\begin{aligned}
-3 &= -2(4) + b \\
-3 &= -8 + b \\
5 &= b
\end{aligned}$$

The equation of the perpendicular line is $y = -2x + 5$ or $2x + y = 5$.

2. First, find the slope between (6, -1) and (1, 3).

$$m = \frac{-1-3}{6-1} = \frac{-4}{5} = -\frac{4}{5}$$

From this, the slope of the perpendicular line will be $\frac{5}{4}$. Now, use (-8, 7) to find the $y-$intercept.

$$\begin{aligned}
7 &= \frac{5}{4}(-8) + b \\
7 &= -10 + b \\
17 &= b
\end{aligned}$$

The equation of the perpendicular line is $y = \frac{5}{4}x + 17$.

3. To determine if the two lines are parallel or perpendicular, we need to change them both into slope-intercept form.

$$\begin{aligned} x-4y &= 8 & 2x+8y &= -32 \\ -4y &= -x+8 \quad \textit{and} & 8y &= -2x-32 \\ y &= \frac{1}{4}x-2 & y &= -\frac{1}{4}x-4 \end{aligned}$$

Now, just look at the slopes. One is $\frac{1}{4}$ and the other is $-\frac{1}{4}$. They are not the same, so they are not parallel. To be perpendicular, the slopes need to be reciprocals, which they are not. Therefore, these two lines are not parallel or perpendicular.

Vocabulary

Perpendicular
: When two lines intersect to form a right, or 90°, angle. The product of the slopes of two perpendicular lines is -1.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is perpendicular to $x-y=-5$.
2. Passes through (-6, -2) and is perpendicular to $y=4$.
3. Passes through (4, 5) and is perpendicular to $y=-\frac{1}{3}x-1$.
4. Passes through (1, -9) and is perpendicular to $x=8$.
5. Passes through (0, 6) and perpendicular to $x-4y=10$.
6. Passes through (-12, 4) and is perpendicular to $y=-3x+5$.
7. Passes through the $x-$intercept of $2x-3y=6$ and perpendicular to $x+6y=-3$.
8. Passes through (7, -8) and is perpendicular to $2x+5y=14$.
9. Passes through (1, 3) and is perpendicular to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (3, -10) and is perpendicular to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is perpendicular to the line that passes through (-15, 7) and (-3, 3).

Are the pairs of lines parallel, perpendicular or neither?

12. $4x+2y=5$ and $5x-10y=-20$
13. $9x+12y=8$ and $6x+8y=-1$
14. $5x-5y=20$ and $x+y=7$
15. $8x-4y=12$ and $4x-y=-15$

2.20 Graphing Lines

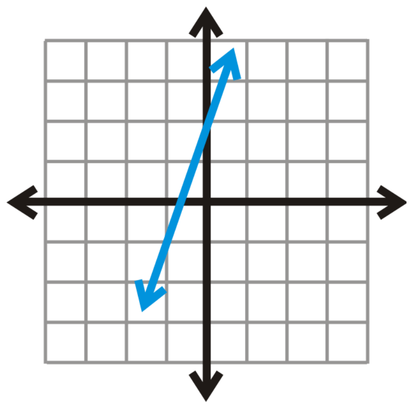
Objective

To be able to graph the equation of a line in slope-intercept or standard form.

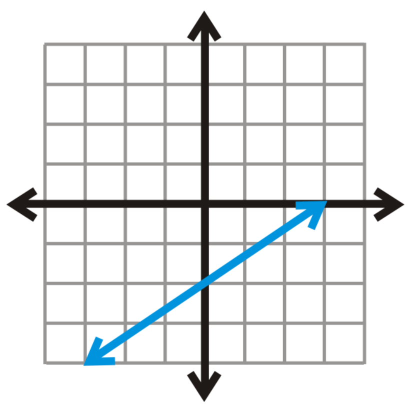
Review Queue

Find the equation of each line below. For the graphs, you may assume the $y-$intercepts are integers.

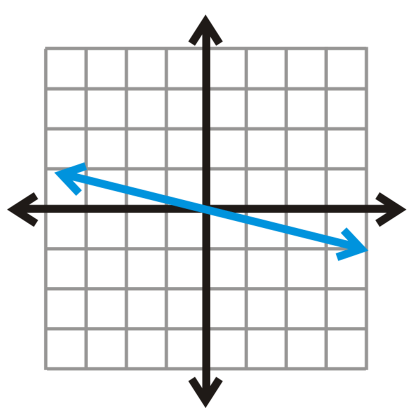
1.



2.



3.



4. What are the x and $y-$intercepts of:

a) $3x-5y=15$

b) $8x-5y=24$

Graph a Line in Slope-Intercept Form

Objective

To graph a line in slope-intercept form.

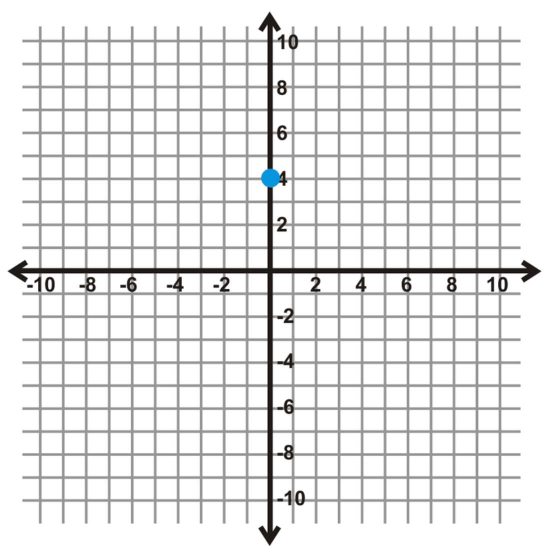
Guidance

From the previous lesson, we know that the equation of a line is $y=mx+b$, where m is the slope and b is the $y-$intercept. From these two pieces of information we can graph any line.

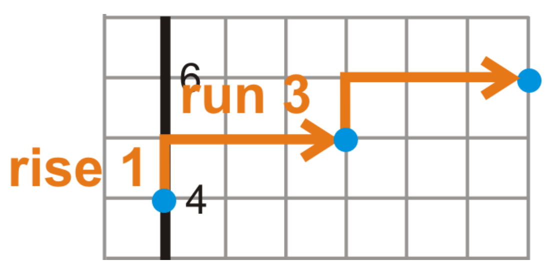
Example A

Graph $y=\frac{1}{3}x+4$ on the Cartesian plane.

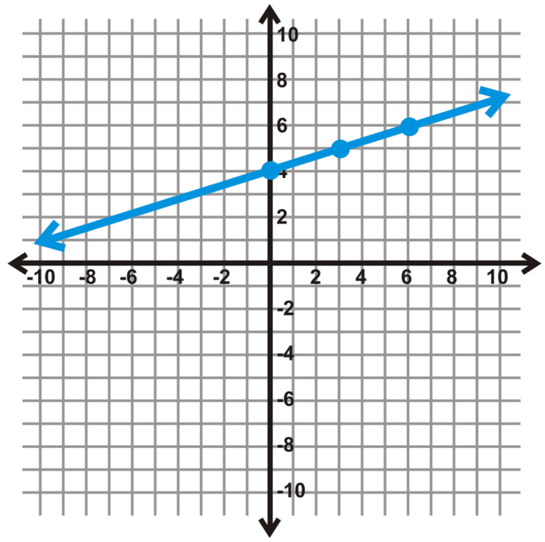
Solution: First, the Cartesian plane is the $x-y$ plane. Typically, when graphing lines, draw each axis from -10 to 10. To graph this line, you need to find the slope and $y-$intercept. By looking at the equation, $\frac{1}{3}$ is the slope and 4, or (0, 4), is the $y-$intercept. To start graphing this line, plot the $y-$intercept on the $y-$axis.



Now, we need to use the slope to find the next point on the line. Recall that the slope is also $\frac{rise}{run}$, so for $\frac{1}{3}$, we will rise 1 and run 3 from the $y-$intercept. Do this a couple of times to get at least three points.



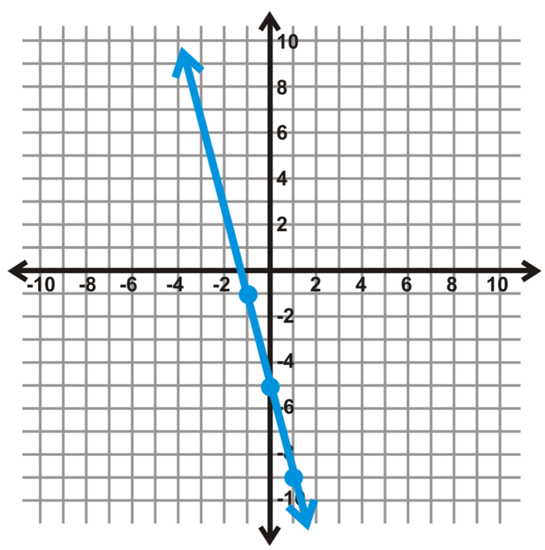
Now that we have three points, connect them to form the line $y=\frac{1}{3}x+4$.



Example B

Graph $y = -4x - 5$.

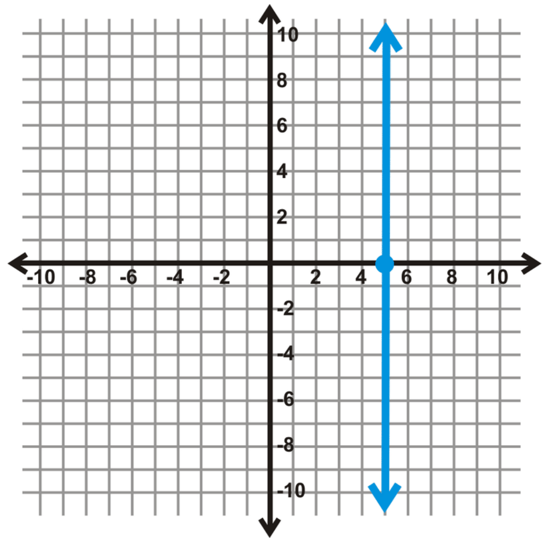
Solution: Now that the slope is negative, the vertical distance will "fall" instead of rise. Also, because the slope is a whole number, we need to put it over 1. Therefore, for a slope of -4, the line will fall 4 and run 1 OR rise 4 and run backward 1. Start at the $y-$intercept, and then use the slope to find a few more points.



Example C

Graph $x = 5$.

Solution: Any line in the form $x = a$ is a vertical line. To graph any vertical line, plot the value, in this case 5, on the $x-$axis. Then draw the vertical line.



To graph a horizontal line, $y = b$, it will be the same process, but plot the value given on the $y-$axis and draw a horizontal line.

Guided Practice

Graph the following lines.

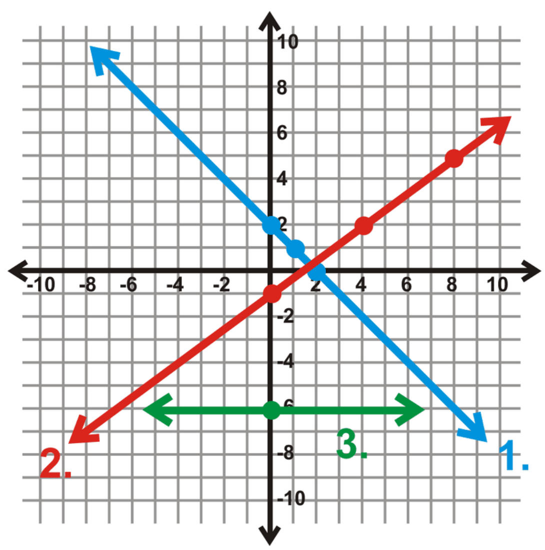
1. $y = -x + 2$

2. $y = \frac{3}{4}x - 1$

3. $y = -6$

Answers

All the answers are on the same grid below.



1. Plot (0, 2) and the slope is -1, which means you fall 1 and run 1.

2. Plot (0, -1) and then rise 3 and run 4 to the next point, (4, 2).

3. Plot -6 on the $y-$axis and draw a horizontal line.

Problem Set

Graph the following lines in the Cartesian plane.

1. $y=-2x-3$
2. $y=x+4$
3. $y=\frac{1}{3}x-1$
4. $y=9$
5. $y=-\frac{2}{5}x+7$
6. $y=\frac{2}{4}x-5$
7. $y=-5x-2$
8. $y=-x$
9. $y=4$
10. $x=-3$
11. $y=\frac{3}{2}x+3$
12. $y=-\frac{1}{6}x-8$
13. Graph $y=4$ and $x=-6$ on the same set of axes. Where do they intersect?
14. If you were to make a general rule for the lines $y=b$ and $x=a$, where will they always intersect?
15. The cost per month, C (in dollars), of placing an ad on a website is $C=0.25x+50$, where x is the number of times someone clicks on your link. How much would it cost you if 500 people clicked on your link?

Graph a Line in Standard Form

Objective

To graph a line in standard form.

Guidance

When a line is in standard form, there are two different ways to graph it. The first is to change the equation to slope-intercept form and then graph as shown in the previous concept. The second is to use standard form to find the x and $y-$intercepts of the line and connect the two. Here are a few examples.

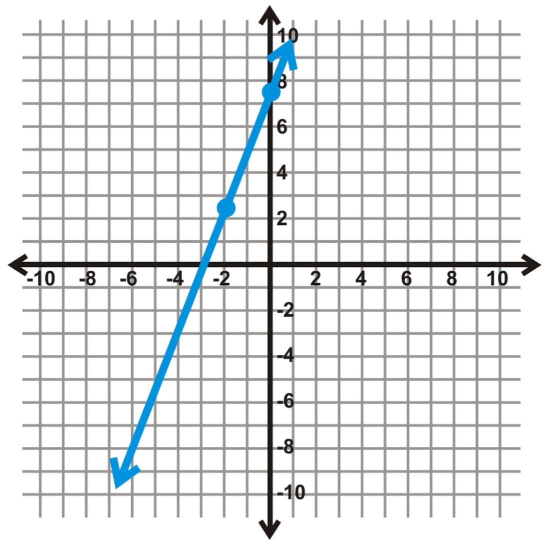
Example A

Graph $5x-2y=-15$.

Solution: Let's use approach #1; change the equation to slope-intercept form.

$$\begin{aligned}5x-2y&=-15\\-2y&=-5x-15\\y&=\frac{5}{2}x+\frac{15}{2}\end{aligned}$$

The $y-$intercept is $\left(0,\frac{15}{2}\right)$. Change the improper fraction to a decimal and approximate it on the graph, (0, 7.5). Then use slope triangles. If you find yourself running out of room "rising 5" and "running 2," you could also "fall 5" and "run backwards 2" to find a point on the other side of the $y-$intercept.



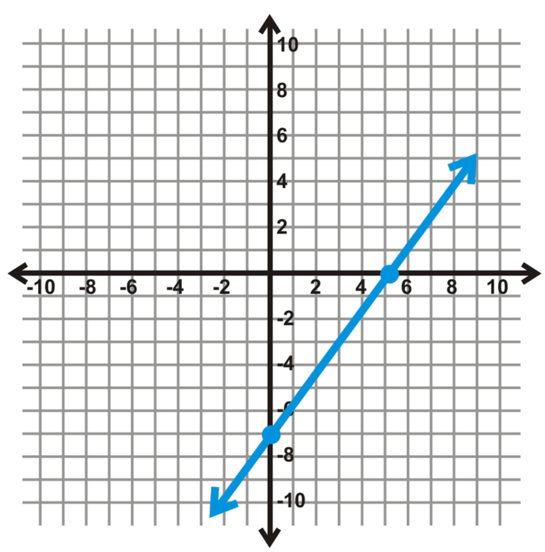
Example B

Graph $4x - 3x = 21$.

Solution: Let's use approach #2; find the x and $y-$intercepts (from *Standard Form of a Line* concept). Recall that the other coordinate will be zero at these points. Therefore, for the $x-$intercept, plug in zero for y and for the $y-$intercept, plug in zero for x.

$$\begin{aligned} 4x - 3(0) &= 21 \\ 4x &= 21 \\ x &= \frac{21}{4} \ \textit{or}\ 5.25 \end{aligned} \qquad\qquad \begin{aligned} 4(0) - 3y &= 21 \\ -3y &= 21 \\ y &= -7 \end{aligned}$$

Now, plot each on their respective axes and draw a line.



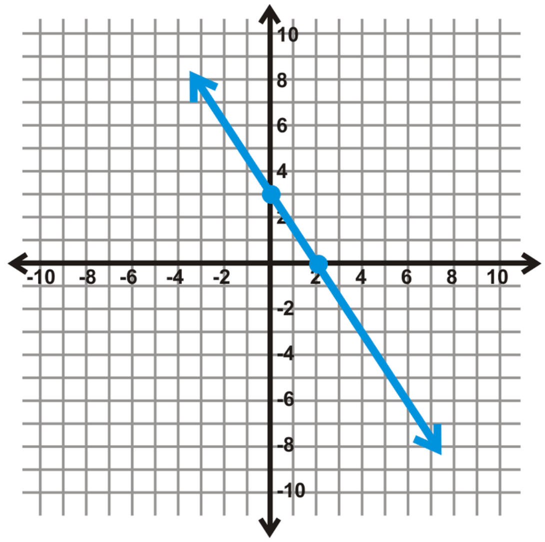
Guided Practice

1. Graph $4x + 6y = 18$ by changing it into slope-intercept form.

2. Graph $5x - 3y = 30$ by plotting the intercepts.

Answers

1. Change $4x+6y=18$ into slope-intercept form by solving for y, then graph.

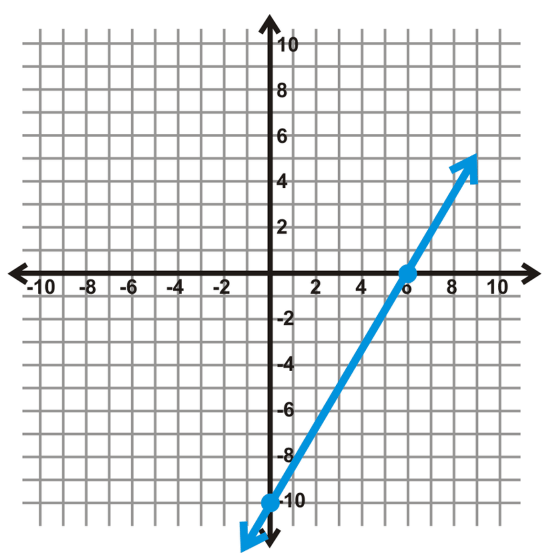


$$\begin{aligned} 4x+6y &= 18 \\ 6y &= -4x+18 \\ y &= -\frac{2}{3}x+3 \end{aligned}$$

2. Substitute in zero for x, followed by y and solve each equation.

$$\begin{aligned} 5(0)-3y &= 30 \\ -3y &= 30 \\ y &= -10 \end{aligned} \qquad\qquad \begin{aligned} 5x-3(0) &= 30 \\ 5x &= 30 \\ x &= 6 \end{aligned}$$

Now, plot each on their respective axes and draw a line.



Problem Set

Graph the following lines by changing the equation to slope-intercept form.

1. $-2x+y=5$
2. $3x+8y=16$
3. $4x-2y=10$
4. $6x+5y=-20$
5. $9x-6y=24$
6. $x+4y=-12$

Graph the following lines by finding the intercepts.

7. $2x+3y=12$
8. $-4x+5y=30$
9. $x-2y=8$
10. $7x+y=-7$
11. $6x+10y=15$
12. $4x-8y=-28$
13. **Writing** Which method do you think is easier? Why?
14. **Writing** Which method would you use to graph $x=-5$? Why?

2.21 Graphing Linear Inequalities in Two Variables

Objective

To graph a linear inequality in two variables on the Cartesian plane.

Review Queue

Graph the following lines on the same set of axes.

1. $y = \frac{1}{4}x - 2$

2. $x - y = -6$

3. $2x + 5y = 15$

Solve the following inequalities. Graph the answer on a number line.

4. $4x - 5 \leq 11$

5. $\frac{3}{5}x > -12$

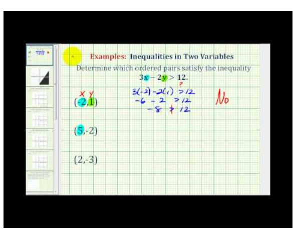
6. $-6x + 11 \geq -13$

Testing Solutions for Linear Inequalities in Two Variables

Objective

To determine if an ordered pair is a solution to a linear inequality in two variables.

Watch This



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James Sousa: Ex: Determine if Ordered Pairs Satisfy a Linear Inequality

Guidance

A **linear inequality** is very similar to the equation of a line, but with an inequality sign. They can be written in one of the following ways:

$$Ax + By < C \qquad Ax + By > C \qquad Ax + By \leq C \qquad Ax + By \geq C$$

Notice that these inequalities are very similar to the standard form of a line. We can also write a linear inequality in slope-intercept form.

$$y < mx + b \qquad y > mx + b \qquad y \leq mx + b \qquad y \geq mx + b$$

In all of these general forms, the A, B, C, m, and b represent the exact same thing they did with lines.

An ordered pair, or point, is a **solution** to a linear inequality if it makes the inequality true when the values are substituted in for x and y.

Example A

Which ordered pair is a solution to $4x - y > -12$?

a) (6, -5)

b) (-3, 0)

c) (-5, 4)

Solution: Plug in each point to see if they make the inequality true.

a)

$$\begin{aligned} 4(6) - (-5) &> -12 \\ 24 + 5 &> -12 \\ 29 &> -12 \end{aligned}$$

b)

$$\begin{aligned} 4(-3) - 0 &> -12 \\ -12 &\not> -12 \end{aligned}$$

c)

$$\begin{aligned} 4(-5) - 4 &> -12 \\ -20 - 4 &> -12 \\ -24 &\not> -12 \end{aligned}$$

Of the three points, a) is the only one where the inequality holds true. b) is not true because the inequality sign is only "greater than," not "greater than or equal to."

Vocabulary

Linear Inequality
An inequality, usually in two variables, of the form $Ax + By < C, Ax + By > C, Ax + By \leq C$, or $Ax + By \geq C$.

Solution
An ordered pair that satisfies a given inequality.

Guided Practice

1. Which inequality is (-7, 1) a solution for?

a) $y < 2x - 1$

b) $4x - 3y \geq 9$

c) $y > -4$

2. List three possible solutions for $5x - y \leq 3$.

Answers

1. Plug (-7, 1) in to each equation. With c), only use the $y-$value.

a)

$$\begin{aligned}1 &< 2(-7)-1\\ 1 &\not< -15\end{aligned}$$

b)

$$\begin{aligned}4(-7)-3(1) &\geq 9\\ -28-3 &\geq 9\\ -31 &\not\geq 9\end{aligned}$$

c) $1 > -4$

(-7, 1) is only a solution to $y > -4$.

2. To find possible solutions, plug in values to the inequality. There are infinitely many solutions. Here are three: (-1, 0), (-4, 3), and (1, 6).

$$\begin{aligned}5(-1)-0 &\leq 3\\ -5 &\leq 3\end{aligned} \qquad \begin{aligned}5(-4)-3 &\leq 3\\ -17 &\leq 3\end{aligned} \qquad \begin{aligned}5(1)-6 &\leq 3\\ -1 &\leq 3\end{aligned}$$

Problem Set

Using the four inequalities below, determine which point is a solution for each one. There may be more than one correct answer. If the answer is none, write *none of these*.

A) $y \leq \frac{2}{3}x-5$

B) $5x+4y > 20$

C) $x-y \geq -5$

D) $y > -4x+1$

1. (9, -1)
2. (0, 0)
3. (-1, 6)
4. (-3, -10)

Determine which inequality each point is a solution for. There may be more than one correct answer. If the answer is none, write *none of these*.

A) (-5, 1)

B) (4, 2)

C) (-12, -7)

D) (8, -9)

5. $2x-3y > 8$
6. $y \leq -x-4$
7. $y \geq 6x+7$
8. $8x+3y < -3$
9. Is (-6, -8) a solution to $y < \frac{1}{2}x-6$?
10. Is (10, 1) a solution to $y \geq -7x+1$?

For problems 11 and 12, find three solutions for each inequality.

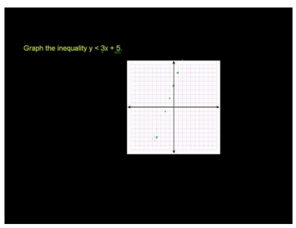
11. $5x - y > 12$
12. $y \leq -2x + 9$

Graphing Inequalities in Two Variables

Objective

To graph a linear inequality on the Cartesian plane.

Watch This



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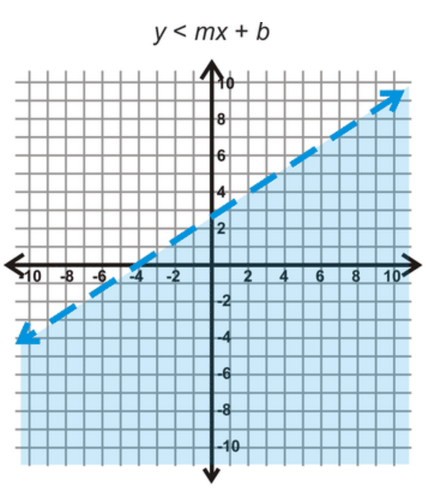
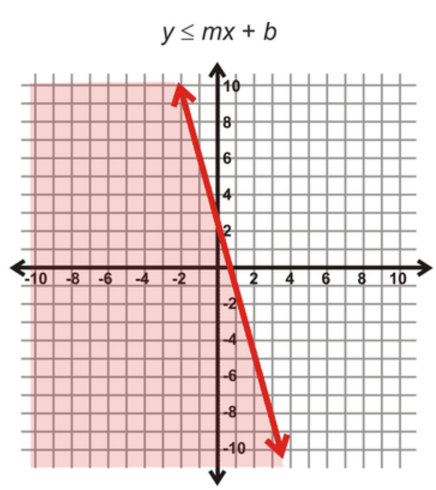
Khan Academy: Graphing linear inequalities in two variables 2

Guidance

Graphing linear inequalities is very similar to graphing lines. First, you need to change the inequality into slope-intercept form. At this point, we will have a couple of differences. If the inequality is in the form $y < mx + b$ or $y > mx + b$, the line will be dotted or dashed because it is not a part of the solution. If the line is in the form $y \leq mx + b$ or $y \geq mx + b$, the line will be solid to indicate that it is included in the solution.

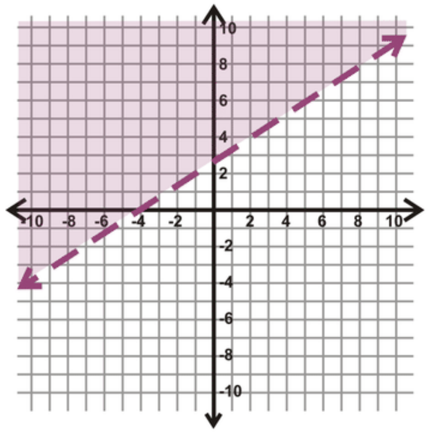
The second difference is the shading. Because these are inequalities, not just the line is the solution. Depending on the sign, there will be shading above or below the line. If the inequality is in the form $y < mx + b$ or $y \leq mx + b$, the shading will be below the line, in reference to the $y-$axis.



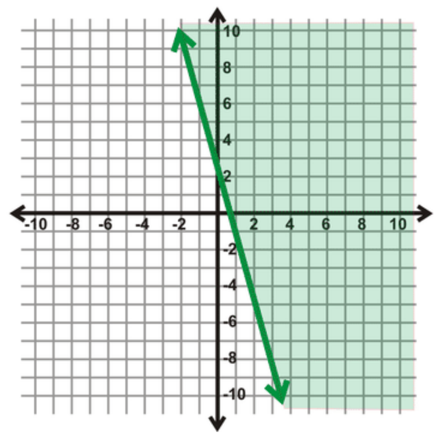


If the inequality is in the form $y > mx + b$ or $y \geq mx + b$, the shading will be above the line.

$y > mx + b$



$y \geq mx + b$



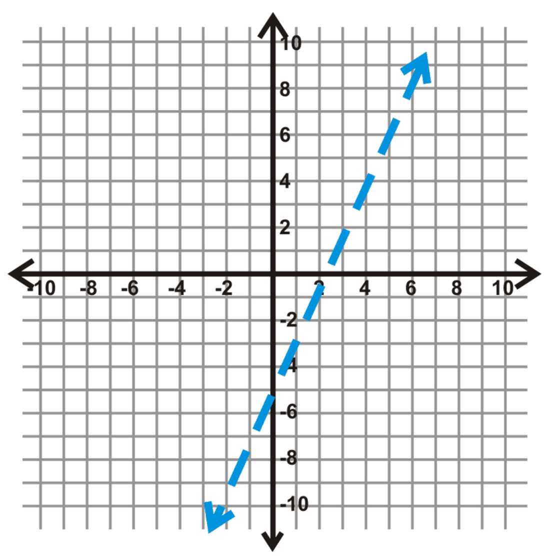
Example A

Graph $4x - 2y < 10$.

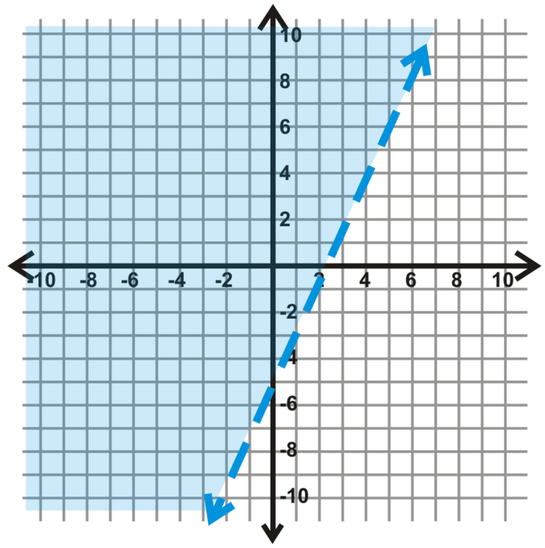
Solution: First, change the inequality into slope-intercept form. Remember, that if you have to divide or multiply by a negative number, you must flip the inequality sign.

$$4x - 2y < 10$$
$$-2y < -4x + 10 \quad \text{Flip the inequality sign because we divided by -2.}$$
$$y > 2x - 5$$

Now, graph the inequality as if it was a line. Plot $y = 2x - 5$ like in the *Graphing Lines in Slope-Intercept Form* concept. However, the line will be dashed because of the "greater than" sign.



Now, we need to determine the shading. You can use one of two methods to do this. The first way is to use the graphs and forms from above. The equation, in slope-intercept form, matches up with the purple dashed line and shading. Therefore, we should shade above the dashed blue line.

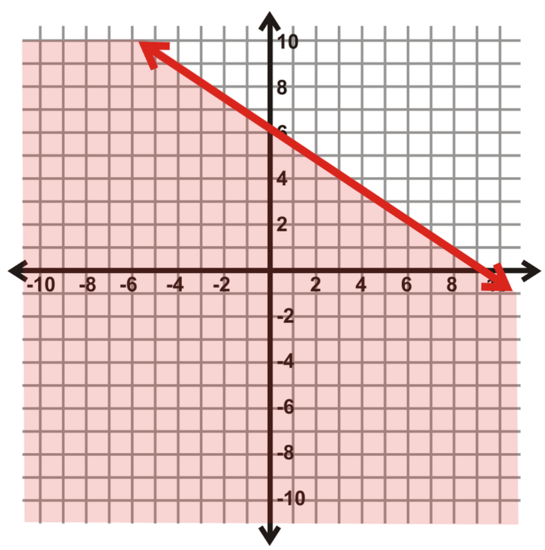


The alternate method would be to test a couple of points to see if they work. If a point is true, then the shading is over that side of the line. If we pick (-5, 0), the inequality yields $-20 < 10$, which tells us that our shading is correct.

Example B

Graph $y \leq -\frac{2}{3}x + 6$.

Solution: This inequality is already in slope-intercept form. So, graph the line, which will be solid, and then determine the shading. Looking at the example graphs above, this inequality should look like the red inequality, so shade below the line.

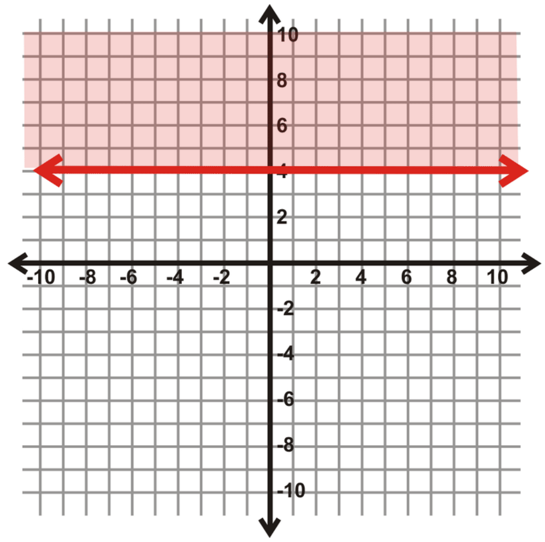


Test a point to make sure our shading is correct. An easy point in the shaded region is (0, 0). Plugging this into the inequality, we get $0 \leq 6$, which is true.

Example C

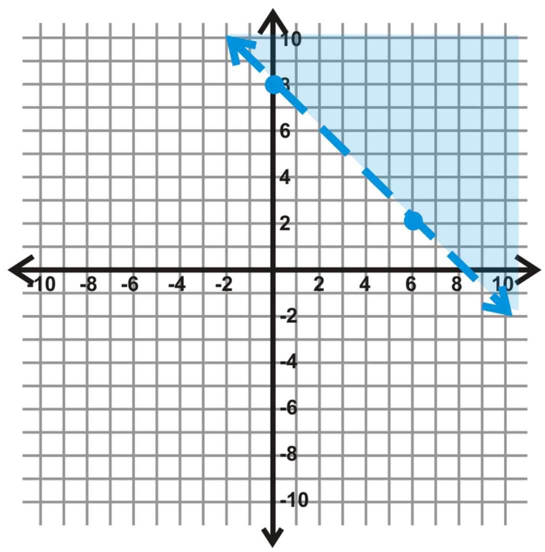
Graph $y \geq 4$.

Solution: Treat this inequality like you are graphing a horizontal line. We will draw a solid line at $y = 4$ and then shade above because of the "$\geq$" sign.



Example D

Determine the linear inequality that is graphed below.



Solution: Find the equation of the line portion just like you did in the *Find the Equation of a Line in Slope-Intercept Form* concept. The given points on the line are (0, 8) and (6, 2) (from the points drawn on the graph). This means that the $y-$intercept is (0, 8). Then, using slope triangles we fall 6 and run 6 to get to (6, 2). This means the slope is $\frac{-6}{6}$ or -1. Because we have a dotted line and the shading is above, our sign will be the >sign. Putting it all together, the equation of our linear inequality is $y > -x + 8$.

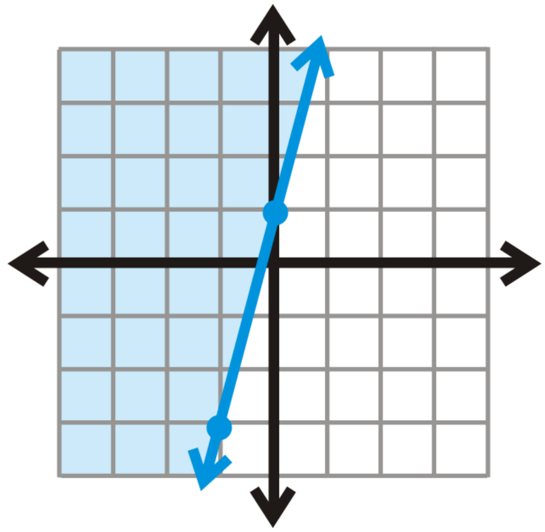
*When finding the equation of an inequality, like above, it is easiest to find the equation in slope-intercept form. To determine which inequality sign to use, look at the shading along the $y-$axis. If the shaded $y-$values get larger, the line will be in the form or $y > mx + b$ or $y \geq mx + b$. If they get smaller, the line will be in the form $y < mx + b$ or $y \leq mx + b$.

Guided Practice

1. Graph $3x - 4y > 20$.

2. Graph $x < -1$.

3. What is the equation of the linear inequality?

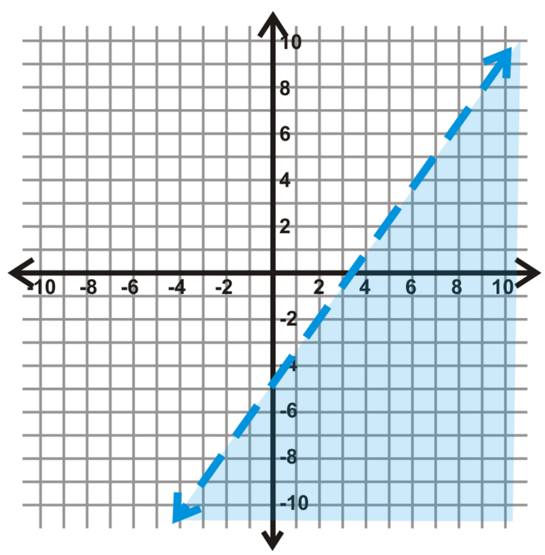


Answers

1. First, change the inequality into slope-intercept form.

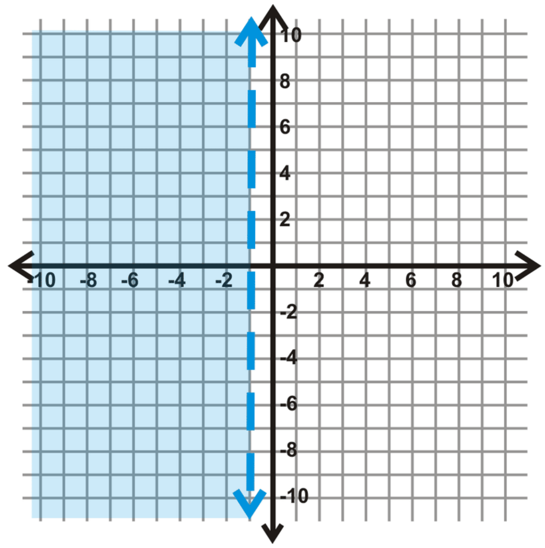
$$\begin{aligned} 3x - 4y &> 20 \\ -4y &> -3x + 20 \\ y &< \frac{3}{4}x - 5 \end{aligned}$$

Now, we need to determine the type of line and shading. Because the sign is "<," the line will be dashed and we will shade below.



Test a point in the shaded region to make sure we are correct. If we test (6, -6) in the original inequality, we get $42 > 20$, which is true.

2. To graph this line on the $x-y$ plane, recall that all vertical lines have the form $x = a$. Therefore, we will have a vertical *dashed* line at -1. Then, the shading will be to the left of the dashed line because that is where x will be less than the value of the line.



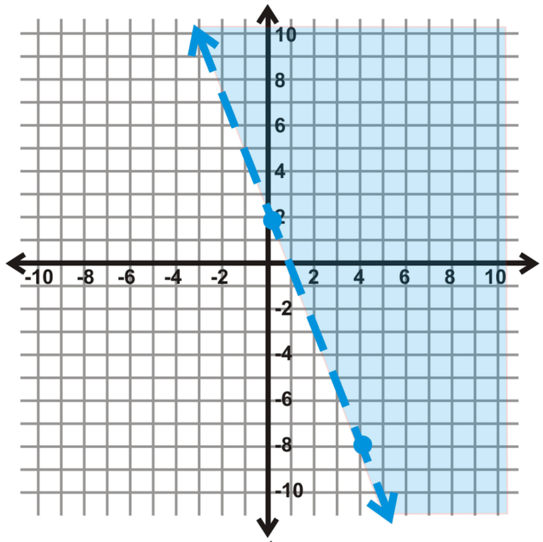
3. Looking at the line, the $y-$intercept is (0, 1). Using a slope triangle, to count down to the next point, we would fall 4, and run backward 1. This means that the slope is $\frac{-4}{-1} = 4$. The line is solid and the shading is above, so we will use the $\geq$ sign. Our inequality is $y \geq 4x+1$.

Problem Set

Graph the following inequalities.

1. $y > x-5$
2. $3x-2y \geq 4$
3. $y < -3x+8$
4. $x+4y \leq 16$
5. $y < -2$
6. $y < -\frac{1}{2}x-3$
7. $x \geq 6$
8. $8x+4y \geq -20$
9. $-4x+y \leq 7$
10. $5x-3y \geq -24$
11. $y > 5x$
12. $y \leq 0$

Determine the equation of each linear inequality below.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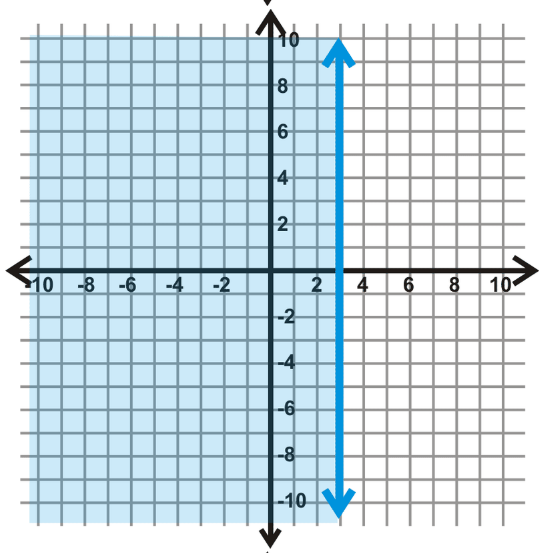
-10

13.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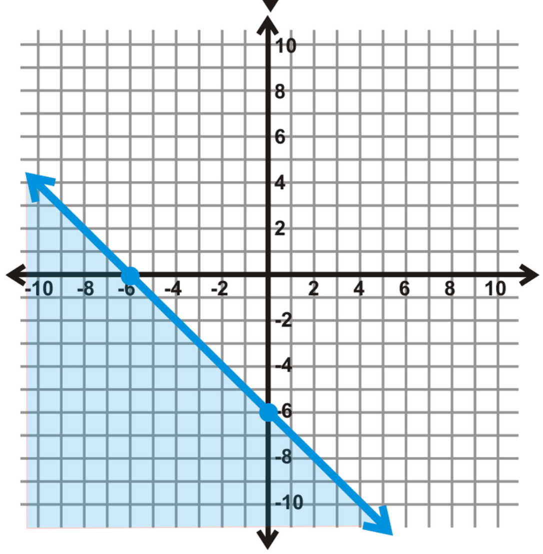
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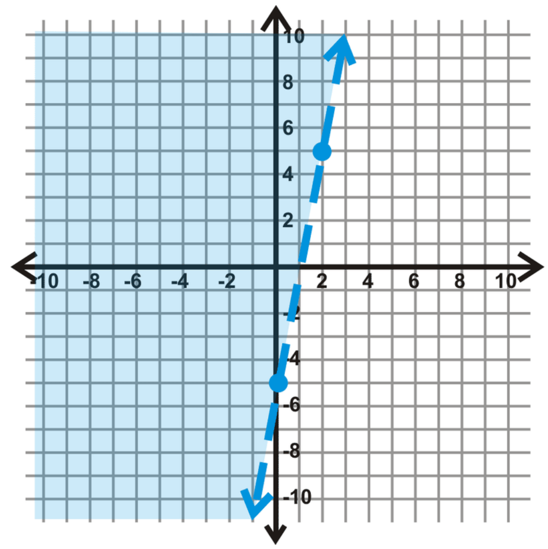
16. 

2.22 Solving Quadratics by Factoring

Objective

To factor and solve any quadratic equation that is considered "factorable."

Review Queue

Solve the following equations.

1. $5x-12=2x+9$
2. $\frac{1}{3}x+\frac{5}{2}=-\frac{1}{3}x-\frac{7}{2}$
3. Solve the system of equations using any method:

$$\begin{aligned} 2x-y&=12 \\ -3x+2y&=-19 \end{aligned}$$

4. Find two numbers whose sum is 10 and product is 16.

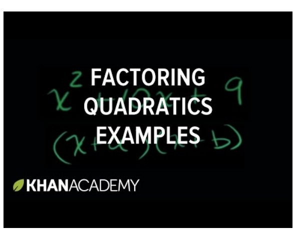
Factoring When the First Coefficient Equals 1

Objective

To factor a quadratic equation in the form x^2+bx+c.

Watch This

Watch the first few examples in this video, until about 12:40.



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Khan Academy: Factoring Quadratic Expressions

Guidance

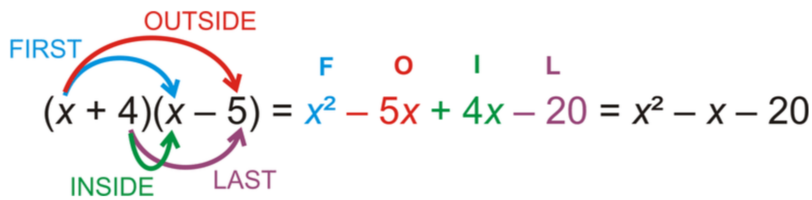
In this chapter we will be discussing quadratic equations. A **quadratic equation** has the form ax^2+bx+c, where $a \neq 0$ (If $a=0$, then the equation would be linear). For all quadratic equations, the 2 is the largest and only exponent. A quadratic equation can also be called a **trinomial** when all three terms are present.

There are four ways to solve a quadratic equation. The easiest is **factoring**. In this concept, we are going to focus on factoring when $a=1$ or when there is no number in front of x^2. First, let's start with a review of multiplying two factors together.

Example A

Multiply $(x+4)(x-5)$.

Solution: Even though this is not a quadratic, the product of the two **factors** will be. Remember from previous math classes that a factor is a number that goes evenly into a larger number. For example, 4 and 5 are factors of 20. So, to determine the larger number that $(x+4)$ and $(x-5)$ go into, we need to multiply them together. One method for multiplying two polynomial factors together is called FOIL. To do this, you need to multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and the LAST terms together and then combine like terms.



Therefore $(x+4)(x-5) = x^2 - x - 20$. We can also say that $(x+4)$ and $(x-5)$ are factors of $x^2 - x - 20$.

More Guidance

Now, we will "undo" the multiplication of two factors by factoring. In this concept, we will only address quadratic equations in the form $x^2 + bx + c$, or when $a = 1$.

Investigation: Factoring $x^2 + bx + c$

1. From the previous example, we know that $(x+m)(x+n) = x^2 + bx + c$. FOIL $(x+m)(x+n)$.

$$(x+m)(x+n) \Rightarrow x^2 + \underbrace{nx + mx}_{bx} + \underbrace{mn}_{c}$$

2. This shows us that the **constant** term, or c, is equal to the product of the constant numbers inside each factor. It also shows us that the **coefficient** in front of x, or b, is equal to the sum of these numbers.

3. Group together the first two terms and the last two terms. Find the Greatest Common Factor, or GCF, for each pair.

$$(x^2 + nx) + (mx + mn)$$
$$x(x+n) + m(x+n)$$

4. Notice that what is inside both sets of parenthesis in Step 3 is the same. This number, $(x+n)$, is the GCF of $x(x+n)$ and $m(x+n)$. You can pull it out in front of the two terms and leave the $x+m$.

$$x(x+n) + m(x+n)$$
$$(x+n)(x+m)$$

We have now shown how to go from FOIL-ing to factoring and back. Let's apply this idea to an example.

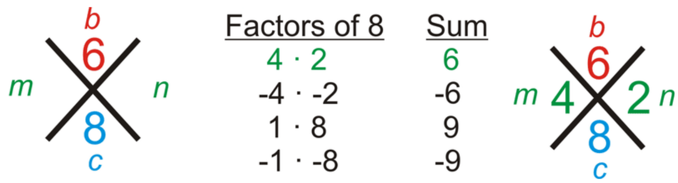
Example B

Factor $x^2 + 6x + 8$.

Solution: Let's use the investigation to help us.

$$x^2 + 6x + 8 = (x+m)(x+n)$$

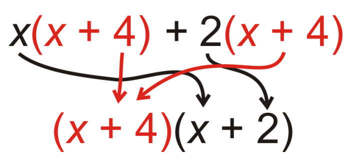
So, from Step 2, b will be equal to the sum of m and n and c will be equal to their product. Applying this to our problem, $6 = m+n$ and $8 = mn$. To organize this, use an "X". Place the sum in the top and the product in the bottom.



The green pair above is the only one that also adds up to 6. Now, move on to Step 3 from our investigation. We need to rewrite the $x-$term, or b, as a sum of m and n.

$$\begin{aligned}
&x^2+6x+8\\
&x^2+4x+2x+8\\
&(x^2+4x)+(2x+8)\\
&x(x+4)+2(x+4)
\end{aligned}$$

Moving on to Step 4, we notice that the $(x+4)$ term is the same. Pull this out and we are done.



Therefore, the factors of x^2+6x+8 are $(x+4)(x+2)$. You can FOIL this to check your answer.

Example C

Factor $x^2+12x-28$.

Solution: We can approach this problem in exactly the same way we did Example B. This time, we will not use the "X." What are the factors of -28 that also add up to 12? Let's list them out to see:

$$-4\cdot 7, 4\cdot -7, 2\cdot -14, -2\cdot 14, 1\cdot -28, -1\cdot 28$$

The **red** pair above is the one that works. Notice that we only listed the factors of *negative* 28.

$$\begin{aligned}
&x^2+12x-28\\
&x^2-2x+14x-28\\
&(x^2-2x)+(14x-28)\\
&x(x-2)+14(x-2)\\
&(x-2)(x+14)
\end{aligned}$$

By now, you might have a couple questions:

1. Does it matter which $x-$term you put first? NO, order does not matter. In the previous example, we could have put $14x$ followed by $-2x$. We would still end up with the same answer.
2. Can I skip the "expanded" part (Steps 3 and 4 in the investigation)? YES and NO. Yes, if $a = 1$ No, if $a \neq 1$ (the next concept). If $a = 1$, then $x^2 + bx + c = (x+m)(x+n)$ such that $m+n = b$ and $mn = c$. Consider this a shortcut.

Example D

Factor $x^2 - 4x$.

Solution: This is an example of a quadratic that is not a trinomial because it only has two terms, also called a **binomial**. There is no c, or constant term. To factor this, we need to look for the GCF. In this case, the largest number that can be taken out of both terms is an x.

$$x^2 - 4x = x(x-4)$$

Therefore, the factors are x and $x-4$.

Guided Practice

1. Multiply $(x-3)(x+8)$.

Factor the following quadratics, if possible.

2. $x^2 - 9x + 20$

3. $x^2 + 7x - 30$

4. $x^2 + x + 6$

5. $x^2 + 10x$

Answers

1. FOIL-ing our factors together, we get:

$$(x-3)(x+8) = x^2 + 8x - 3x - 24 = x^2 + 5x - 24$$

2. Using the "X," we have:

From the shortcut above, $-4 + -5 = -9$ and $-4 \cdot -5 = 20$.

$$x^2 - 9x + 20 = (x-4)(x-5)$$

3. Let's list out all the factors of -30 and their sums. The sums are in red.

$$-10 \cdot 3\ (-7), -3 \cdot 10\ (7), -2 \cdot 15\ (13), -15 \cdot 2\ (-13), -1 \cdot 30\ (29), -30 \cdot 1\ (-29)$$

From this, the factors of -30 that add up to 7 are -3 and 10. $x^2+7x-30=(x-3)(x+10)$

4. There are no factors of 6 that add up to 1. If we had -6, then the trinomial would be factorable. But, as is, this is not a factorable trinomial.

5. The only thing we can do here is to take out the GCF. $x^2+10x=x(x+10)$

Vocabulary

Quadratic Equation
: An equation where the largest exponent is a 2 and has the form ax^2+bx+c, $a \neq 0$.

Trinomial
: A quadratic equation with three terms.

Binomial
: A quadratic equation with two terms.

Factoring
: A way to break down a quadratic equation into smaller factors.

Factor
: A number that goes evenly into a larger number.

FOIL
: A method used to multiply together two factors. You multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and LAST terms and then combine any like terms.

Coefficient
: The number in front of a variable.

Constant
: A number that is added or subtracted within an equation.

Problem Set

Multiply the following factors together.

1. $(x+2)(x-8)$
2. $(x-9)(x-1)$
3. $(x+7)(x+3)$

Factor the following quadratic equations. If it cannot be factored, write *not factorable*. You can use either method presented in the examples.

4. x^2-x-2
5. $x^2+2x-24$
6. x^2-6x
7. x^2+6x+9
8. $x^2+8x-10$
9. $x^2-11x+30$
10. $x^2+13x-30$

11. $x^2+11x+28$
12. $x^2-8x+12$
13. $x^2-7x-44$
14. $x^2-8x-20$
15. x^2+4x+3
16. $x^2-5x+36$
17. $x^2-5x-36$
18. x^2+x

Challenge Fill in the X's below with the correct numbers.

19. 

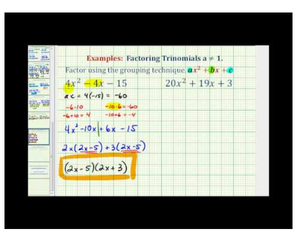
20. 

Factoring When the First Coefficient Doesn't Equal 1

Objective

To multiply factors and factor quadratic equations in the form ax^2+bx+c by expanding the $x-$term.

Watch This



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James Sousa: Ex: Factor Trinomials When A is NOT Equal to 1 - Grouping Method

Guidance

When we add a number in front of the x^2 term, it makes factoring a little trickier. We still follow the investigation from the previous section, but *cannot* use the shortcut. First, let's try FOIL-ing when the coefficients in front of the $x-$terms are not 1.

Example A

Multiply $(3x-5)(2x+1)$

Solution: We can still use FOIL.

FIRST $3x \cdot 2x = 6x^2$

OUTSIDE $3x \cdot 1 = 3x$

INSIDE $-5 \cdot 2x = -10x$

LAST $-5 \cdot 1 = -5$

Combining all the terms together, we get: $6x^2 + 3x - 10x - 5 = 6x^2 - 7x - 5$.

Now, let's work backwards and factor a trinomial to get two factors. Remember, you can always check your work by multiplying the final factors together.

Example B

Factor $6x^2 - x - 2$.

Solution: This is a factorable trinomial. When there is a coefficient, or number in front of x^2, you must follow all the steps from the investigation in the previous concept; no shortcuts. Also, m and n no longer have a product of c and a sum of b. This would not take the coefficient of x^2 into account. What we need to do is multiply together a and c (from $ax^2 + bx + c$) and then find the two numbers whose product is ac and sum is b. Let's use the X to help us organize this.

Now, we can see, we need the two factors of -12 that also add up to -1.

TABLE 2.8:

Factors	Sum
$-1, 12$	11
1, -12	-11
2, -6	-4
-2, 6	4
$3, -4$	-1
-3, 4	1

The factors that work are 3 and -4. Now, take these factors and rewrite the $x-$term expanded using 3 and -4 (Step 3 from the investigation in the previous concept).

$$6x^2 - x - 2$$

$$6x^2 - 4x + 3x - 2$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$(6x^2 - 4x) + (3x - 2)$$
$$2x(3x - 2) + 1(3x - 2)$$

Just like in the investigation, what is in the parenthesis is *the same*. We now have two terms that both have $(3x-2)$ as factor. Pull this factor out.

$$2x(3x-2)+1(3x-2)$$
$$(3x-2)(2x+1)$$

The factors of $6x^2-x-2$ are $(3x-2)(2x+1)$. You can FOIL these to check your answer.

Example C

Factor $4x^2+8x-5$.

Solution: Let's make the steps from Example B a little more concise.

1. Find ac and the factors of this number that add up to b.

$4\cdot -5=-20$ The factors of -20 that add up to 8 are 10 and -2.

2. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 1.

$$4x^2+8x-5$$
$$4x^2+10x-2x-5$$

3. Group the first two and second two terms together, find the GCF and factor again.

$$(4x^2+10x)+(-2x-5)$$
$$2x(2x+5)-1(2x+5)$$
$$(2x+5)(2x-1)$$

Alternate Method: What happens if we list $-2x$ before $10x$ in Step 2?

$$4x^2-2x+10x-5$$
$$(4x^2-2x)(10x-5)$$
$$2x(2x-1)+5(2x-1)$$
$$(2x-1)(2x+5)$$

This tells us it does not matter which $x-$term we list first in Step 2 above.

Example D

Factor $12x^2-22x-20$.

Solution: Let's use the steps from Example C, but we are going to add an additional step at the beginning.

1. Look for any common factors. Pull out the GCF of all three terms, if there is one.

$$12x^2-22x-20=2(6x^2-11x-10)$$

This will make it much easier for you to factor what is inside the parenthesis.

2. Using what is inside the parenthesis, find ac and determine the factors that add up to b.

$$6 \cdot -10 = -60 \rightarrow -15 \cdot 4 = -60, \ -15+4 = -11$$

The factors of -60 that add up to -11 are -15 and 4.

3. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 2.

$$\begin{aligned}&2(6x^2 - 11x - 10)\\&2(6x^2 - 15x + 4x - 10)\end{aligned}$$

4. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&2(6x^2 - 15x + 4x - 10)\\&2\left[(6x^2 - 15x) + (4x - 10)\right]\\&2\left[3x(2x-5) + 2(2x-5)\right]\\&2(2x-5)(3x+2)\end{aligned}$$

Guided Practice

1. Multiply $(4x-3)(3x+5)$.

Factor the following quadratics, if possible.

2. $15x^2 - 4x - 3$

3. $3x^2 + 6x - 12$

4. $24x^2 - 30x - 9$

5. $4x^2 + 4x - 48$

Answers

1. FOIL: $(4x-3)(3x+5) = 12x^2 + 20x - 9x - 15 = 12x^2 + 11x - 15$

2. Use the steps from the examples above. There is no GCF, so we can find the factors of ac that add up to b.

$15 \cdot -3 = -45$ The factors of -45 that add up to -4 are -9 and 5.

$$\begin{aligned}&15x^2 - 4x - 3\\&(15x^2 - 9x) + (5x - 3)\\&3x(5x-3) + 1(5x-3)\\&(5x-3)(3x+1)\end{aligned}$$

3. $3x^2 + 6x - 12$ has a GCF of 3. Pulling this out, we have $3(x^2 + 2x - 6)$. There is no number in front of x^2, so we see if there are any factors of -6 that add up to 2. There are not, so this trinomial is not factorable.

4. $24x^2 - 30x - 9$ also has a GCF of 3. Pulling this out, we have $3(8x^2 - 10x - 3)$. $ac = -24$. The factors of -24 than add up to -10 are -12 and 2.

$$\begin{aligned}
&3(8x^2 - 10x - 3)\\
&3\left[(8x^2 - 12x) + (2x - 3)\right]\\
&3\left[4x(2x-3) + 1(2x-3)\right]\\
&3(2x-3)(4x+1)
\end{aligned}$$

5. $4x^2 + 4x - 48$ has a GCF of 4. Pulling this out, we have $4(x^2 + x - 12)$. This trinomial does not have a number in front of x^2, so we can use the shortcut from the previous concept. What are the factors of -12 that add up to 1?

$$\begin{aligned}
&4(x^2 + x - 12)\\
&4(x+4)(x-3)
\end{aligned}$$

Problem Set

Multiply the following expressions.

1. $(2x-1)(x+5)$
2. $(3x+2)(2x-3)$
3. $(4x+1)(4x-1)$

Factor the following quadratic equations, if possible. If they cannot be factored, write *not factorable*. Don't forget to look for any GCFs first.

4. $5x^2 + 18x + 9$
5. $6x^2 - 21x$
6. $10x^2 - x - 3$
7. $3x^2 + 2x - 8$
8. $4x^2 + 8x + 3$
9. $12x^2 - 12x - 18$
10. $16x^2 - 6x - 1$
11. $5x^2 - 35x + 60$
12. $2x^2 + 7x + 3$
13. $3x^2 + 3x + 27$
14. $8x^2 - 14x - 4$
15. $10x^2 + 27x - 9$
16. $4x^2 + 12x + 9$
17. $15x^2 + 35x$
18. $6x^2 - 19x + 15$
19. Factor $x^2 - 25$. What is b?
20. Factor $9x^2 - 16$. What is b? What types of numbers are a and c?

Factoring Special Quadratics

Objective

To factor perfect square trinomials and the difference of squares.

Watch This

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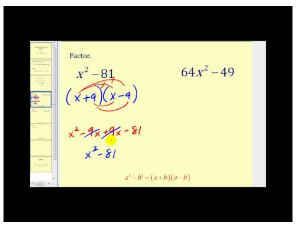
Khan Academy: U09_L2_TI_we1 Factoring Special Products 1

Then, watch the first part of this video, until about 3:10



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James Sousa: Factoring a Difference of Squares

Guidance

There are a couple of special quadratics that, when factored, have a pattern.

Investigation: Multiplying $(a+b)^2$

1. Rewrite $(a+b)^2$ as the product of two factors. Expand $(a+b)^2$. $(a+b)^2 = (a+b)(a+b)$
2. FOIL your answer from Step 1. This is a **perfect square trinomial.** $a^2+2ab+b^2$
3. $(a-b)^2$ also produces a perfect square trinomial. $(a-b)^2 = a^2-2ab+b^2$
4. Apply the formula above to factoring $9x^2-12x+4$. First, find a and b.

$$\begin{aligned} a^2 &= 9x^2, \; b^2 = 4 \\ a &= 3x, \quad b = 2 \end{aligned}$$

5. Now, plug a and b into the appropriate formula.

$$\begin{aligned} (3x-2)^2 &= (3x)^2 - 2(3x)(2) + 2^2 \\ &= 9x^2 - 12x + 4 \end{aligned}$$

Investigation: Multiplying $(a-b)(a+b)$

1. FOIL $(a-b)(a+b)$.

$$\begin{aligned} (a-b)(a+b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

2. This is a **difference of squares.** The difference of squares will always factor to be $(a+b)(a-b)$.

3. Apply the formula above to factoring $25x^2-16$. First, find a and b.

$$\begin{aligned} a^2 &= 25x^2,\ b^2 = 16 \\ a &= 5x, \qquad b = 4 \end{aligned}$$

4. Now, plug a and b into the appropriate formula. $(5x-4)(5x+4) = (5x)^2-4^2$

**It is important to note that if you forget these formulas or do not want to use them, you can still factor all of these quadratics the same way you did in the previous two concepts.

Example A

Factor x^2-81.

Solution: Using the formula from the investigation above, we need to first find the values of a and b.

$$\begin{aligned} x^2-81 &= a^2-b^2 \\ a^2 &= x^2,\ b^2 = 81 \\ a &= x, \qquad b = 9 \end{aligned}$$

Now, plugging x and 9 into the formula, we have $x^2-81 = (x-9)(x+9)$. To solve for a and b, we found the **square root** of each number. Recall that the square root is a number that, when multiplied by itself, produces another number. This other number is called a **perfect square**.

Alternate Method

Rewrite x^2-81 so that the middle term is present. $x^2+0x-81$

Using the method from the previous two concepts, what are the two factors of -81 that add up to 0? 9 and -9

Therefore, the factors are $(x-9)(x+9)$.

Example B

Factor $36x^2+120x+100$.

Solution: First, check for a GCF.

$$4(9x^2+30x+25)$$

Now, double-check that the quadratic equation above fits into the perfect square trinomial formula.

$$\begin{array}{ccc} a^2 = 9x^2 & b^2 = 25 & \\ \sqrt{a^2} = \sqrt{9x^2} & \sqrt{b^2} = \sqrt{25} & 2ab = 30x \\ a = 3x & b = 5 & 2(3x)(5) = 30x \end{array}$$

Using a and b above, the equation factors to be $4(3x+5)^2$. If you did not factor out the 4 in the beginning, the formula will still work. a would equal $6x$ and b would equal 10, so the factors would be $(6x+10)^2$. If you expand and find the GCF, you would have $(6x+10)^2 = (6x+10)(6x+10) = 2(3x+5)2(3x+5) = 4(3x+5)^2$.

Alternate Method

First, find the GCF. $4(9x^2+30x+25)$

Then, find ac and expand b accordingly. $9\cdot 25=225$, the factors of 225 that add up to 30 are 15 and 15.

$$\begin{aligned}
&4(9x^2+30x+25)\\
&4(9x^2+15x+15x+25)\\
&4\left[(9x^2+15x)+(15x+25)\right]\\
&4\left[3x(3x+5)+5(3x+5)\right]\\
&4(3x+5)(3x+5)\ or\ 4(3x^2+5)
\end{aligned}$$

Again, notice that if you do not use the formula discovered in this concept, you can still factor and get the correct answer.

Example C

Factor $48x^2-147$.

Solution: At first glance, this does not look like a difference of squares. 48 nor 147 are square numbers. But, if we take a 3 out of both, we have $3(16x^2-49)$. 16 and 49 are both square numbers, so now we can use the formula.

$$\begin{aligned}
16x^2&=a^2 \qquad & 49&=b^2\\
4x&=a & 7&=b
\end{aligned}$$

The factors are $(4x-7)(4x+7)$.

Guided Practice

Factor the following quadratic equations.

1. x^2-4

2. $2x^2-20x+50$

3. $81x^2+144+64$

Answers

1. $a=x$ and $b=2$. Therefore, $x^2-4=(x-2)(x+2)$.

2. Factor out the GCF, 2. $2(x^2-10x+25)$. This is now a perfect square trinomial with $a=x$ and $b=5$.

$$2(x^2-10x+25)=2(x-5)^2.$$

3. This is a perfect square trinomial and no common factors. Solve for a and b.

$$\begin{aligned}
81x^2&=a^2 \qquad & 64&=b^2\\
9x&=a & 8&=b
\end{aligned}$$

The factors are $(9x+8)^2$.

Vocabulary

Perfect Square Trinomial
A quadratic equation in the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$.

Difference of Squares
A quadratic equation in the form a^2-b^2.

Square Root
A number, that when multiplied by itself produces another number. 3 is the square root of 9.

Perfect Square
A number that has a square root that is an integer. 25 is a perfect square.

Problem Set

1. List the perfect squares that are less than 200.
2. Why do you think there is no *sum of squares* formula?

Factor the following quadratics, if possible.

3. x^2-1
4. x^2+4x+4
5. $16x^2-24x+9$
6. $-3x^2+36x-108$
7. $144x^2-49$
8. $196x^2+140x+25$
9. $100x^2+1$
10. $162x^2+72x+8$
11. $225-x^2$
12. $121-132x+36x^2$
13. $5x^2+100x-500$
14. $256x^2-676$
15. **Error Analysis** Spencer is given the following problem: Multiply $(2x-5)^2$. Here is his work:

$$(2x-5)^2=(2x)^2-5^2=4x^2-25$$

His teacher tells him the answer is $4x^2-20x+25$. What did Spencer do wrong? Describe his error and correct the problem.

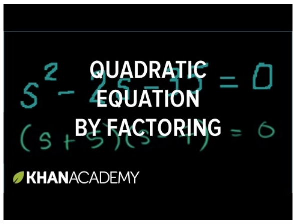
Solving Quadratics by Factoring

Objective

To solve factorable quadratic equations for x.

Watch This

Watch the first part of this video, until about 4:40.



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Khan Academy: Solving Quadratic Equations by Factoring.avi

Guidance

In this lesson we have not actually solved for x. Now, we will apply factoring to solving a quadratic equation. It adds one additional step to the end of what you have already been doing. Let's go through an example.

Example A

Solve $x^2 - 9x + 18 = 0$ by factoring.

Solution: The only difference between this problem and previous ones from the concepts before is the addition of the $=$ sign. Now that this is present, we need to solve for x. We can still factor the way we always have. Because $a = 1$, determine the two factors of 18 that add up to -9.

$$\begin{aligned} x^2 - 9x + 18 &= 0 \\ (x-6)(x-3) &= 0 \end{aligned}$$

Now, we have two factors that, when multiplied, equal zero. Recall that when two numbers are multiplied together and one of them is zero, the product is always zero.

Zero-Product Property: If $ab = 0$, then $a = 0$ or $b = 0$.

This means that $x - 6 = 0$ OR $x - 3 = 0$. Therefore, $x = 6$ or $x = 3$. There will always be the same number of solutions as factors.

Check your answer:

$$\begin{aligned} 6^2 - 9(6) + 18 = 0 \quad &or \quad 3^2 - 9(3) + 18 = 0 \\ 36 - 54 + 18 = 0 \quad & \quad\quad 9 - 27 + 18 = 0 \boxed{\checkmark} \end{aligned}$$

Example B

Solve $6x^2 + x - 4 = 11$ by factoring.

Solution: At first glance, this might not look factorable to you. However, before we factor, we must combine like terms. Also, the Zero-Product Property tells us that in order to solve for the factors, one side of the equation must be zero.

$$\begin{aligned} 6x^2 + x - 4 &= \cancel{11} \\ \underline{\quad\quad -11} &\underline{= -\cancel{11}} \\ 6x^2 + x - 15 &= 0 \end{aligned}$$

Now, factor. The product of ac is -90. What are the two factors of -90 that add up to 1? 10 and -9. Expand the $x-$term and factor.

$$\begin{aligned}6x^2+x-15&=0\\6x^2-9x+10x-15&=0\\3x(2x-3)+5(2x-3)&=0\\(2x-3)(3x+5)&=0\end{aligned}$$

Lastly, set each factor equal to zero and solve.

$$\begin{aligned}2x-3&=0 & 3x+5&=0\\2x&=3 \quad \mathit{or} & 3x&=-5\\x&=\frac{3}{2} & x&=-\frac{5}{3}\end{aligned}$$

Check your work:

$$\begin{aligned}6\left(\frac{3}{2}\right)^2+\frac{3}{2}-4&=11 & 6\left(-\frac{5}{3}\right)^2-\frac{5}{3}-4&=11\\6\cdot\frac{9}{4}+\frac{3}{2}-4&=11 \quad \mathit{or} & 6\cdot\frac{25}{9}-\frac{5}{3}-4&=11\ \boxed{\checkmark}\\\frac{27}{2}+\frac{3}{2}-4&=11 & \frac{50}{3}-\frac{5}{3}-4&=11\\15-4&=11 & 15-4&=11\end{aligned}$$

Example C

Solve $10x^2-25x=0$ by factoring.

Solution: Here is an example of a quadratic equation without a constant term. The only thing we can do is take out the GCF.

$$\begin{aligned}10x^2-25x&=0\\5x(2x-5)&=0\end{aligned}$$

Set the two factors equal to zero and solve.

$$\begin{aligned}5x&=0 & 2x-5&=0\\x&=0 \quad \mathit{or} & 2x&=5\\ & & x&=\frac{5}{2}\end{aligned}$$

Check:

$$\begin{aligned}10(0)^2-25(0)&=0 & 10\left(\frac{5}{2}\right)^2-25\left(\frac{5}{2}\right)&=0\\0&=0 \quad \mathit{or} & 10\cdot\frac{25}{4}-\frac{125}{2}&=0\ \boxed{\checkmark}\\ & & \frac{125}{2}-\frac{125}{2}&=0\end{aligned}$$

Guided Practice

Solve the following equations by factoring.

1. $4x^2 - 12x + 9 = 0$

2. $x^2 - 5x = 6$

3. $8x - 20x^2 = 0$

4. $12x^2 + 13x + 7 = 12 - 4x$

Answers

1. $ac = 36$. The factors of 36 that also add up to -12 are -6 and -6. Expand the $x-$term and factor.

$$\begin{aligned}4x^2 - 12x + 9 &= 0\\4x^2 - 6x - 6x + 9 &= 0\\2x(2x-3) - 3(2x-3) &= 0\\(2x-3)(2x-3) &= 0\end{aligned}$$

The factors are the same. When factoring a perfect square trinomial, the factors will always be the same. In this instance, the solutions for x will also be the same. Solve for x.

$$\begin{aligned}2x - 3 &= 0\\2x &= 3\\x &= \frac{3}{2}\end{aligned}$$

When the two factors are the same, we call the solution for x a **double root** because it is the solution twice.

2. Here, we need to get everything on the same side of the equals sign in order to factor.

$$\begin{aligned}x^2 - 5x &= 6\\x^2 - 5x - 6 &= 0\end{aligned}$$

Because there is no number in front of x^2, we need to find the factors of -6 that add up to -5.

$$(x-6)(x+1) = 0$$

Solving each factor for x, we get that $x = 6$ or $x = -1$.

3. Here there is no constant term. Find the GCF to factor.

$$\begin{aligned}8x - 20x^2 &= 0\\4x(2-5x) &= 0\end{aligned}$$

Solve each factor for x.

$$\begin{aligned} 4x &= 0 & 2-5x &= 0 \\ x &= 0 \quad or & 2 &= 5x \\ & & \frac{2}{5} &= x \end{aligned}$$

4. This problem is slightly more complicated than #2. Combine all like terms onto the same side of the equals sign so that one side is zero.

$$\begin{aligned} 12x^2+13x+7 &= 12-4x \\ 12x^2+17x-5 &= 0 \end{aligned}$$

$ac = -60$. The factors of -60 that add up to 17 are 20 and -3. Expand the $x-$term and factor.

$$\begin{aligned} 12x^2+17x-5 &= 0 \\ 12x^2+20x-3x-5 &= 0 \\ 4x(3x+5)-1(3x+5) &= 0 \\ (3x+5)(4x-1) &= 0 \end{aligned}$$

Solve each factor for x.

$$\begin{aligned} 3x+5 &= 0 & 4x-1 &= 0 \\ 3x &= -5 \quad or & 4x &= 1 \\ x &= -\frac{5}{3} & x &= \frac{1}{4} \end{aligned}$$

Vocabulary

Solution
The answer to an equation. With quadratic equations, solutions can also be called *zeros* or *roots*.

Double Root
A solution that is repeated twice.

Problem Set

Solve the following quadratic equations by factoring, if possible.

1. $x^2+8x-9=0$
2. $x^2+6x=0$
3. $2x^2-5x=12$
4. $12x^2+7x-10=0$
5. $x^2=9$
6. $30x+25=-9x^2$
7. $2x^2+x-5=0$

8. $16x = 32x^2$
9. $3x^2 + 28x = -32$
10. $36x^2 - 48 = 1$
11. $6x^2 + x = 4$
12. $5x^2 + 12x + 4 = 0$

Challenge Solve these quadratic equations by factoring. They are all factorable.

13. $8x^2 + 8x - 5 = 10 - 6x$
14. $-18x^2 = 48x + 14$
15. $36x^2 - 24 = 96x - 39$
16. **Real Life Application** George is helping his dad build a fence for the backyard. The total area of their backyard is 1600 square feet. The width of the house is half the length of the yard, plus 7 feet. How much fencing does George's dad need to buy?

2.23 Solving Quadratics by Using Square Roots

Objective

Reviewing simplifying square roots and to solve a quadratic equation by using square roots.

Review Queue

1. What is $\sqrt{64}$? Can there be more than one answer?
2. What two numbers should $\sqrt{18}$ be between? How do you know?
3. Find $\sqrt{18}$ on your calculator.
4. Solve $x^2 - 25 = 0$ by factoring.

Simplifying Square Roots

Objective

Simplifying, adding, subtracting and multiplying square roots.

Guidance

Before we can solve a quadratic equation using square roots, we need to review how to simplify, add, subtract, and multiply them. Recall that the **square root** is a number that, when multiplied by itself, produces another number. 4 is the square root of 16, for example. -4 is also the square root of 16 because $(-4)^2 = 16$. The symbol for square root is the **radical** sign, or $\sqrt{\ }$. The number under the radical is called the **radicand**.

If the square root of an integer is not another integer, it is an irrational number.

Example A

Find $\sqrt{50}$ using:

a) A calculator.

b) By simplifying the square root.

Solution:

a) To plug the square root into your graphing calculator, typically there is a $\sqrt{\ }$ or SQRT button. Depending on your model, you may have to enter 50 before or after the square root button. Either way, your answer should be $\sqrt{50} = 7.071067811865\ldots$ In general, we will round to the hundredths place, so 7.07 is sufficient.

b) To simplify the square root, the square numbers must be "pulled out." Look for factors of 50 that are square numbers: 4, 9, 16, 25... 25 is a factor of 50, so break the factors apart.

$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. *This is the most accurate answer*.

<u>**Radical Rules**</u>

1. $\sqrt{ab} = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ Any two radicals can be multiplied together.
2. $x\sqrt{a} \pm y\sqrt{a} = x \pm y\sqrt{a}$ The radicands must be the same in order to add or subtract.
3. $\left(\sqrt{a}\right)^2 = \sqrt{a}^2 = a$ The square and square root cancel each other out.

Example B

Simplify $\sqrt{45}+\sqrt{80}-2\sqrt{5}$.

Solution: At first glance, it does not look like we can simplify this. But, we can simplify each radical by pulling out the perfect squares.

$$\sqrt{45}=\sqrt{9\cdot 5}=3\sqrt{5}$$
$$\sqrt{80}=\sqrt{16\cdot 5}=4\sqrt{5}$$

Rewriting our expression, we have: $3\sqrt{5}+4\sqrt{5}-2\sqrt{5}$ and all the radicands are the same. Using the Order of Operations, our answer is $5\sqrt{5}$.

Example C

Simplify $2\sqrt{35}\cdot 4\sqrt{7}$.

Solution: Multiply across.

$$2\sqrt{35}\cdot 4\sqrt{7}=2\cdot 4\sqrt{35\cdot 7}=8\sqrt{245}$$

Now, simplify the radical. $8\sqrt{245}=8\sqrt{49\cdot 5}=8\cdot 7\sqrt{5}=56\sqrt{5}$

Guided Practice

Simplify the following radicals.

1. $\sqrt{150}$
2. $2\sqrt{3}-\sqrt{6}+\sqrt{96}$
3. $\sqrt{8}\cdot\sqrt{20}$

Answers

1. Pull out all the square numbers.

$$\sqrt{150}=\sqrt{25\cdot 6}=5\sqrt{6}$$

Alternate Method: Write out the prime factorization of 150.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}$$

Now, pull out any number that has a pair. Write it *once* in front of the radical and multiply together what is left over under the radical.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}=5\sqrt{6}$$

2. Simplify $\sqrt{96}$ to see if anything can be combined. We will use the alternate method above.

$$\sqrt{96}=\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3}=2\cdot 2\sqrt{6}=4\sqrt{6}$$

Rewrite the expression: $2\sqrt{3}-\sqrt{6}+4\sqrt{6}=2\sqrt{3}+3\sqrt{6}$. This is fully simplified. $\sqrt{3}$ and $\sqrt{6}$ cannot be combined because they do not have the same value under the radical.

3. This problem can be done two different ways.

1^{st} Method: Multiply radicals, then simplify the answer.

$$\sqrt{8}\cdot\sqrt{20}=\sqrt{160}=\sqrt{16\cdot 10}=4\sqrt{10}$$

2^{nd} Method: Simplify radicals, then multiply.

$$\sqrt{8}\cdot\sqrt{20}=\left(\sqrt{4\cdot 2}\right)\cdot\left(\sqrt{4\cdot 5}\right)=2\sqrt{2}\cdot 2\sqrt{5}=2\cdot 2\sqrt{2\cdot 5}=4\sqrt{10}$$

Depending on the complexity of the problem, either method will work. Pick whichever method you prefer.

Vocabulary

Square Root
A number, that when multiplied by itself, produces another number.

Perfect Square
A number that has an integer for a square root.

Radical
The $\sqrt{}$, or square root, sign.

Radicand
The number under the radical.

Problem Set

Find the square root of each number by using the calculator. Round your answer to the nearest hundredth.

1. 56
2. 12
3. 92

Simplify the following radicals. If it cannot be simplified further, write *cannot be simplified*.

4. $\sqrt{18}$
5. $\sqrt{75}$
6. $\sqrt{605}$
7. $\sqrt{48}$
8. $\sqrt{50}\cdot\sqrt{2}$
9. $4\sqrt{3}\cdot\sqrt{21}$
10. $\sqrt{6}\cdot\sqrt{20}$
11. $\left(4\sqrt{5}\right)^2$
12. $\sqrt{24}\cdot\sqrt{27}$
13. $\sqrt{16}+2\sqrt{8}$

14. $\sqrt{28}+\sqrt{7}$
15. $-8\sqrt{3}-\sqrt{12}$
16. $\sqrt{72}-\sqrt{50}$
17. $\sqrt{6}+7\sqrt{6}-\sqrt{54}$
18. $8\sqrt{10}-\sqrt{90}+7\sqrt{5}$

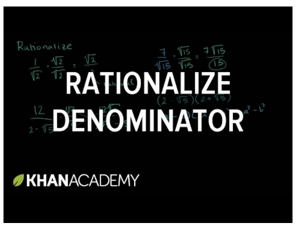
Dividing Square Roots

Objective

To divide radicals and rationalize the denominator.

Watch This

Watch the first part of this video, until about 3:15.



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Khan Academy: How to Rationalize a Denominator

Guidance

Dividing radicals can be a bit more difficult that the other operations. The main complication is that you cannot leave any radicals in the denominator of a fraction. For this reason we have to do something called **rationalizing the denominator**, where you multiply the top and bottom of a fraction by the same radical that is in the denominator. This will cancel out the radicals and leave a whole number.

Radical Rules

4. $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

5. $\frac{\sqrt{a}}{\sqrt{b}}\cdot\frac{\sqrt{b}}{\sqrt{b}}=\frac{\sqrt{ab}}{b}$

Example A

Simplify $\sqrt{\frac{1}{4}}$.

Solution: Break apart the radical by using Rule #4.

$$\sqrt{\frac{1}{4}}=\frac{\sqrt{1}}{\sqrt{4}}=\frac{1}{2}$$

Example B

Simplify $\frac{2}{\sqrt{3}}$.

Solution: This might look simplified, but radicals cannot be in the denominator of a fraction. This means we need to apply Rule #5 to get rid of the radical in the denominator, or rationalize the denominator. Multiply the top and bottom of the fraction by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example C

Simplify $\sqrt{\frac{32}{40}}$.

Solution: Reduce the fraction, and then apply the rules above.

$$\sqrt{\frac{32}{40}} = \sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Guided Practice

Simplify the following expressions using the Radical Rules learned in this concept and the previous concept.

1. $\sqrt{\frac{1}{2}}$

2. $\sqrt{\frac{64}{50}}$

3. $\frac{4\sqrt{3}}{\sqrt{6}}$

Answers

1. $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. $\sqrt{\frac{64}{50}} = \sqrt{\frac{32}{25}} = \frac{\sqrt{16 \cdot 2}}{5} = \frac{4\sqrt{2}}{5}$

3. The only thing we can do is rationalize the denominator by multiplying the numerator and denominator by $\sqrt{6}$ and then simplify the fraction.

$$\frac{4\sqrt{3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{18}}{6} = \frac{4\sqrt{9 \cdot 2}}{6} = \frac{12\sqrt{2}}{6} = 2\sqrt{2}$$

Vocabulary

Rationalize the denominator
The process used to get a radical out of the denominator of a fraction.

Problem Set

Simplify the following fractions.

1. $\sqrt{\frac{4}{25}}$

2. $\sqrt{\frac{96}{121}}$
3. $\frac{5\sqrt{2}}{\sqrt{10}}$
4. $\frac{6}{\sqrt{15}}$
5. $\sqrt{\frac{60}{35}}$
6. $8\frac{\sqrt{18}}{\sqrt{30}}$
7. $\frac{12}{\sqrt{6}}$
8. $\sqrt{\frac{208}{143}}$
9. $\frac{21\sqrt{3}}{2\sqrt{14}}$

Challenge Use all the Radical Rules you have learned in the last two concepts to simplify the expressions.

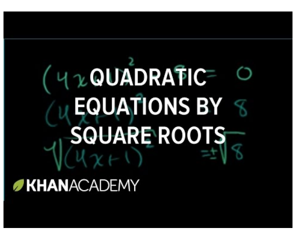
10. $\sqrt{\frac{8}{12}} \cdot \sqrt{15}$
11. $\sqrt{\frac{32}{45}} \cdot \frac{6\sqrt{20}}{\sqrt{5}}$
12. $\frac{\sqrt{24}}{\sqrt{2}} + \frac{8\sqrt{26}}{\sqrt{8}}$

Solving Quadratics Using Square Roots

Objective

To use the properties of square roots to solve certain types of quadratic equations.

Watch This



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Khan Academy: Solving Quadratics by Square Roots

Guidance

Now that you are familiar with square roots, we will use them to solve quadratic equations. Keep in mind, that square roots cannot be used to solve every type of quadratic. In order to solve a quadratic equation by using square roots, an $x-$term *cannot* be present. Solving a quadratic equation by using square roots is very similar to solving a linear equation. In the end, you must isolate the x^2 or whatever is being squared.

Example A

Solve $2x^2 - 3 = 15$.

Solution: Start by isolating the x^2.

$$\begin{aligned}2x^2-3&=15\\2x^2&=18\\x^2&=9\end{aligned}$$

At this point, you can take the square root of both sides.

$$\begin{aligned}\sqrt{x^2}&=\pm\sqrt{9}\\x&=\pm 3\end{aligned}$$

Notice that x has two solutions; 3 or -3. When taking the square root, always put the $\pm$ (plus or minus sign) in front of the square root. This indicates that the positive or negative answer will be the solution.

Check:

$$\begin{array}{ccc}2(3)^2-3=15 & & 2(-3)^2-3=15\\ 2\cdot 9-3=15 & or & 2\cdot 9-3=15\ \boxed{\checkmark}\\ 18-3=15 & & 18-3=15\end{array}$$

Example B

Solve $\frac{x^2}{16}+3=27$.

Solution: Isolate x^2 and then take the square root.

$$\begin{aligned}\frac{x^2}{16}+3&=27\\\frac{x^2}{16}&=24\\x^2&=384\\x&=\pm\sqrt{384}=\pm 8\sqrt{6}\end{aligned}$$

Example C

Solve $3(x-5)^2+7=43$.

Solution: In this example, x is not the only thing that is squared. Isolate the $(x-5)^2$ and then take the square root.

$$\begin{aligned}3(x-5)^2+7&=43\\3(x-5)^2&=36\\(x-5)^2&=12\\x-5&=\pm\sqrt{12}\ or\ \pm 2\sqrt{3}\end{aligned}$$

Now that the square root is gone, add 5 to both sides.

$$x-5=\pm 2\sqrt{3}$$
$$x=5\pm 2\sqrt{3}$$

$x=5+2\sqrt{3}$ or $5-2\sqrt{3}$. We can estimate these solutions as decimals; 8.46 or 1.54. Remember, that the most accurate answer includes the radical numbers.

Guided Practice

Solve the following quadratic equations.

1. $\frac{2}{3}x^2-14=38$

2. $11+x^2=4x^2+5$

3. $(2x+1)^2-6=19$

Answers

1. Isolate x^2 and take the square root.

$$\frac{2}{3}x^2-14=38$$
$$\frac{2}{3}x^2=52$$
$$x^2=78$$
$$x=\pm\sqrt{78}$$

2. Combine all like terms, then isolate x^2.

$$11+x^2=4x^2+5$$
$$-3x^2=-6$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

3. Isolate what is being squared, take the square root, and then isolate x.

$$(2x+1)^2-6=19$$
$$(2x+1)^2=25$$
$$2x+1=\pm 5$$
$$2x=-1\pm 5$$
$$x=\frac{-1\pm 5}{2}\rightarrow x=\frac{-1+5}{2}=2\ or\ x=\frac{-1-5}{2}=-3$$

Problem Set

Solve the following quadratic equations. Reduce answers as much as possible. No decimals.

1. $x^2=144$

2. $5x^2 - 4 = 16$
3. $8 - 10x^2 = -22$
4. $(x+2)^2 = 49$
5. $6(x-5)^2 + 1 = 19$
6. $\frac{3}{4}x^2 - 19 = 26$
7. $x^2 - 12 = 36 - 2x^2$
8. $9 - \frac{x^2}{3} = -33$
9. $-4(x+7)^2 = -52$
10. $2(3x+4)^2 - 5 = 45$
11. $\frac{1}{3}(x-10)^2 - 8 = 16$
12. $\frac{(x-1)^2}{6} - \frac{8}{3} = \frac{7}{2}$

Use either factoring or solving by square roots to solve the following quadratic equations.

13. $x^2 - 16x + 55 = 0$
14. $2x^2 - 9 = 27$
15. $6x^2 + 23x = -20$
16. **Writing** Write a set of hints that will help you remember when you should solve an equation by factoring and by square roots. Are there any quadratics that can be solved using either method?
17. Solve $x^2 - 9 = 0$ by factoring and by using square roots. Which do you think is easier? Why?
18. Solve $(3x-2)^2 + 1 = 17$ by using square roots. Then, solve $3x^2 - 4x - 4 = 0$ by factoring. What do you notice? What can you conclude?
19. **Real Life Application** The *aspect ratio* of a TV screen is the ratio of the screen's width to its height. For HDTVs, the aspect ratio is 16:9. What is the width and height of a 42 inch screen TV? (42 inches refers to the length of the screen's diagonal.) HINT: Use the Pythagorean Theorem. Round your answers to the nearest hundredth.
20. **Real Life Application** When an object is dropped, its speed continually increases until it reaches the ground. This scenario can be modeled by the equation $h = -16t^2 + h_0$, where h is the height, t is the time (in seconds), and $h_0$is the initial height of the object. Round your answers to the nearest hundredth.
    a. If you drop a ball from 200 feet, what is the height after 2 seconds?
    b. After how many seconds will the ball hit the ground?

2.24 Complex Numbers

Objective

To define and use complex and imaginary numbers. Then, solve quadratic equations with imaginary solutions.

Review Queue

1. Can you find $\sqrt{-25}$? Why or why not?
2. Simplify $\sqrt{192} \cdot \sqrt{27}$
3. Simplify $\sqrt{\frac{12}{15}}$
4. Solve $4(x-6)^2 - 7 = 61$

Defining Complex Numbers

Objective

To define, discover the "powers of i," and add and subtract complex and imaginary numbers.

Watch This

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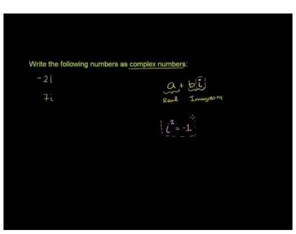
Khan Academy: Introduction to i and Imaginary Numbers

Then, watch this video.



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Khan Academy: Complex Numbers

Guidance

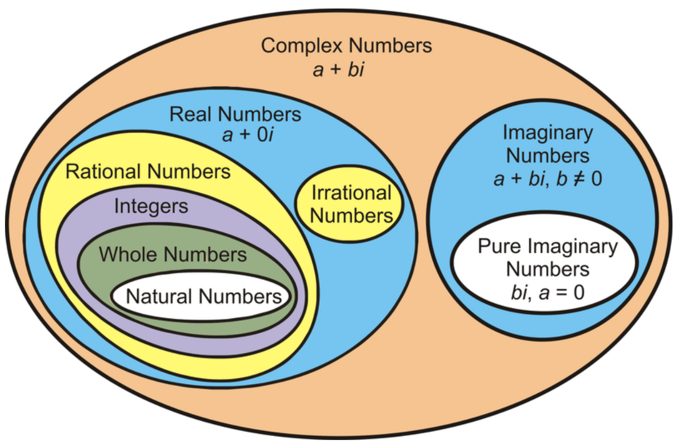
Before this concept, all numbers have been real numbers. 2, -5, $\sqrt{11}$, and $\frac{1}{3}$ are all examples of real numbers. Look at #1 from the Review Queue. With what we have previously learned, we cannot find $\sqrt{-25}$ because you cannot take the square root of a negative number. There is no real number that, when multiplied by itself, equals -25. Let's simplify $\sqrt{-25}$.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1}$$

In order to take the square root of a negative number we are going to assign $\sqrt{-1}$ a variable, i. i represents an **imaginary number**. Now, we can use i to take the square root of a negative number.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1} = 5i$$

All **complex numbers** have the form $a+bi$, where a and b are real numbers. a is the **real part** of the complex number and b is the **imaginary part**. If $b=0$, then a is left and the number is a **real number.** If $a=0$, then the number is only bi and called a **pure imaginary number**. If $b \neq 0$ and $a \neq 0$, the number will be an imaginary number.



Example A

Find $\sqrt{-162}$.

Solution: First pull out the i. Then, simplify $\sqrt{162}$.

$$\sqrt{-162} = \sqrt{-1} \cdot \sqrt{162} = i\sqrt{162} = i\sqrt{81 \cdot 2} = 9i\sqrt{2}$$

Investigation: Powers of i

In addition to now being able to take the square root of a negative number, i also has some interesting properties. Try to find i^2, i^3, and i^4.

1. Write out i^2 and simplify. $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1}^2 = -1$
2. Write out i^3 and simplify. $i^3 = i^2 \cdot i = -1 \cdot i = -i$
3. Write out i^4 and simplify. $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$
4. Write out i^5 and simplify. $i^5 = i^4 \cdot i = 1 \cdot i = i$
5. Write out i^6 and simplify. $i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$
6. Do you see a pattern? Describe it and try to find i^{19}.

You should see that the powers of i repeat every 4 powers. So, all the powers that are divisible by 4 will be equal to 1. To find i^{19}, divide 19 by 4 and determine the remainder. That will tell you what power it is the same as.

$$i^{19} = i^{16} \cdot i^3 = 1 \cdot i^3 = -i$$

Example B

Find:

a) i^{32}

b) i^{50}

c) i^7

Solution:

a) 32 is divisible by 4, so $i^{32} = 1$.

b) $50 \div 4 = 12$, with a remainder of 2. Therefore, $i^{50} = i^2 = -1$.

c) $7 \div 4 = 1$, with a remainder of 3. Therefore, $i^7 = i^3 = -i$

Example C

Simplify the complex expressions.

a) $(6-4i)+(5+8i)$

b) $9-(4+i)+(2-7i)$

Solution: To add or subtract complex numbers, you need to combine like terms. Be careful with negatives and properly distributing them. Your answer should always be in **standard form**, which is $a+bi$.

a) $(6-4i)+(5+8i) = 6-4i+5+8i = 11+4i$

b) $9-(4+i)+(2-7i) = 9-4-i+2-7i = 7-8i$

Guided Practice

Simplify.

1. $\sqrt{-49}$

2. $\sqrt{-125}$

3. i^{210}

4. $(8-3i)-(12-i)$

Answers

1. Rewrite $\sqrt{-49}$ in terms of i and simplify the radical.

$$\sqrt{-49} = i\sqrt{49} = 7i$$

2. Rewrite $\sqrt{-125}$ in terms of i and simplify the radical.

$$\sqrt{-125} = i\sqrt{125} = i\sqrt{25\cdot 5} = 5i\sqrt{5}$$

3. $210 \div 4 = 52$, with a remainder of 2. Therefore, $i^{210} = i^2 = -1$.

4. Distribute the negative and combine like terms.

$$(8-3i)-(12-i)=8-3i-12+i=-4-2i$$

Vocabulary

Imaginary Numbers
Any number with an i associated with it. Imaginary numbers have the form $a+bi$ or bi.

Complex Numbers
All real and imaginary numbers. Complex numbers have the **standard form** $a+bi$, where a or b can be zero. a is the **real part** and bi is the **imaginary part**.

Pure Imaginary Numbers
An imaginary number without a real part, only bi.

Problem Set

Simplify each expression and write in standard form.

1. $\sqrt{-9}$
2. $\sqrt{-242}$
3. $6\sqrt{-45}$
4. $-12i\sqrt{98}$
5. $\sqrt{-32}\cdot\sqrt{-27}$
6. $7i\sqrt{-126}$
7. i^8
8. $16i^{22}$
9. $-9i^{65}$
10. i^{365}
11. $2i^{91}$
12. $\sqrt{-\frac{16}{80}}$
13. $(11-5i)+(6-7i)$
14. $(14+2i)-(20+9i)$
15. $(8-i)-(3+4i)+15i$
16. $-10i-(1-4i)$
17. $(0.2+1.5i)-(-0.6+i)$
18. $6+(18-i)-(2+12i)$
19. $-i+(19+22i)-(8-14i)$
20. $18-(4+6i)+(17-9i)+24i$

Multiplying and Dividing Complex Numbers

Objective

To multiply and divide complex numbers.

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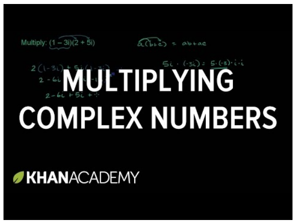

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Khan Academy: Multiplying Complex Numbers

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Khan Academy: Dividing Complex Numbers

Guidance

When multiplying complex numbers, FOIL the two numbers together (see *Factoring when* $a = 1$ concept) and then combine like terms. At the end, there will be an i^2 term. Recall that $i^2 = -1$ and continue to simplify.

Example A

Simplify:

a) $6i(1-4i)$

b) $(5-2i)(3+8i)$

Solution:

a) Distribute the $6i$ to both parts inside the parenthesis.

$$6i(1-4i) = 6i - 24i^2$$

Substitute $i^2 = -1$ and simplify further.

$$\begin{aligned} &= 6i - 24(-1) \\ &= 24 + 6i \end{aligned}$$

Remember to always put the real part first.

b) FOIL the two terms together.

$$\begin{aligned} (5-2i)(3+8i) &= 15 + 40i - 6i - 16i^2 \\ &= 15 + 34i - 16i^2 \end{aligned}$$

Substitute $i^2 = -1$ and simplify further.

$$
\begin{aligned}
&= 15 + 34i - 16(-1) \\
&= 15 + 34i + 16 \\
&= 31 + 34i
\end{aligned}
$$

More Guidance

Dividing complex numbers is a bit more complicated. Similar to irrational numbers, complex numbers cannot be in the denominator of a fraction. To get rid of the complex number in the denominator, we need to multiply by the **complex conjugate**. If a complex number has the form $a+bi$, then its complex conjugate is $a-bi$. For example, the complex conjugate of $-6+5i$ would be $-6-5i$. Therefore, rather than dividing complex numbers, we multiply by the complex conjugate.

Example B

Simplify $\frac{8-3i}{6i}$.

Solution: In the case of dividing by a pure imaginary number, you only need to multiply the top and bottom by that number. Then, use multiplication to simplify.

$$
\begin{aligned}
\frac{8-3i}{6i} \cdot \frac{6i}{6i} &= \frac{48i - 18i^2}{36i^2} \\
&= \frac{18 + 48i}{-36} \\
&= \frac{18}{-36} + \frac{48}{-36} i \\
&= -\frac{1}{2} - \frac{4}{3} i
\end{aligned}
$$

When the complex number contains fractions, write the number in standard form, keeping the real and imaginary parts separate. Reduce both fractions separately.

Example C

Simplify $\frac{3-5i}{2+9i}$.

Solution: Now we are dividing by $2+9i$, so we will need to multiply the top and bottom by the complex conjugate, $2-9i$.

$$
\begin{aligned}
\frac{3-5i}{2+9i} \cdot \frac{2-9i}{2-9i} &= \frac{6 - 27i - 10i + 45i^2}{4 - 18i + 18i - 81i^2} \\
&= \frac{6 - 37i - 45}{4 + 81} \\
&= \frac{-39 - 37i}{85} \\
&= -\frac{39}{85} - \frac{37}{85} i
\end{aligned}
$$

Notice, by multiplying by the complex conjugate, the denominator becomes a real number and you can split the fraction into its real and imaginary parts.

In both Examples B and C, substitute $i^2 = -1$ to simplify the fraction further. ***Your final answer should never have any power of*** i ***greater than 1.***

Guided Practice

1. What is the complex conjugate of $7-5i$?

Simplify the following complex expressions.

2. $(7-4i)(6+2i)$

3. $\frac{10-i}{5i}$

4. $\frac{8+i}{6-4i}$

Answers

1. $7+5i$

2. FOIL the two expressions.

$$\begin{aligned}(7-4i)(6+2i) &= 42+14i-24i-8i^2 \\ &= 42-10i+8 \\ &= 50-10i\end{aligned}$$

3. Multiply the numerator and denominator by $5i$.

$$\begin{aligned}\frac{10-i}{5i}\cdot\frac{5i}{5i} &= \frac{50i-5i^2}{25i^2} \\ &= \frac{5+50i}{-25} \\ &= \frac{5}{-25}+\frac{50}{-25}i \\ &= -\frac{1}{5}-2i\end{aligned}$$

4. Multiply the numerator and denominator by the complex conjugate, $6+4i$.

$$\begin{aligned}\frac{8+i}{6-4i}\cdot\frac{6+4i}{6+4i} &= \frac{48+32i+6i+4i^2}{36+24i-24i-16i^2} \\ &= \frac{48+38i-4}{36+16} \\ &= \frac{44+38i}{52} \\ &= \frac{44}{52}+\frac{38}{52}i \\ &= \frac{11}{13}+\frac{19}{26}i\end{aligned}$$

Vocabulary

Complex Conjugate

The "opposite" of a complex number. If a complex number has the form $a+bi$, its complex conjugate is $a-bi$. When multiplied, these two complex numbers will produce a real number.

Problem Set

Simplify the following expressions. Write your answers in standard form.

1. $i(2-7i)$
2. $8i(6+3i)$
3. $-2i(11-4i)$
4. $(9+i)(8-12i)$
5. $(4+5i)(3+16i)$
6. $(1-i)(2-4i)$
7. $4i(2-3i)(7+3i)$
8. $(8-5i)(8+5i)$
9. $\frac{4+9i}{3i}$
10. $\frac{6-i}{12i}$
11. $\frac{7+12i}{-5i}$
12. $\frac{4-2i}{6-6i}$
13. $\frac{2-i}{2+i}$
14. $\frac{10+8i}{2+4i}$
15. $\frac{14+9i}{7-20i}$

Solving Quadratic Equations with Complex Number Solutions

Objective

To apply what we have learned about complex numbers and solve quadratic equations with complex number solutions.

Guidance

When you solve a quadratic equation, there will always be two answers. Until now, we thought the answers were always real numbers. In actuality, there are quadratic equations that have imaginary solutions as well. The possible solutions for a quadratic are:

2 real solutions

$$x^2-4=0$$
$$x=-2,2$$

Double root

$$x^2+4x+4=0$$
$$x=-2,-2$$

2 imaginary solutions

$$x^2+4=0$$
$$x=-2i,2i$$

Example A

Solve $3x^2+27=0$.

Solution: First, factor out the GCF.

$$3(x^2+9)=0$$

Now, try to factor x^2+9. Rewrite the quadratic as x^2+0x+9 to help. There are no factors of 9 that add up to 0. Therefore, this is not a factorable quadratic. Let's solve it using square roots.

$$\begin{aligned}3x^2+27&=0\\3x^2&=-27\\x^2&=-9\\x&=\pm\sqrt{-9}=\pm 3i\end{aligned}$$

Quadratic equations with imaginary solutions are never factorable.

Example B

Solve $(x-8)^2=-25$

Solution: Solve using square roots.

$$\begin{aligned}(x-8)^2&=-25\\x-8&=\pm 5i\\x&=8\pm 5i\end{aligned}$$

Example C

Solve $2(3x-5)+10=-30$.

Solution: Solve using square roots.

$$\begin{aligned}2(3x-5)^2+10&=-30\\2(3x-5)^2&=-40\\(3x-5)^2&=-20\\3x-5&=\pm 2i\sqrt{5}\\3x&=5\pm 2i\sqrt{5}\\x&=\frac{5}{3}\pm\frac{2\sqrt{5}}{3}i\end{aligned}$$

Guided Practice

1. Solve $4(x-5)^2+49=0$.

2. Solve $-\frac{1}{2}(3x+8)^2-16=2$.

Answers

Both of these quadratic equations can be solved by using square roots.

1.

$$
\begin{aligned}
4(x-5)^2+49&=0\\
4(x-5)^2&=-49\\
(x-5)^2&=-\frac{49}{4}\\
x-5&=\pm\frac{7}{2}i\\
x&=5\pm\frac{7}{2}i
\end{aligned}
$$

2.

$$
\begin{aligned}
-\frac{1}{2}(3x+8)^2-16&=2\\
-\frac{1}{2}(3x+8)^2&=18\\
(3x+8)^2&=-36\\
3x+8&=\pm 6i\\
3x&=-8\pm 6i\\
x&=-\frac{8}{3}\pm 2i
\end{aligned}
$$

Problem Set

Solve the following quadratic equations.

1. $(x+1)^2=-121$
2. $5x^2+16=-29$
3. $14-4x^2=38$
4. $(x-9)^2-2=-82$
5. $-3(x+6)^2+1=37$
6. $4(x-5)^2-3=-59$
7. $(2x-1)^2+5=-23$
8. $-(6x+5)^2=72$
9. $7(4x-3)^2-15=-68$
10. If a quadratic equation has $4-i$ as a solution, what must the other solution be?
11. If a quadratic equation has $6+2i$ as a solution, what must the other solution be?
12. **Challenge** Recall that the factor of a quadratic equation has the form $(x\pm m)$ where m is any number. Find the quadratic equation that has the solution $3+2i$.

2.25 Completing the Square

Objective

We will introduce another technique to solve quadratic equations, called completing the square.

Review Queue

Solve the following equations. Use the appropriate method.

1. $x^2 - 18x + 32 = 0$
2. $2(x-4)^2 = -54$
3. $4x^2 - 5x - 6 = 0$
4. $x^2 - 162 = 0$

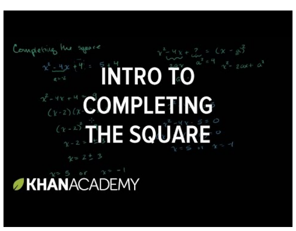
Completing the Square When the First Coefficient Equals 1

Objective

Learning how to complete the square for quadratic equations in the form $x^2 + bx + c = 0$.

Watch This

Watch the first part of this video, until about 5:25.



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Khan Academy: Solving Quadratic Equations by Completing the Square

Guidance

Completing the square is another technique used to solve quadratic equations. When completing the square, the goal is to make a perfect square trinomial and factor it.

Example A

Solve $x^2 - 8x - 1 = 10$.

Solution:

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right. This is only for organizational purposes, but it really helps. Leave a little space after the $x-$term.

$$x^2 - 8x = 11$$

2. Now, "complete the square." Determine what number would make a perfect square trinomial with x^2-8x+c. To do this, divide the $x-$term by 2 and square that number, or $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2=\left(\frac{8}{2}\right)^2=4^2=16$$

3. Add this number to *both sides* in order to keep the equation balanced.

$$x^2-8x+16=11+16$$

4. Factor the left side to the **square of a binomial** and simplify the right.

$$(x-4)^2=27$$

5. Solve by using square roots.

$$\begin{aligned} x-4&=\pm 3\sqrt{3}\\ x&=4\pm 3\sqrt{3}\end{aligned}$$

Completing the square enables you to solve any quadratic equation using square roots. Through this process, we can make an unfactorable quadratic equation solvable, like the one above. It can also be used with quadratic equations that have imaginary solutions.

Example B

Solve $x^2+12x+37=0$

Solution: First, this is not a factorable quadratic equation. Therefore, the only way we know to solve this equation is to complete the square. Follow the steps from Example A.

1. Organize the polynomial, x's on the left, constant on the right.

$$x^2+12x=-37$$

2. Find $\left(\frac{b}{2}\right)^2$ and add it to both sides.

$$\begin{aligned}\left(\frac{b}{2}\right)^2=\left(\frac{12}{2}\right)^2&=6^2=36\\ x^2+12x+36&=-37+36\end{aligned}$$

3. Factor the left side and solve.

$$\begin{aligned}(x+6)^2&=-1\\ x+6&=\pm i\\ x&=-6\pm i\end{aligned}$$

Example C

Solve $x^2-11x-15=0$.

Solution: This is not a factorable equation. Use completing the square.

1. Organize the polynomial, x's on the left, constant on the right.

$$x^2-11x=15$$

2. Find $\left(\frac{b}{2}\right)^2$ and add it to both sides.

$$\begin{aligned}\left(\frac{b}{2}\right)^2=\left(\frac{11}{2}\right)^2&=\frac{121}{4}\\x^2-11x+\frac{121}{4}&=15+\frac{121}{4}\end{aligned}$$

3. Factor the left side and solve.

$$\begin{aligned}\left(x-\frac{11}{2}\right)^2&=\frac{60}{4}+\frac{121}{4}\\\left(x-\frac{11}{2}\right)^2&=\frac{181}{4}\\x-\frac{11}{2}&=\pm\frac{\sqrt{181}}{2}\\x&=\frac{11}{2}\pm\frac{\sqrt{181}}{2}\end{aligned}$$

Guided Practice

1. Find the value of c that would make (x^2-2x+c) a perfect square trinomial. Then, factor the trinomial.

Solve the following quadratic equations by completing the square.

2. $x^2+10x+21=0$

3. $x-5x=12$

Answers

1. $c=\left(\frac{b}{2}\right)^2=\left(\frac{2}{2}\right)^2=1^2=1$. The factors of x^2-2x+1 are $(x-1)(x-1)$ or $(x-1)^2$.

2. Use the steps from the examples above.

$$
\begin{aligned}
x^2+10x+21&=0\\
x^2+10x&=-21\\
x^2+10x+\left(\frac{10}{2}\right)^2&=-21+\left(\frac{10}{2}\right)^2\\
x^2+10x+25&=-21+25\\
(x+5)^2&=4\\
x+5&=\pm 2\\
x&=-5\pm 2\\
x&=-7,-3
\end{aligned}
$$

3. Use the steps from the examples above.

$$
\begin{aligned}
x^2-5x&=12\\
x^2-5x+\left(\frac{5}{2}\right)^2&=12+\left(\frac{5}{2}\right)^2\\
x^2-5x+\frac{25}{4}&=\frac{48}{4}+\frac{25}{4}\\
\left(x-\frac{5}{2}\right)^2&=\frac{73}{4}\\
x-\frac{5}{2}&=\pm\frac{\sqrt{73}}{2}\\
x&=\frac{5}{2}\pm\frac{\sqrt{73}}{2}
\end{aligned}
$$

Vocabulary

Binomial
A mathematical expression with two terms.

Square of a Binomial
A binomial that is squared.

Complete the Square
The process used to solve unfactorable quadratic equations.

Problem Set

Determine the value of c that would complete the perfect square trinomial.

1. x^2+4x+c
2. x^2-2x+c
3. $x^2+16x+c$

Rewrite the perfect square trinomial as a square of a binomial.

4. x^2+6x+9
5. $x^2-7x+\frac{49}{4}$
6. $x^2-\frac{1}{2}x+\frac{1}{16}$

Solve the following quadratic equations by completing the square.

7. $x^2+6x-15=0$
8. $x^2+10x+29=0$
9. $x^2-14x+9=-60$
10. $x^2+3x+18=-2$
11. $x^2-9x-5=23$
12. $x^2-20x=60$

Solve the following quadratic equations by factoring, square roots, or completing the square.

13. $x^2+x-30=0$
14. $x^2-18x+90=0$
15. $x^2+15x+56=0$
16. $x^2+3x-24=12$
17. $(x-2)^2-20=-45$
18. $x^2+24x+44=-19$
19. Solve $x^2+7x-44=0$ by factoring and completing the square. Which method do you prefer?
20. **Challenge** Solve $x^2+\frac{17}{8}x-2=-9$.

Completing the Square When the First Coefficient Doesn't Equal 1

Objective

Learning how to complete the square for quadratic equations in the form $ax^2+bx+c=0$.

Guidance

When there is a number in front of x^2, it will make completing the square a little more complicated. See how the steps change in Example A.

Example A

Solve $3x^2-9x+11=0$

Solution:

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right.

$$3x^2-9x=-11$$

2. Pull out a from everything on the left side. Even if b is not divisible by a, the coefficient of x^2 needs to be 1 in order to complete the square.

$$3(x^2-3x+\_\_\_)=-11$$

3. Now, complete the square. Determine what number would make a perfect square trinomial.

To do this, divide the $x-$term by 2 and square that number, or $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

4. Add this number to the interior of the parenthesis on the left side. On the right side, you will need to add $a \cdot \left(\frac{b}{2}\right)^2$ to keep the equation balanced.

$$3\left(x^2 - 3x + \frac{9}{4}\right) = -11 + \frac{27}{4}$$

5. Factor the left side and simplify the right.

$$3\left(x - \frac{3}{2}\right)^2 = -\frac{17}{4}$$

6. Solve by using square roots.

$$\begin{aligned}\left(x - \frac{3}{2}\right)^2 &= -\frac{17}{12}\\ x - \frac{3}{2} &= \pm\frac{i\sqrt{17}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\\ x &= \frac{3}{2} \pm \frac{\sqrt{51}}{6}i\end{aligned}$$

Be careful with the addition of step 2 and the changes made to step 4. A very common mistake is to add $\left(\frac{b}{2}\right)^2$ to both sides, without multiplying by a for the right side.

Example B

Solve $4x^2 + 7x - 18 = 0$.

Solution: Let's follow the steps from Example A.

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right.

$$4x^2 - 7x = 18$$

2. Pull out a from everything on the left side.

$$4\left(x^2 + \frac{7}{4}x + \_\_\right) = 18$$

3. Now, complete the square. Find $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{8}\right)^2 = \frac{49}{64}$$

4. Add this number to the interior of the parenthesis on the left side. On the right side, you will need to add $a \cdot \left(\frac{b}{2}\right)^2$ to keep the equation balanced.

$$4\left(x^2 + \frac{7}{4}x + \frac{49}{64}\right) = 18 + \frac{49}{16}$$

5. Factor the left side and simplify the right.

$$4\left(x + \frac{7}{8}\right)^2 = \frac{337}{16}$$

6. Solve by using square roots.

$$\left(x + \frac{7}{8}\right)^2 = \frac{337}{64}$$
$$x + \frac{7}{8} = \pm\frac{\sqrt{337}}{8}$$
$$x = -\frac{7}{8} \pm \frac{\sqrt{337}}{8}$$

Guided Practice

Solve the following quadratic equations by completing the square.

1. $5x^2 + 29x - 6 = 0$

2. $8x^2 - 32x + 4 = 0$

Answers

Use the steps from the examples above to solve for x.

1.

$$\begin{aligned}
5x^2+29x-6&=0\\
5\left(x^2+\frac{29}{5}x\right)&=6\\
5\left(x^2+\frac{29}{5}x+\frac{841}{100}\right)&=6+\frac{841}{20}\\
5\left(x+\frac{29}{10}\right)^2&=\frac{961}{20}\\
\left(x+\frac{29}{10}\right)^2&=\frac{961}{100}\\
x+\frac{29}{10}&=\pm\frac{31}{10}\\
x&=-\frac{29}{10}\pm\frac{31}{10}\\
x&=-6,\frac{1}{5}
\end{aligned}$$

2.

$$\begin{aligned}
8x^2-32x+4&=0\\
8(x^2-4x)&=-4\\
8(x^2-4x+4)&=-4+32\\
8(x-2)^2&=28\\
(x-2)^2&=\frac{7}{2}\\
x-2&=\pm\frac{\sqrt{7}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\
x&=2\pm\frac{\sqrt{14}}{2}
\end{aligned}$$

Problem Set

Solve the quadratic equations by completing the square.

1. $6x^2-12x-7=0$
2. $-4x^2+24x-100=0$
3. $5x^2-30x+55=0$
4. $2x^2-x-6=0$
5. $\frac{1}{2}x^2+7x+8=0$
6. $-3x^2+4x+15=0$

Solve the following equations by factoring, using square roots, or completing the square.

7. $4x^2-4x-8=0$
8. $2x^2+9x+7=0$
9. $-5(x+4)^2-19=26$
10. $3x^2+30x-5=0$

11. $9x^2 - 15x - 6 = 0$
12. $10x^2 + 40x + 88 = 0$

Problems 13-15 build off of each other.

13. **Challenge** Complete the square for $ax^2 + bx + c = 0$. Follow the steps from Examples A and B. Your final answer should be in terms of a, b, and c.
14. For the equation $8x^2 + 6x - 5 = 0$, use the formula you found in #13 to solve for x.
15. Is the equation in #14 factorable? If so, factor and solve it.
16. **Error Analysis** Examine the worked out problem below.

$$\begin{aligned}
4x^2 - 48x + 11 &= 0 \\
4(x^2 - 12x + ___) &= -11 \\
4(x^2 - 12x + 36) &= -11 + 36 \\
4(x-6)^2 &= 25 \\
(x-6)^2 &= \frac{25}{4} \\
x - 6 &= \pm\frac{5}{2} \\
x &= 6 \pm \frac{5}{2} \rightarrow \frac{17}{2}, \frac{7}{2}
\end{aligned}$$

Plug the answers into the original equation to see if they work. If not, find the error and correct it.

2.26 The Quadratic Formula

Objective

To derive and use the Quadratic Formula to solve quadratic equations and determine how many solutions an equation has.

Review Queue

Solve each equation by completing the square.

1. $x^2 - 4x + 20 = 0$

2. $4x^2 - 12x - 33 = 0$

Solve each equation by factoring.

3. $12x^2 + 31x + 20 = 0$

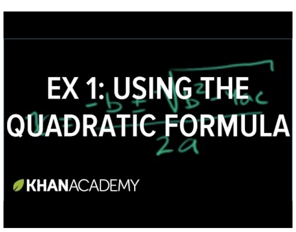
4. $5x^2 - 30x - 23 = x^2 - 77$

Deriving and Using the Quadratic Formula

Objective

Deriving the Quadratic Formula and using it to solve any quadratic equation.

Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/60110

Khan Academy: Quadratic Formula 1

Guidance

The last way to solve a quadratic equation is the **Quadratic Formula.** This formula is derived from completing the square for the equation $ax^2 + bx + c = 0$ (see #13 from the Problem Set in the previous concept). We will derive the formula here.

Investigation: Deriving the Quadratic Formula

Walk through each step of completing the square of $ax^2 + bx + c = 0$.

1. Move the constant to the right side of the equation. $ax^2 + bx = -c$

2. "Take out" a from everything on the left side of the equation. $a\left(x^2 + \frac{b}{a}x\right) = -c$

3. Complete the square using $\frac{b}{a}$. $\left(\frac{b}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$

4. Add this number to both sides. Don't forget on the right side, you need to multiply it by a (to account for the a

outside the parenthesis). $a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)=-c+\frac{b^2}{4a}$

5. Factor the quadratic equation inside the parenthesis and give the right hand side a common denominator. $a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$

6. Divide both sides by a. $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

7. Take the square root of both sides. $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

8. Subtract $\frac{b}{2a}$ from both sides to get x by itself. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This formula will enable you to solve any quadratic equation as long as you know a, b, and c (from $ax^2+bx+c=0$).

Example A

Solve $9x^2-30x+26=0$ using the Quadratic Formula.

Solution: First, make sure one side of the equation is zero. Then, find a, b, and c. $a=9, b=-30, c=26$. Now, plug in the values into the formula and solve for x.

$$\begin{aligned}
x&=\frac{-(-30)\pm\sqrt{(-30)^2-4(9)(26)}}{2(9)}\\
&=\frac{30\pm\sqrt{900-936}}{18}\\
&=\frac{30\pm\sqrt{-36}}{18}\\
&=\frac{30\pm 6i}{18}\\
&=\frac{5}{3}\pm\frac{1}{3}i
\end{aligned}$$

Example B

Solve $2x^2+5x-15=-x^2+7x+2$ using the Quadratic Formula.

Solution: Let's get everything onto the left side of the equation.

$$\begin{aligned}
&2x^2+5x-15=-x^2+7x+2\\
&3x^2-2x-13=0
\end{aligned}$$

Now, use $a=3, b=-2$, and $c=-13$ and plug them into the Quadratic Formula.

$$\begin{aligned}
x&=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-13)}}{2(3)}\\
&=\frac{2\pm\sqrt{4+156}}{6}\\
&=\frac{2\pm\sqrt{160}}{6}\\
&=\frac{2\pm 4\sqrt{10}}{3}
\end{aligned}$$

Example C

Solve $x^2+20x+51=0$ by factoring, completing the square, and the Quadratic Formula.

Solution: While it might not look like it, 51 is not a prime number. Its factors are 17 and 3, which add up to 20.

$$\begin{aligned} x^2+20x+51&=0\\ (x+17)(x+13)&=0\\ x&=-17,-3 \end{aligned}$$

Now, solve by completing the square.

$$\begin{aligned} x^2+20x+51&=0\\ x^2+20x&=-51\\ x^2+20x+100&=-51+100\\ (x+10)^2&=49\\ x+10&=\pm 7\\ x&=-10\pm 7\rightarrow -17,-3 \end{aligned}$$

Lastly, let's use the Quadratic Formula. $a=1, b=20, c=51$.

$$\begin{aligned} x&=\frac{-20\pm\sqrt{20^2-4(1)(51)}}{2(1)}\\ &=\frac{-20\pm\sqrt{400-204}}{2}\\ &=\frac{-20\pm\sqrt{196}}{2}\\ &=\frac{-20\pm 14}{2}\\ &=-17,-3 \end{aligned}$$

Notice that no matter how you solve this, or any, quadratic equation, the answer will always be the same.

Guided Practice

1. Solve $-6x^2+15x-22=0$ using the Quadratic Formula.

2. Solve $2x^2-x-15=0$ using all three methods.

Answers

1. $a=-6, b=15$, and $c=-22$

$$\begin{aligned} x&=\frac{-15\pm\sqrt{15^2-4(-6)(-22)}}{2(-6)}\\ &=\frac{-15\pm\sqrt{225-528}}{-12}\\ &=\frac{-15\pm i\sqrt{303}}{-12}\\ &=\frac{5}{4}\pm\frac{\sqrt{303}}{12}i \end{aligned}$$

2. Factoring: $ac = -30$. The factors of -30 that add up to -1 are -6 and 5. Expand the $x-$term.

$$\begin{aligned}
2x^2 - 6x + 5x - 15 &= 0 \\
2x(x-3) + 5(x-3) &= 0 \\
(x-3)(2x+5) &= 0 \\
x &= 3, -\frac{5}{2}
\end{aligned}$$

Complete the square

$$\begin{aligned}
2x^2 - x - 15 &= 0 \\
2x^2 - x &= 15 \\
2\left(x^2 - \frac{1}{2}x\right) &= 15 \\
2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) &= 15 + \frac{1}{8} \\
2\left(x - \frac{1}{4}\right)^2 &= \frac{121}{8} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{121}{16} \\
x - \frac{1}{4} &= \pm\frac{11}{4} \\
x &= \frac{1}{4} \pm \frac{11}{4} \to 3, -\frac{5}{2}
\end{aligned}$$

Quadratic Formula

$$\begin{aligned}
x &= \frac{1 \pm \sqrt{1^2 - 4(2)(-15)}}{2(2)} \\
&= \frac{1 \pm \sqrt{1+120}}{4} \\
&= \frac{1 \pm \sqrt{121}}{4} \\
&= \frac{1 \pm 11}{4} \\
&= \frac{12}{4}, -\frac{10}{4} \to 3, -\frac{5}{2}
\end{aligned}$$

Vocabulary

Quadratic Formula: For any quadratic equation in the form $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Problem Set

Solve the following equations using the Quadratic Formula.

1. $x^2 + 8x + 9 = 0$

2. $4x^2 - 13x - 12 = 0$
3. $-2x^2 + x + 5 = 0$
4. $7x^2 - 11x + 12 = 0$
5. $3x^2 + 4x + 5 = 0$
6. $x^2 - 14x + 49 = 0$

Choose any method to solve the equations below.

7. $x^2 + 5x - 150 = 0$
8. $8x^2 - 2x - 3 = 0$
9. $-5x^2 + 18x - 24 = 0$
10. $10x^2 + x - 2 = 0$
11. $x^2 - 16x + 4 = 0$
12. $9x^2 - 196 = 0$

Solve the following equations using all three methods.

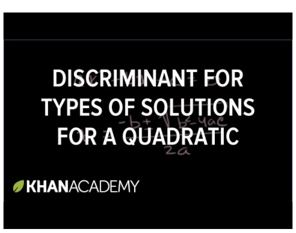
13. $4x^2 + 20x + 25 = 0$
14. $x^2 - 18x - 63 = 0$
15. **Writing** Explain when you would use the different methods to solve different types of equations. Would the type of answer (real or imaginary) help you decide which method to use? Which method do you think is the easiest?

Using the Discriminant

Objective

Using the discriminant of the Quadratic Formula to determine how many real solutions an equation has.

Watch This



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Khan Academy: Discriminant for Types of Solutions for a Quadratic

Guidance

From the previous concept, the Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression under the radical, $b^2 - 4ac$, is called the **discriminant.** You can use the discriminant to determine the number and type of solutions an equation has.

Investigation: Solving Equations with Different Types of Solutions

1. Solve $x^2 - 8x - 20 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{144}}{2} \\ &= \frac{8 \pm 12}{2} \rightarrow 10, -2 \end{aligned}$$

2. Solve $x^2 - 8x + 6 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{0}}{2} \\ &= \frac{8 \pm 0}{2} \rightarrow 4 \end{aligned}$$

3. Solve $x^2 - 8x + 20 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{-16}}{2} \\ &= \frac{8 \pm 4i}{2} \rightarrow 4 \pm 2i \end{aligned}$$

4. Look at the values of the discriminants from Steps 1-3. How do they differ? How does that affect the final answer?

From this investigation, we can conclude:

- If $b^2 - 4ac > 0$, then the equation has two real solutions.
- If $b^2 - 4ac = 0$, then the equation has one real solution; a double root.
- If $b^2 - 4ac < 0$, then the equation has two imaginary solutions.

Example A

Determine the type of solutions $4x^2 - 5x + 17 = 0$ has.

Solution: Find the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(4)(17) \\ &= 25 - 272 \end{aligned}$$

At this point, we know the answer is going to be negative, so there is no need to continue (unless we were solving the problem). This equation has two imaginary solutions.

Example B

Solve the equation from Example A to prove that it does have two imaginary solutions.

Solution: Use the Quadratic Formula.

$$x = \frac{5 \pm \sqrt{25 - 272}}{8} = \frac{5 \pm \sqrt{-247}}{8} = \frac{5}{8} \pm \frac{\sqrt{247}}{8} i$$

Guided Practice

1. Use the discriminant to determine the type of solutions $-3x^2-8x+16=0$ has.

2. Use the discriminant to determine the type of solutions $25x^2-80x+64=0$ has.

3. Solve the equation from #1.

Answers

1.

$$\begin{aligned} b^2-4ac &= (-8)^2-4(-3)(16) \\ &= 64+192 \\ &= 256 \end{aligned}$$

This equation has two real solutions.

2.

$$\begin{aligned} b^2-4ac &= (-80)^2-4(25)(64) \\ &= 6400-6400 \\ &= 0 \end{aligned}$$

This equation has one real solution.

3. $x=\frac{8\pm\sqrt{256}}{-6}=\frac{8\pm16}{-6}=-4,\frac{4}{3}$

Vocabulary

Discriminant
The value under the radical in the Quadratic Formula, b^2-4ac. The discriminant tells us number and type of solution(s) a quadratic equation has.

Problem Set

Determine the number and type of solutions each equation has.

1. $x^2-12x+36=0$
2. $5x^2-9=0$
3. $2x^2+6x+15=0$
4. $-6x^2+8x+21=0$
5. $x^2+15x+26=0$
6. $4x^2+x+1=0$

Solve the following equations using the Quadratic Formula.

7. $x^2-17x-60=0$
8. $6x^2-20=0$
9. $2x^2+5x+11=0$

Challenge Determine the values for c that make the equation have a) two real solutions, b) one real solution, and c) two imaginary solutions.

10. $x^2+2x+c=0$
11. $x^2-6x+c=0$
12. $x^2+12x+c=0$
13. What is the discriminant of $x^2+2kx+4=0$? Write your answer in terms of k.
14. For what values of k will the equation have a) two real solutions, b) one real solution, and c) two imaginary solutions?

2.27 Solving Linear Systems by Substitution

Objective

To solve linear systems in two variables by using substitution to make an equation in one variable that can be solved.

Review Queue

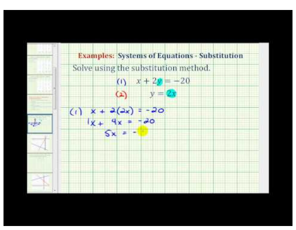
1. Substitute $x = 3$ into the expression $\frac{3(x-2)+5x}{x}$ and evaluate.
2. Solve for y: $5x - 3y = 15$
3. Solve for x: $2x + 14y = 42$

Solving Systems with One Solution Using Substitution

Objective

Solve consistent, independent systems using the substitution method.

Watch This



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James Sousa: Ex 1: Solve a System of Equations Using Substitution

Guidance

In the substitution method we will be looking at the two equations and deciding which variable is easiest to solve for so that we can write one of the equations as $x =$ or $y =$. Next we will replace either the x or the y accordingly in the *other* equation. The result will be an equation with only one variable that we can solve

Example A

Solve the system using substitution:

$$\begin{aligned} 2x + y &= 12 \\ -3x - 5y &= -11 \end{aligned}$$

Solution: The first step is to look for a variable that is easy to isolate. In other words, does one of the variables have a coefficient of 1? Yes, that variable is the y in the first equation. So, start by isolating or solving for y: $y = -2x + 12$

This expression can be used to replace the y in the other equation and solve for x:

$$\begin{aligned} -3x-5(-2x+12) &= -11 \\ -3x+10x-60 &= -11 \\ 7x-60 &= -11 \\ 7x &= 49 \\ x &= 7 \end{aligned}$$

Now that we have found x, we can use this value in our expression to find y:

$$\begin{aligned} y &= -2(7)+12 \\ y &= -14+12 \\ y &= -2 \end{aligned}$$

Recall that the solution to a linear system is a point in the coordinate plane where the two lines intersect. Therefore, our answer should be written as a point: (7, -2). You can check your answer by substituting this point into both equations to make sure that it satisfies them:

$$\begin{aligned} 2(7)+-2 &= 14-2 = 12 \\ -3(7)-5(-2) &= -21+10 = -11 \quad \boxed{\checkmark} \end{aligned}$$

Example B

Solve the system using substitution:

$$\begin{aligned} 2x+3y &= 13 \\ x+5y &= -4 \end{aligned}$$

Solution: In the last example, y was the easiest variable to isolate. Is that the case here? No, this time, x is the variable with a coefficient of 1. It is easy to fall into the habit of always isolating y since you have done it so much to write equation in slope-intercept form. Try to avoid this and look at each system to see which variable is easiest to isolate. Doing so will help reduce your work.

Solving the second equation for x gives: $x = -5y-4$.

This expression can be used to replace the x in the other equation and solve for y:

$$\begin{aligned} 2(-5y-4)+3y &= 13 \\ -10y-8+3y &= 13 \\ -7y-8 &= 13 \\ -7y &= 21 \\ y &= -3 \end{aligned}$$

Now that we have found y, we can use this value in our expression to find x:

$$x = -5(-3) - 4$$
$$x = 15 - 4$$
$$x = 11$$

So, the solution to this system is (11, -3). Don't forget to check your answer:

$$2(11) + 3(-3) = 22 - 9 = 13$$
$$11 + 5(-3) = 11 - 15 = -4 \quad \boxed{\checkmark}$$

Example C

Solve the system using substitution:

$$4x + 3y = 4$$
$$6x - 2y = 19$$

Solution: In this case, none of the variables have a coefficient of 1. So, we can just pick on to solve for. Let's solve for the x in equation 1:

$$4x = -3y + 4$$
$$x = -\frac{3}{4}y + 1$$

Now, this expression can be used to replace the x in the other equation and solve for y:

$$6\left(-\frac{3}{4}y + 1\right) - 2y = 19$$
$$-\frac{18}{4}y + 6 - 2y = 19$$
$$-\frac{9}{2}y - \frac{4}{2}y = 13$$
$$-\frac{13}{2}y = 13$$
$$\left(-\frac{2}{13}\right)\left(-\frac{13}{2}\right)y = 13\left(-\frac{2}{13}\right)$$
$$y = -2$$

Now that we have found y, we can use this value in our expression find x:

$$x = \left(-\frac{3}{4}\right)(-2) + 1$$
$$x = \frac{6}{4} + 1$$
$$x = \frac{3}{2} + \frac{2}{2}$$
$$x = \frac{5}{2}$$

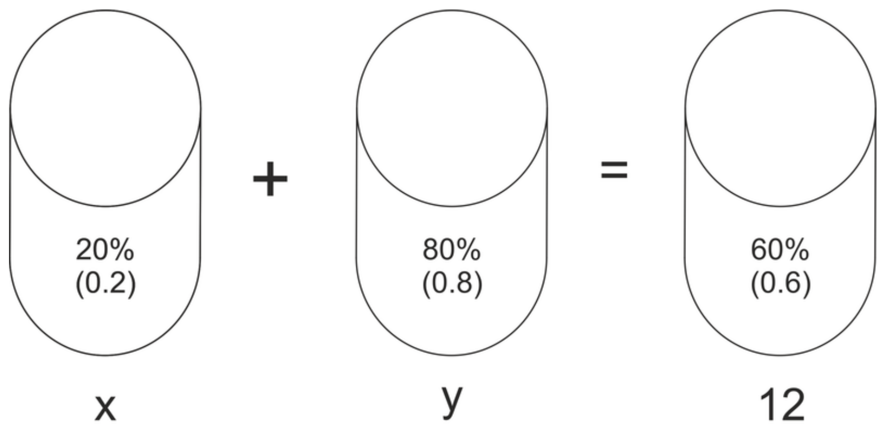
So, the solution is $\left(\frac{5}{2},-2\right)$. Check your answer:

$$4\left(\frac{5}{2}\right)+3(-2)=10-6=4$$
$$6\left(\frac{5}{2}\right)-2(-2)=15+4=19 \quad \boxed{\checkmark}$$

Example D

Rex and Carl are making a mixture in science class. They need to have 12 ounces of a 60% saline solution. To make this solution they have a 20% saline solution and an 80% saline solution. How many ounces of each do they need to make the correct mixture?

Solution: This type of word problem can be daunting for many students. Let's try to make it easier by organizing our information into a "picture" equation as shown below:



In this picture, we can see that we will be mixing x ounces of the 20% solution with y ounces of the 80% solution to get 12 ounces of the 60% solution. The two equations are thus:

$$0.2x+0.8y=0.6(12)$$
$$x+y=12$$

Now we can solve the system using substitution. Solve for y in the second equation to get:$y=12-x$.

Now, substitute and solve in the first equation:

$$0.2x+0.8(12-x)=0.6(12)$$
$$0.2x+9.6-0.8x=7.2$$
$$-0.6x=-2.4$$
$$x=4$$

Now we can find y:

$$y=12-x$$
$$y=12-4$$
$$y=8$$

Therefore, Rex and Carl need 4 ounces of the 20% saline solution and 8 ounces of the 80% saline solution to make the correct mixture.

Guided Practice

Solve the following systems using the substitution method.

1.

$$\begin{aligned}3x+4y&=-13\\x&=-2y-9\end{aligned}$$

2.

$$\begin{aligned}-2x-5y&=-39\\x+3y&=24\end{aligned}$$

3.

$$\begin{aligned}y&=\frac{1}{2}x-21\\y&=-2x+9\end{aligned}$$

Answers

1. In this problem, the second equation is already solved for x so we can use that in the first equation to find y:

$$\begin{aligned}3(-2y-9)+4y&=-13\\-6y-27+4y&=-13\\-2y-27&=-13\\-2y&=14\\y&=-7\end{aligned}$$

Now we can find x:

$$\begin{aligned}&x=-2(-7)-9\\&x=14-9\\&x=5\end{aligned}$$

Therefore the solution is (5, -7).

2. This time the x in the second equation is the easiest variable to isolate: $x=-3y+24$. Let's use this in the first expression to find y:

$$\begin{aligned}-2(-3y+24)-5y&=-39\\6y-48-5y&=-39\\y-48&=-39\\y&=9\end{aligned}$$

Now we can find x:

$$\begin{aligned}x &= -3(9)+24\\ x &= -27+24\\ x &= -3\end{aligned}$$

Therefore the solution is (-3, 9).

3. In this case, both equations are equal to y. Since $y = y$, by the Reflexive Property of Equality, we can let the right hand sides of the equations be equal too. This is still a substitution problem; it just looks a little different.

$$\begin{aligned}\frac{1}{2}x-21 &= -2x+9\\ 2\left(\frac{1}{2}x-21\right. &= \left.-2x+9\right)\\ x-42 &= -4x+18\\ 5x &= 60\\ x &= 12\end{aligned}$$

Now we can find y:

$$\begin{aligned}y &= \frac{1}{2}(12)-21 & & & y &= -2(12)+9\\ y &= 6-21 & &or & y &= -24+9\\ y &= -15 & & & y &= -15\end{aligned}$$

Therefore our solution is (12, -15).

Problem Set

Solve the following systems using substitution. Remember to check your answers.

1.

$$\begin{aligned}x+3y &= -1\\ 2x+9y &= 7\end{aligned}$$

2.

$$\begin{aligned}7x+y &= 6\\ x-2y &= -12\end{aligned}$$

3.

$$\begin{aligned}5x+2y &= 0\\ y &= x-7\end{aligned}$$

4.

$$\begin{aligned}2x-5y&=21\\x&=-6y+2\end{aligned}$$

5.

$$\begin{aligned}y&=x+3\\y&=2x-1\end{aligned}$$

6.

$$\begin{aligned}x+6y&=1\\-2x-11y&=-4\end{aligned}$$

7.

$$\begin{aligned}2x+y&=18\\-3x+11y&=-27\end{aligned}$$

8.

$$\begin{aligned}2x+3y&=5\\5x+7y&=8\end{aligned}$$

9.

$$\begin{aligned}-7x+2y&=9\\5x-3y&=3\end{aligned}$$

10.

$$\begin{aligned}2x-6y&=-16\\-6x+10y&=8\end{aligned}$$

11.

$$\begin{aligned}2x-3y&=-3\\8x+6y&=12\end{aligned}$$

12.

$$\begin{aligned}5x+y&=-3\\y&=15x+9\end{aligned}$$

Set up and solve a system of linear equations to answer each of the following word problems.

13. Alicia and Sarah are at the supermarket. Alicia wants to get peanuts from the bulk food bins and Sarah wants to get almonds. The almonds cost \$6.50 per pound and the peanuts cost \$3.50 per pound. Together they buy 1.5 pounds of nuts. If the total cost is \$6.75, how much did each girl get? Set up a system to solve using substitution.
14. Marcus goes to the department store to buy some new clothes. He sees a sale on t-shirts (\$5.25) and shorts (\$7.50). Marcus buys seven items and his total, before sales tax, is \$43.50. How many of each item did he buy?
15. Jillian is selling tickets for the school play. Student tickets are \$3 and adult tickets are \$5. If 830 people buy tickets and the total revenue is \$3104, how many students attended the play?

Solving Systems with No or Infinitely Many Solutions Using Substitution

Objective

To understand how a system with no solution and a system with infinitely many solutions are discovered using the substitution method.

Guidance

When a system has no solution or an infinite number of solutions and we attempt to find a single, unique solution using an algebraic method, such as substitution, the variables will cancel out and we will have an equation consisting of only constants. If the equation is untrue as seen below in Example A, then the system has no solution. If the equation is always true, as seen in Example B, then there are infinitely many solutions.

Example A

Solve the system using substitution:

$$\begin{aligned} 3x-2y&=7 \\ y&=\frac{3}{2}x+5 \end{aligned}$$

Solution: Since the second equation is already solved for y, we can use this in the first equation to solve for x:

$$\begin{aligned} 3x-2\left(\frac{3}{2}x+5\right)&=7 \\ 3x-3x+5&=7 \\ 5&\neq 7 \end{aligned}$$

Since the substitution above resulted in the elimination of the variable, x, and an untrue equation involving only constants, the system has no solution. The lines are parallel and the system is inconsistent.

Example B

Solve the system using substitution:

$$\begin{aligned} -2x+5y&=-2 \\ 4x-10y&=4 \end{aligned}$$

Solution: We can solve for x in the first equation as follows:

$$\begin{aligned} -2x&=-5y-2 \\ x&=\frac{5}{2}y+1 \end{aligned}$$

Now, substitute this expression into the second equation and solve for y:

$$
\begin{aligned}
4\left(\frac{5}{2}y+1\right)-10y&=4\\
10y+4-10y&=4\\
4&=4\\
(0&=0)
\end{aligned}
$$

In the process of solving for y, the variable is cancelled out and we are left with only constants. We can stop at the step where 4 = 4 or continue and subtract 4 on each side to get 0 = 0. Either way, this is a true statement. As a result, we can conclude that this system has an infinite number of solutions. The lines are coincident and the system is consistent and dependent.

Guided Practice

Solve the following systems using substitution. If there is no unique solution, state whether there is no solution or infinitely many solutions.

1.

$$
\begin{aligned}
y&=\frac{2}{5}x-3\\
2x-5y&=15
\end{aligned}
$$

2.

$$
\begin{aligned}
-x+7y&=5\\
3x-21y&=-5
\end{aligned}
$$

3.

$$
\begin{aligned}
3x-5y&=0\\
-2x+6y&=0
\end{aligned}
$$

Answers

1. Substitute the first equation into the second and solve for x:

$$
\begin{aligned}
2x-5\left(\frac{2}{5}x-3\right)&=15\\
2x-2x+15&=15\\
15&=15\\
(0&=0)
\end{aligned}
$$

Since the result is a true equation, the system has infinitely many solutions.

2. Solve the first equation for x to get: $x=7y-5$. Now, substitute this into the second equation to solve for y:

$$
\begin{aligned}
3(7y-5)-21y&=-5\\
21y-15-21y&=-5\\
-15&\neq-5
\end{aligned}
$$

Since the result is an untrue equation, the system has no solution.

3. Solving the second equation for x we get: $x = 3y$. Now, we can substitute this into the first equation to solve for y:

$$\begin{aligned}3(3y)-5y&=0\\9y-5y&=0\\4y&=0\\y&=0\end{aligned}$$

Now we can use this value of y to find x:

$$\begin{aligned}x&=3y\\x&=3(0)\\x&=0\end{aligned}$$

Therefore, this system has a solution at (0, 0). After solving systems that result in 0 = 0, it is easy to get confused by a result with zeros for the variables. It is perfectly okay for the intersection of two lines to occur at (0, 0).

Problem Set

Solve the following systems using substitution.

1. $$\begin{aligned}17x-3y&=5\\y&=3x+1\end{aligned}$$

2. $$\begin{aligned}4x-14y&=21\\y&=\frac{2}{7}x+7\end{aligned}$$

3. $$\begin{aligned}-24x+9y&=12\\8x-3y&=-4\end{aligned}$$

4. $$\begin{aligned}y&=-\frac{3}{4}x+9\\6x+8y&=72\end{aligned}$$

5. $$\begin{aligned}2x+7y&=12\\y&=-\frac{2}{3}x+4\end{aligned}$$

6.

$$2x = -6y + 11$$
$$y = -\frac{1}{3}x + 7$$

7.

$$\frac{1}{2}x - \frac{4}{5}y = 8$$
$$5x - 8y = 50$$

8.

$$-6x + 16y = 38$$
$$x = \frac{8}{3}y - \frac{19}{3}$$

9.

$$x = y$$
$$5x + 3y = 0$$

10.

$$\frac{1}{2}x + 3y = -15$$
$$y = x - 5$$

11.

$$\frac{2}{3}x + \frac{1}{6}y = 2$$
$$y = -4x + 12$$

12.

$$16x - 4y = 3$$
$$y = 4x + 7$$

2.28 Solving Linear Systems by Linear Combinations (Elimination)

Objective

Solve systems by combining the two equations (or multiples of the two equations) to eliminate a variable.

Review Queue

1. Solve the system by graphing:

$$y = \frac{2}{3}x - 3$$
$$y = -\frac{1}{2}x + 4$$

2. Solve the system by substitution:

$$x + 3y = 2$$
$$4x + y = -25$$

3. Find the least common multiple of the following pairs of numbers:

a) 3 and 5

b) 2 and 10

c) 6 and 15

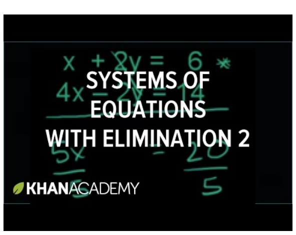
d) 4 and 10

Solving Systems Without Multiplying

Objective

Solve systems by adding the two equations together to eliminate a variable.

Watch This



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Khan Academy: Addition Elimination Method 1

Guidance

In this lesson we will be looking at systems in which the two equations contain coefficients of one variable that are **additive inverses (opposites)** of one another.

Example A

Solve the system using Linear Combination:

$$\begin{aligned} 2x - 3y &= -9 \\ 5x + 3y &= 30 \end{aligned}$$

Solution: Notice that the coefficients of the y terms are opposites. When we add the two equations together, these terms will be eliminated because their sum is $0y = 0$.

$$\begin{aligned} 2x - \cancel{3y} &= -9 \\ + \underline{5x + \cancel{3y}} &\underline{= 30} \\ 7x &= 21 \end{aligned}$$

Now we can solve for x:

$$\begin{aligned} 7x &= 21 \\ x &= 3 \end{aligned}$$

Now that we have found x, we can plug this value into either equation to find y:

$$\begin{aligned} 2(3) - 3y &= -9 & & & 5(3) + 3y &= 30 \\ 6 - 3y &= -9 & & & 15 + 3y &= 30 \\ -3y &= -15 & & or & 3y &= 15 \\ y &= 5 & & & y &= 5 \end{aligned}$$

The solution is therefore: (3, 5).

Remember to check your answer:

$$\begin{aligned} 2(3) - 3(5) &= 6 - 15 = -9 \\ 5(3) + 3(5) &= 15 + 15 = 30 \quad \boxed{\checkmark} \end{aligned}$$

Example B

Solve the system using Linear Combination:

$$\begin{aligned} x + 4y &= 2 \\ -x - 5y &= -3 \end{aligned}$$

Solution: Notice that the coefficients of the x terms are opposites. When we add the two equations together, these terms will be eliminated because their sum is $0x = 0$.

$$\begin{aligned} \cancel{x}+4y&=2 \\ +\underline{-\cancel{x}-5y}&\underline{=-3} \\ -y&=-1 \end{aligned}$$

Now we can solve for y:

$$\begin{aligned} -y&=-1 \\ y&=1 \end{aligned}$$

Now that we have found y, we can plug this value into either equation to find x:

$$\begin{aligned} x+4(1)&=2 & -x-5(1)&=-3 \\ x+4&=2 & -x-5&=-3 \\ x&=-2 \quad or & -x&=2 \\ & & x&=-2 \end{aligned}$$

The solution is therefore: (-2, 1).

Remember to check your answer:

$$\begin{aligned} -2+4(1)&=-2+4=2 \\ -(-2)-5(1)&=2-5=-3 \quad \boxed{\checkmark} \end{aligned}$$

Example C

Solve the system using Linear Combination:

$$\begin{aligned} 2x+y&=2 \\ -3x+y&=-18 \end{aligned}$$

Solution: In this case the coefficients of the y terms are the same, not opposites. One way to solve this system using linear combination would be to subtract the second equation from the first instead of adding it. Sometimes subtraction results in more errors, however, particularly when negative numbers are involved. Instead of subtracting, multiply the second equation by -1 and then add them together.

$$\begin{aligned} -1(-3x+y&=-18) \\ 3x-y&=18 \end{aligned}$$

Essentially, we changed all of the signs of the terms in this equation.

Now we can add the two equations together to eliminate y:

$$\begin{array}{r} 2x + \cancel{y} = 2 \\ + \underline{3x - \cancel{y} = 18} \\ 5x = 20 \end{array}$$

Now we can solve for x:

$$\begin{aligned} 5x &= 20 \\ x &= 4 \end{aligned}$$

Now that we have found x, plug this value into either equation to find y:

$$\begin{aligned} 2(4) + y &= 2 & \quad & -3(4) + y = -18 \\ 8 + y &= 2 & \quad or \quad & -12 + y = -18 \\ y &= -6 & & y = -6 \end{aligned}$$

The solution is therefore: (4, -6).

Remember to check your answer:

$$\begin{aligned} 2(4) + (-6) &= 8 - 6 = 2 \\ -3(4) + (-6) &= -12 - 6 = -18 \quad \boxed{\checkmark} \end{aligned}$$

Guided Practice

Solve the following systems using Linear Combinations.

1.

$$\begin{aligned} 4x + 5y &= 8 \\ -2x - 5y &= 6 \end{aligned}$$

2.

$$\begin{aligned} 2x + 3y &= 3 \\ 2x - y &= 23 \end{aligned}$$

3.

$$\begin{aligned} 2x + 3y &= -6 \\ y &= 2x - 2 \end{aligned}$$

Answers

1. First we can add the two equations together to eliminate y and solve for x:

$$\begin{aligned} 4x+\cancel{5y}&=8 \\ +\underline{-2x-\cancel{5y}}&\underline{=6} \\ 2x&=14 \\ x&=7 \end{aligned}$$

Substitute x into one equation to find y:

$$\begin{aligned} 4(7)+5y&=8 \\ 28+5y&=8 \\ 5y&=-20 \\ y&=-4 \end{aligned}$$

Solution: (7, -4)

2. This time we need to begin by multiplying the second equation by -1 to get $-2x+y=-23$. Now we can add the two equations together to eliminate x and solve for y:

$$\begin{aligned} 2x+\cancel{3y}&=3 \\ +\underline{-\cancel{2x}+y}&\underline{=-23} \\ 4y&=-20 \\ y&=-5 \end{aligned}$$

Substitute y into one equation to find x:

$$\begin{aligned} 2x+(3-5)&=3 \\ 2x-15&=3 \\ 2x&=18 \\ x&=9 \end{aligned}$$

Solution: (9, -5)

3. In this example, the second equation is not written in standard form. We must first rewrite this equation in standard form so that the variable will align vertically when we add the equations together. The second equation should be $-2x+y=-2$ after we subtract $2x$ from both sides. Now we can add the two equations together to eliminate x and solve for y:

$$\begin{aligned} \cancel{2x}+3y&=-6 \\ +\quad\underline{-\cancel{2x}+y}&\underline{=-2} \\ 4y&=-8 \\ y&=-2 \end{aligned}$$

Substitute y into one equation to find x:

$$\begin{aligned}2x+3(-2)&=-6\\2x-6&=-6\\2x&=0\\x&=0\end{aligned}$$

Solution: (0, -2)

Vocabulary

Additive Inverse (Opposite)
The additive inverse of a number is the number multiplied by -1. When a number and it's additive inverse are added together the result is zero (the identity for addition). Examples of opposites: 2 and -2, -3 and 3, 15 and -15, $\frac{1}{2}$ and $-\frac{1}{2}$.

Problem Set

Solve the following systems using linear combinations.

1.
$$\begin{aligned}4x+2y&=-6\\-5x-2y&=4\end{aligned}$$

2.
$$\begin{aligned}-3x+5y&=-34\\3x-y&=14\end{aligned}$$

3.
$$\begin{aligned}x+y&=-1\\x-y&=21\end{aligned}$$

4.
$$\begin{aligned}2x+8y&=-4\\-2x+3y&=15\end{aligned}$$

5.
$$\begin{aligned}8x-12y&=24\\-3x+12y&=21\end{aligned}$$

6.
$$\begin{aligned}x+3y&=-2\\-x-2y&=4\end{aligned}$$

7.

$$\begin{aligned}5x+7y&=2\\5x+3y&=38\end{aligned}$$

8.

$$\begin{aligned}12x-2y&=2\\5x-2y&=-5\end{aligned}$$

9.

$$\begin{aligned}2x+y&=25\\x+y&=5\end{aligned}$$

10.

$$\begin{aligned}\frac{1}{2}x+3y&=-3\\y&=\frac{1}{2}x-5\end{aligned}$$

11.

$$\begin{aligned}3x+5y&=10\\y&=-3x-10\end{aligned}$$

12.

$$\begin{aligned}6x+3y&=-3\\3y&=-7x+1\end{aligned}$$

13.

$$\begin{aligned}4x-2y&=5\\-4x+2y&=11\end{aligned}$$

14.

$$\begin{aligned}9x+2y&=0\\-9x-3y&=0\end{aligned}$$

15.

$$\begin{aligned}11x+7y&=12\\-11x&=7y-12\end{aligned}$$

Set up and solve a linear system of equations to solve the following word problems.

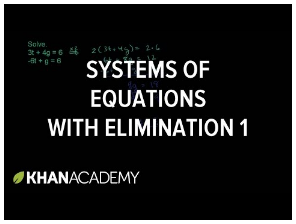
16. The sum of two numbers is 15 and their difference is 3. Find the two numbers.
17. Jessica and Maria got to the supermarket to buy fruit. Jessica buys 5 apples and 6 oranges and her total before tax is \$3.05. Maria buys 7 apples and 6 oranges and her total before tax is \$3.55. What is the price of each fruit? *Hint: Let x be the price of one apple and y be the price of one orange*.

Solving Systems by Multiplying One Equation to Cancel a Variable

Objective

Solve systems using linear combinations by multiplying one equation by a constant to eliminate a variable.

Watch This



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Khan Academy: Solving systems by elimination2

Guidance

It is not always the case that the coefficients of one variable in a system of linear equations are the same or opposites. When this is not the case, we may be able to multiply one of the equations by a constant so that we have opposite coefficients of one variable and can proceed to solve the system as we have previously done.

Example A

Solve the system using linear combinations:

$$\begin{aligned} 4x+y&=0 \\ x-3y&=26 \end{aligned}$$

Solution: Here we have coefficients of y that are opposite signs (one is positive and one is negative). We can get opposite values if we multiply the first equation by 3. Be careful; make sure you multiply the entire equation, including the constant, by 3:

$$\begin{aligned} 3(4x+y&=0) \\ 12x+3y&=0 \end{aligned}$$

Now we can use this new equation in our system to eliminate y and solve for x:

$$\begin{aligned} 12x+\cancel{3y}&=0 \\ +\quad \underline{x-\cancel{3y}}&\underline{=26} \\ 13x&=26 \\ x&=2 \end{aligned}$$

Now, find y:

$$\begin{aligned} 4(2)+y&=0 \\ 8+y&=0 \\ y&=-8 \end{aligned}$$

Solution: (2, -8)

Check your answer:

$$\begin{aligned} 4(2)+(-8) &= 8-8=0 \\ (2)-3(-8) &= 2+24=26 \quad \boxed{\checkmark} \end{aligned}$$

* Note that we could have used the other equation to find y in the final step.

** From the beginning, we could have multiplied the second equation by -4 instead to cancel out the x variable.

Example B

Solve the system using linear combinations:

$$\begin{aligned} 2x+5y &= 1 \\ y &= -3x+21 \end{aligned}$$

Solution: For this system, we must first rewrite the second equation in standard form so that we can see how the coefficients compare. If we add $3x$ to both sides we get:

$$\begin{aligned} 2x+5y &= 1 \\ 3x+y &= 21 \end{aligned}$$

Now, we can see that if we multiply the second equation by -5, the coefficients of y will be opposites.

$$\begin{aligned} 2x+\cancel{5y} &= 1 \\ + \quad \underline{-15x-\cancel{5y}} &\underline{= -105} \\ -13x &= -104 \\ x &= 8 \end{aligned}$$

Now, find y:

$$\begin{aligned} y &= -3(8)+21 \\ y &= -24+21 \\ y &= -3 \end{aligned}$$

Solution: (8, -3)

Check your answer:

$$\begin{aligned} 2(8)+5(-3) &= 16-15=1 \\ -3 = -3(8)+21 &= -24+21=-3 \quad \boxed{\checkmark} \end{aligned}$$

Example C

Solve the system using linear combinations:

$$\begin{aligned} 4x-6y &= -12 \\ y &= \frac{2}{3}x+2 \end{aligned}$$

Solution: Again, we need to rearrange the second equation in this system to get it in standard form. We can do this by subtracting $\frac{2}{3}x$ on both sides to get the following system:

$$\begin{aligned} 4x-6y &= -12 \\ -\frac{2}{3}x+y &= 2 \end{aligned}$$

Multiply the second equation by 6 to eliminate y:

$$\begin{aligned} 6\left(-\frac{2}{3}x+y=2\right) \\ -4x+6y=12 \end{aligned}$$

And add it to the first equation.

$$\begin{aligned} 4x-\cancel{6y} &= -12 \\ + \quad \underline{-4x+\cancel{6y} = 12} \\ 0x &= 0 \\ 0 &= 0 \end{aligned}$$

Here, both variables were eliminated and we wound up with 0 = 0. Recall that this is a true statement and thus this system has infinite solutions.

Guided Practice

Solve the following systems using linear combinations.

1.

$$\begin{aligned} 3x+12y &= -3 \\ -x-5y &= 0 \end{aligned}$$

2.

$$\begin{aligned} 0.75x+5y &= 0 \\ 0.25x-9y &= 0 \end{aligned}$$

3.

$$\begin{aligned} x-3y &= 5 \\ y &= \frac{1}{3}x+8 \end{aligned}$$

Answers

1. In this problem we can just multiply the second equation by 3 to get coefficients of x which are opposites: $3(-x-5y=0) \Rightarrow -3x-15y=0$

$$\begin{aligned}
\cancel{3x}+12y&=-3\\
\underline{-\cancel{3x}-15y}&\underline{=0}\\
-3y&=-3\\
y&=1
\end{aligned}$$

Now we can find x:

$$\begin{aligned}
3x+12(1)&=-3\\
3x+12&=-3\\
3x&=-15\\
x&=-5
\end{aligned}$$

Solution: (-5, 1)

2. For this system, we need to multiply the second equation by -3 to get coefficients of x which are opposites: $-3(0.25x-9y=0) \Rightarrow -0.75x+27y=0$

$$\begin{aligned}
\cancel{0.75x}+5y&=0\\
-\underline{\cancel{0.75x}-9y}&\underline{=0}\\
-4y&=0\\
y&=0
\end{aligned}$$

Now we can find x:

$$\begin{aligned}
0.75x+5(0)&=0\\
0.75x&=0\\
x&=0
\end{aligned}$$

Solution: (0, 0)

3. This time we need to rewrite the second equation in standard form:

$$\begin{aligned}
x-3y&=5\\
-\frac{1}{3}x+y&=8.
\end{aligned}$$

Now we can multiply the second equation by 3 to get coefficients of x that are opposites:

$3\left(-\frac{1}{3}x+y=8\right) \Rightarrow -x+3y=24$, Now our system is:

$$x - 3y = 5$$
$$-x + 3y = 24$$

When we add these equations together, both variables are eliminated and the result is 0 = 29, which is an untrue statement. Therefore, this system has no solution.

Problem Set

Solve the following systems using linear combinations.

1. $$x - 7y = 27$$ $$2x + y = 9$$

2. $$x + 3y = 31$$ $$3x - 5y = -5$$

3. $$10x + y = -6$$ $$-7x - 5y = -13$$

4. $$2x + 4y = 18$$ $$x - 5y = 9$$

5. $$2x + 6y = 8$$ $$3x + 2y = -23$$

6. $$12x - y = 2$$ $$2x + 5y = 21$$

7. $$2x + 4y = 24$$ $$-3x - 2y = -26$$

8. $$3x + 2y = 19$$ $$5x + 4y = 23$$

9.

$$\begin{aligned} 3x - 9y &= 13 \\ x - 3y &= 7 \end{aligned}$$

10.

$$\begin{aligned} 8x + 2y &= -4 \\ 3y &= -16x + 2 \end{aligned}$$

11.

$$\begin{aligned} 3x + 2y &= -3 \\ -6x - 5y &= 4 \end{aligned}$$

12.

$$\begin{aligned} 10x + 6y &= -24 \\ y &= -\frac{5}{3}x - 4 \end{aligned}$$

13.

$$\begin{aligned} \frac{1}{3}x - \frac{2}{3}y &= -8 \\ \frac{1}{2}x - \frac{1}{3}y &= 12 \end{aligned}$$

14.

$$\begin{aligned} 6x - 10y &= -8 \\ y &= -\frac{3}{5}x \end{aligned}$$

15.

$$\begin{aligned} 4x - 14y &= -52 \\ y &= \frac{2}{7}x + 3 \end{aligned}$$

Set up and solve a system of linear equations for each of the following word problems.

16. Lia is making a mixture of Chlorine and water in her science class. She needs to make 13 ml of a 60% chlorine solution from a solution that is 35% chlorine and a second solution which is 75% chlorine. How many milliliters of each solution does she need?
17. Chelsea and Roberto each sell baked goods for their club's fundraiser. Chelsea sells 13 cookies and 7 brownies and collects a total of \$11.75. Roberto sells 10 cookies and 14 brownies and collects a total of \$15.50. How much did they charge for the cookies and the brownies?
18. Mattie wants to plant some flowers in her yard. She has space for 15 plants. She buys pansies and daisies at her local garden center. The pansies are each \$2.75 and the daisies are each \$2.00. How many of each does she buy if she spends a total of \$35.25?

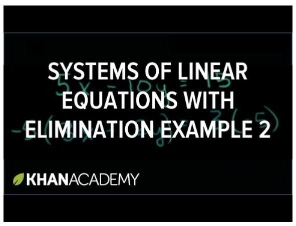
Solving Systems by Multiplying Both Equations to Cancel a Variable

Objective

Solve linear systems using linear combinations in which both equations must be multiplied by a constant to cancel a variable.

Watch This

Watch the second portion of this video, starting around 5:00.



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Khan Academy: Solving Systems of Equations by Multiplication

Guidance

In the linear systems in this lesson, we will need to multiply both equations by a constant in order to have opposite coefficients of one of the variables. In order to determine what numbers to multiply by, we will be finding the least common multiple of the given coefficients. Recall that the least common multiple of two numbers is the smallest number which is divisible by both of the given numbers. For example, 12 is the least common multiple of 4 and 6 because it is the smallest number divisible by both 4 and 6.

Example A

Solve the system using linear combinations:

$$\begin{aligned}2x-5y&=15\\3x+7y&=8\end{aligned}$$

Solution: In this problem we cannot simply multiply one equation by a constant to get opposite coefficients for one of the variables. Here we will need to identify the least common multiple of the coefficients of one of the variables and use this value to determine what to multiple each equation by. If we look at the coefficients of x, 2 and 3, the least common multiple of these numbers is 6. So, we want to have the coefficients of x be 6 and -6 so that they are opposites and will cancel when we add the two equations together. In order to get coefficients of 6 and -6 we can multiply the first equation by 3 and the second equation by -2 (it doesn't matter which one we make negative.)

$$\begin{array}{rcr}3(2x-5y=15) & \Rightarrow & \cancel{6x}-15y=45\\ -2(3x+7y=8) & & +\ \underline{-\cancel{6x}-14y=-16}\\ & & -29y=29\\ & & y=-1\end{array}$$

Now find x:

$$\begin{aligned}2(x)-5(-1)&=15\\2x+5&=15\\2x&=10\\x&=5\end{aligned}$$

Solution: (5, -1)

Check your answer:

$$\begin{aligned}2(5)-5(-1)&=10+5=15\\3(5)+7(-1)&=15-7=8\quad\boxed{\checkmark}\end{aligned}$$

* This problem could also be solved by eliminating the y variables first. To do this, find the least common multiple of the coefficients of y, 5 and 7. The least common multiple is 35, so we would multiply the first equation by 7 and the second equation by 5. Since one of them is already negative, we don't have to multiply by a negative.

Example B

Solve the system using linear combinations:

$$\begin{aligned}7x+20y&=-9\\-2x-3y&=8\end{aligned}$$

Solution: The first step is to decide which variable to eliminate. Either one can be eliminated but sometimes it is helpful to look at what we need to multiply by to eliminate each one and determine which is easier to eliminate. In general, it is easier to work with smaller numbers so in this case it makes sense to eliminate x first. The Least Common Multiple (LCM) of 7 and 2 is 14. To get 14 as the coefficient of each term, we need to multiple the first equation by 2 and the second equation by 7:

$$\begin{array}{lcr}2(7x+20y=-9) & \Rightarrow & \cancel{14x}+40y=-18\\7(-2x-3y=8) & & \underline{-\cancel{14x}-21y=56}\\ & & 19y=38\\ & & y=2\end{array}$$

Now find x:

$$\begin{aligned}-2x-3(2)&=8\\-2x-6&=8\\-2x&=14\\x&=-7\end{aligned}$$

Solution: (-7, 2)

Check your answer:

$$\begin{aligned} 7(-7)+20(2) &= -49+40 = -9 \\ -2(-7)-3(2) &= 14-6 = 8 \quad \boxed{\checkmark} \end{aligned}$$

Example C

Solve the system using linear combinations:

$$\begin{aligned} 14x-6y &= -3 \\ 16x-9y &= -7 \end{aligned}$$

Solution: This time, we will eliminate y. We need to find the LCM of 6 and 9. The LCM is 18, so we will multiply the first equation by 3 and the second equation by -2. Again, it doesn't matter which equation we multiply by a negative value.

$$\begin{aligned} 3(14x-6y=-3) \quad &\Rightarrow \quad +42x-\cancel{18y} = -9 \\ -2(16x-9y=-7) \quad & \quad\quad \underline{-32x+\cancel{18y} = 14} \\ & \quad\quad\quad\quad 10x = 5 \\ & \quad\quad\quad\quad\quad x = 2 \end{aligned}$$

Now find y:

$$\begin{aligned} 14\left(\frac{1}{2}\right)-6y &= -3 \\ 7-6y &= -3 \\ -6y &= -10 \\ y &= \frac{10}{6} = \frac{5}{3} \end{aligned}$$

Solution: $\left(\frac{1}{2}, \frac{5}{3}\right)$

Check your answer:

$$\begin{aligned} 14\left(\frac{1}{2}\right)-6\left(\frac{5}{3}\right) &= 7-10 = -3 \\ 16\left(\frac{1}{2}\right)-9\left(\frac{5}{3}\right) &= 8-15 = -7 \quad \boxed{\checkmark} \end{aligned}$$

Example D

A one pound mix consisting of 30% cashews and 70% pistachios sells for \$6.25. A one pound mix consisting of 80% cashews and 20% pistachios sells for \$7.50. How would a mix consisting of 50% of each type of nut sell for?

Solution: First we need to write a system of linear equations to represent the given information. Let $x =$ the cost of the cashews per pound and let $y =$ the cost of the pistachios per pound. Now we can write two equations to represent the two different mixes of nuts:

$$
\begin{aligned}
0.3x + 0.7y &= 6.25 \\
0.8x + 0.2y &= 7.50
\end{aligned}
$$

Now we can solve this system to determine the cost of each type of nut per pound. If we eliminate y, we will need to multiple the first equation by 2 and the second equation by -7:

$$
\begin{aligned}
2(0.3x - 0.7y = 6.25) \quad &\Rightarrow \quad +0.6x - \cancel{14y} = 12.5 \\
-7(0.8x + 0.2y = 7.50) \quad & \quad \underline{-5.6x + \cancel{14y} = -52.5} \\
& \qquad -5x = -40 \\
& \qquad x = 8
\end{aligned}
$$

Now find y:

$$
\begin{aligned}
0.3(8) + 0.7y &= 6.25 \\
2.4 + 0.7y &= 6.25 \\
0.7y &= 3.85 \\
y &= 5.5
\end{aligned}
$$

So, we have determined that the cost of the cashews is \$8 per pound and the cost of the pistachios is \$5.50 per pound. Now we can determine the cost of the 50% mix as follows:

$0.5(8.00) + 0.5(5.50) = 4.00 + 2.25 = 6.25$ So, the new mix is \$6.25 per pound.

Guided Practice

Solve the following systems using linear combinations:

1.

$$
\begin{aligned}
6x + 5y &= 3 \\
-4x - 2y &= -14
\end{aligned}
$$

2.

$$
\begin{aligned}
9x - 7y &= -19 \\
5x - 3y &= -15
\end{aligned}
$$

3.

$$
\begin{aligned}
15x - 21y &= -63 \\
7y &= 5x + 21
\end{aligned}
$$

Answers

1. We can eliminate either variable here. To eliminate x, we can multiple the first equations by 2 and the second equation by 3 to get 12 - the LCM of 6 and 4.

$$\begin{aligned} 2(6x+5y=3) \quad &\Rightarrow \quad +\cancel{12x}+10y=6 \\ 3(-4x-2y=-14) \quad & \quad \underline{-\cancel{12x}-6y=-42} \\ & \quad 4y=-36 \\ & \quad y=-9 \end{aligned}$$

Now find x:

$$\begin{aligned} 6x+5(-9)&=3 \\ 6x-45&=3 \\ 6x&=48 \\ x&=8 \end{aligned}$$

Solution: (8, -9)

2. Again we can eliminate either variable. To eliminate y, we can multiply the first equation by 3 and the second equation by -7:

$$\begin{aligned} 3(9x-7y=-19) \quad &\Rightarrow \quad +27x-\cancel{21y}=-57 \\ -7(5x-3y=-15) \quad & \quad + \underline{-35x+\cancel{21y}=105} \\ & \quad -8x=48 \\ & \quad x=-6 \end{aligned}$$

Now find y:

$$\begin{aligned} 5(-6)-3y&=-15 \\ -30-3y&=-15 \\ -3y&=15 \\ y&=-5 \end{aligned}$$

Solution: (-6, -5)

3. To start this one we need to get the second equation in standard form. The resulting system will be:

$$\begin{aligned} 15x-21y&=-63 \\ -5x+7y&=21 \end{aligned}$$

This time we just need to multiply the second equation by 3 to eliminate x:

$$\begin{aligned} 15x-21y=-63 \quad &\Rightarrow \quad +\cancel{15x}-21y=-63 \\ 3(-5x+7y=21) \quad & \quad + \underline{-\cancel{15x}+21y=-63} \\ & \quad 0y=0 \\ & \quad 0=0 \end{aligned}$$

Solution: There are infinite solutions.

Problem Set

Solve the systems using linear combinations.

1.

$$\begin{aligned} 17x - 5y &= 4 \\ 2x + 7y &= 46 \end{aligned}$$

2.

$$\begin{aligned} 9x + 2y &= -13 \\ 11x + 5y &= 2 \end{aligned}$$

3.

$$\begin{aligned} 3x + 4y &= -16 \\ 5x + 5y &= -5 \end{aligned}$$

4.

$$\begin{aligned} 2x - 8y &= -8 \\ 3x + 7y &= 45 \end{aligned}$$

5.

$$\begin{aligned} 5x - 10y &= 60 \\ 6x + 3y &= -33 \end{aligned}$$

6.

$$\begin{aligned} 3x + 10y &= -50 \\ -5x - 7y &= 6 \end{aligned}$$

7.

$$\begin{aligned} 11x + 6y &= 30 \\ 13x - 5y &= -25 \end{aligned}$$

8.

$$\begin{aligned} 15x + 2y &= 23 \\ 18x - 9y &= -18 \end{aligned}$$

9.

$$\begin{aligned} 12x + 8y &= 64 \\ 17x - 12y &= 9 \end{aligned}$$

10.

$$\begin{aligned}11x-3y&=12\\33x-36&=9y\end{aligned}$$

11.

$$\begin{aligned}4x+3y&=0\\6x-13y&=35\end{aligned}$$

12.

$$\begin{aligned}18x+2y&=-2\\-12x-3y&=-7\end{aligned}$$

13.

$$\begin{aligned}-6x+11y&=-109\\8x-15y&=149\end{aligned}$$

14.

$$\begin{aligned}8x&=-5y-1\\-32x+20y&=8\end{aligned}$$

15.

$$\begin{aligned}10x-16y&=-12\\-15x+14y&=-27\end{aligned}$$

Set up and solve a system of equations to answer the following questions.

16. A mix of 35% almonds and 65% peanuts sells for \$5.70. A mix of 75% almonds and 25% peanuts sells for \$6.50. How much should a mix of 60% almonds and 40% peanuts sell for?
17. The Robinson family pays \$19.75 at the movie theater for 3 medium popcorns and 4 medium drinks. The Jamison family pays \$33.50 at the same theater for 5 medium popcorns and 7 medium drinks. How much would it cost for a couple to get 2 medium drinks and 2 medium popcorns?
18. A cell phone company charges extra when users exceed their included call time and text message limits. One user paid \$3.24 extra having talked for 240 extra minutes and sending 12 additional texts. A second user talked for 120 extra minutes and sent 150 additional texts and was charged \$4.50 above the regular fee. How much extra would a user be charged for talking 140 extra minutes and sending 200 additional texts?

Determining the Best Method (Graphing, Substitution, Linear Combinations)

Objective

Solve systems of linear equations using the most efficient method.

Guidance

Any of the methods (graphing, substitution, linear combination) learned in this unit can be used to solve a linear system of equations. Sometimes, however it is more efficient to use one method over another based on how the equations are presented. For example

- If both equations are presented in slope intercept form $(y = mx + b)$, then either graphing or substitution would be most efficient.
- If one equation is given in slope intercept form or solved for x, then substitution might be easiest.
- If both equations are given in standard form $(Ax + By = C)$, then linear combinations is usually most efficient.

Example A

Solve the following system:

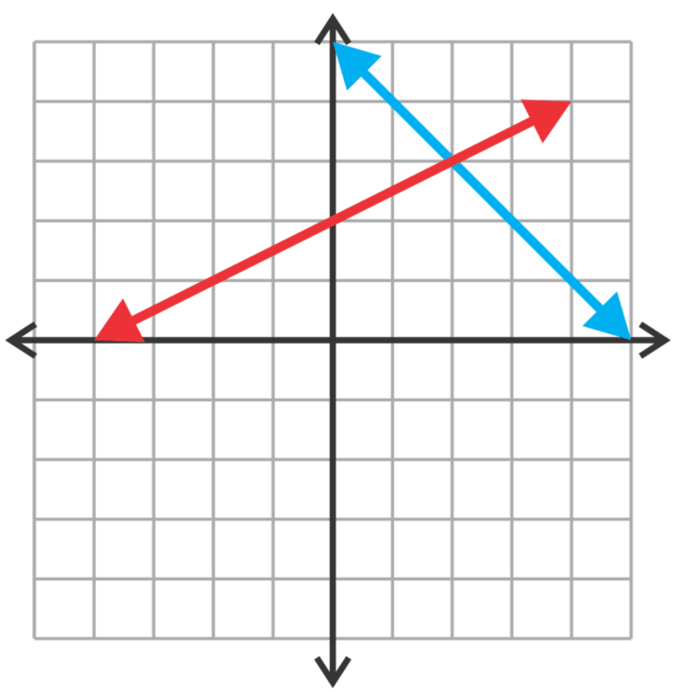
$$y = -x + 5$$
$$y = \frac{1}{2}x + 2$$

Solution: Since both equations are in slope intercept form we could easily graph these lines. The question is whether or not the intersection of the two lines will lie on the "grid" (whole numbers). If not, it is very difficult to determine an answer from a graph. One way to get around this difficulty is to use technology to graph the lines and find their intersection.

The first equation has a $y-$intercept of 5 and slope of -1. It is shown here graphed in **blue**.

The second equation has a $y-$intercept of 2 and a slope of $\frac{1}{2}$. It is shown here graphed in **red**.

The two lines clearly intersect at (2, 3).



Alternate Method: Substitution may be the preferred method for students who would rather solve equations algebraically. Since both of these equations are equal to y, we can let the right hand sides be equal to each other and solve for x:

$$\begin{aligned}
-x+5 &= \frac{1}{2}x+2 \\
2\left(-x+5 = \frac{1}{2}x+2\right) &\leftarrow \text{Multiplying the equation by 2 eliminates the fraction.} \\
-2x+10 &= x+4 \\
6 &= 3x \\
x &= 2
\end{aligned}$$

Now solve for y:

$$\begin{aligned}
y &= -(2)+5 \\
y &= 3
\end{aligned}$$

Solution: (2, 3)

Example B

Solve the system:

$$\begin{aligned}
15x+y &= 24 \\
y &= -4x+2
\end{aligned}$$

Solution: This time one of our equations is already solved for y. It is easiest here to use this expression to substitute into the other equation and solve:

$$\begin{aligned}
15x+(-4x+2) &= 24 \\
15x-4x+2 &= 24 \\
11x &= 22 \\
x &= 2
\end{aligned}$$

Now solve for y:

$$\begin{aligned}
y &= -4(2)+2 \\
y &= -8+2 \\
y &= -6
\end{aligned}$$

Solution: (2, -6)

Check your answer:

$$\begin{aligned}
15(2)+(-6) = 30-6 &= 24 \\
-6 = -4(2)+2 = -8+2 &= -6 \quad \boxed{\checkmark}
\end{aligned}$$

Example C

Solve the system:

$$\begin{aligned} -6x+11y &= 86 \\ 9x-13y &= -115 \end{aligned}$$

Solution: Both equations in this example are in standard form so the easiest method to use here is linear combinations. Since the LCM of 6 and 9 is 18, we will multiply the first equation by 3 and the second equation by 2 to eliminate x first:

$$\begin{aligned} 3(-6x+11y=86) \quad &\Rightarrow \quad -\cancel{18x}+33y=258 \\ 2(9x-13y=-115) \quad & \qquad \underline{\cancel{18x}-26y=-230} \\ & \qquad\qquad 7y=28 \\ & \qquad\qquad y=4 \end{aligned}$$

Now solve for x:

$$\begin{aligned} -6x+11(4) &= 86 \\ -6x+44 &= 86 \\ -6x &= 42 \\ x &= -7 \end{aligned}$$

Solution: (-7, 4)

Check your answer:

$$\begin{aligned} -6(-7)+11(4) &= 42+44=86 \\ 9(-7)-13(4) &= -63-52=-115 \quad \boxed{\checkmark} \end{aligned}$$

Example D

A rental car company, Affordable Autos, charges \$30 per day plus \$0.51 per mile driven. A second car rental company, Cheap Cars, charges \$25 per day plus \$0.57 per mile driven. For a short distance, Cheap Cars offers the better deal. At what point (after how many miles in a single day) does the Affordable Autos rental company offer the better deal? Set up and solve a system of linear equations using technology.

Solution: First set up equations to represent the total cost (for one day's rental) for each company:

Affordable Autos $\Rightarrow y=0.51x+30$

Cheap cars $\Rightarrow y=0.57x+25$

Now graph and solve this system using technology.

You may need to play with the graph window on the calculator to be able to view the intersection. A good window is: $0 \leq x \leq 125$ and $0 \leq y \leq 125$. Once we have the intersection in the viewing window we can go to the Calc menu and select Intersect. Now, select each of the lines and press enter to find the intersection: (83.3, 72.5). So, Affordable Autos has a better deal if we want to drive more than 83.3 miles during our one day rental.

Guided Practice

Solve the following systems using the most efficient method:

1.

$$\begin{aligned} y &= -3x+2 \\ y &= 2x-3 \end{aligned}$$

2.

$$\begin{aligned} 4x+5y &= -5 \\ x &= 2y-11 \end{aligned}$$

3.

$$\begin{aligned} 4x-5y &= -24 \\ -15x+7y &= -4 \end{aligned}$$

Answers

1. This one could be solved by graphing, graphing with technology or substitution. This time we will use substitution. Since both equations are solved for y, we can set them equal and solve for x:

$$\begin{aligned} -3x+2 &= 2x-3 \\ 5 &= 5x \\ x &= 1 \end{aligned}$$

Now solve for y:

$$\begin{aligned} y &= -3(1)+2 \\ y &= -3+2 \\ y &= -1 \end{aligned}$$

Solution: (1, -1)

2. Since the second equation here is solved for x, it makes sense to use substitution:

$$\begin{aligned} 4(2y-11)+5y &= -5 \\ 8y-44+5y &= -5 \\ 13y &= 39 \\ y &= 3 \end{aligned}$$

Now solve for x:

$$\begin{aligned} x &= 2(3)-11 \\ x &= 6-11 \\ x &= -5 \end{aligned}$$

Solution: (-5, 3)

3. This time, both equations are in standard form so it makes the most sense to use linear combinations. We can eliminate y by multiplying the first equation by 7 and the second equation by 5:

$$\begin{aligned}
7(4x-5y=-24) \quad &\Rightarrow \quad 28x-\cancel{35y}=-168 \\
5(-15x+7y=-4) \quad & \quad \underline{-75x+\cancel{35y}=-20} \\
& \quad -47x=-188 \\
& \quad x=4
\end{aligned}$$

Now find y:

$$\begin{aligned}
4(4)-5y&=-24 \\
16-5y&=-24 \\
-5y&=-40 \\
y&=8
\end{aligned}$$

Solution: (4, 8)

Problem Set

Solve the following systems using linear combinations.

1.

$$\begin{aligned}
5x-2y&=-1 \\
8x+4y&=56
\end{aligned}$$

2.

$$\begin{aligned}
3x+y&=-16 \\
-4x-y&=21
\end{aligned}$$

3.

$$\begin{aligned}
7x+2y&=4 \\
y&=-4x+1
\end{aligned}$$

4.

$$\begin{aligned}
6x+5y&=25 \\
x&=2y+24
\end{aligned}$$

5.

$$\begin{aligned}
-8x+10y&=-1 \\
2x-6y&=2
\end{aligned}$$

6.

$$3x+y=18$$
$$-7x+3y=-10$$

7.

$$2x+15y=-3$$
$$-3x-5y=-6$$

8.

$$15x-y=19$$
$$13x+2y=48$$

9.

$$x=-9y-2$$
$$-2x-15y=6$$

10.

$$3x-4y=1$$
$$-2x+3y=1$$

11.

$$x-y=2$$
$$3x-2y=-7$$

12.

$$3x+12y=-18$$
$$y=-\frac{1}{4}x-\frac{3}{2}$$

13.

$$-2x-8y=-2$$
$$x=\frac{1}{2}y+10$$

14.

$$14x+y=3$$
$$-21x-3y=-3$$

15.

$$y=\frac{4}{5}x+7$$
$$8x-10y=2$$

Solve the following word problem by creating and solving a system of linear equations.

16. Jack and James each buy some small fish for their new aquariums. Jack buys 10 clownfish and 7 goldfish for \$28.25. James buys 5 clownfish and 6 goldfish for \$17.25. How much does each type of fish cost?
17. The sum of two numbers is 35. The larger number is one less than three times the smaller number. What are the two numbers?
18. Rachel offers to go to the coffee shop to buy cappuccinos and lattes for her coworkers. She buys a total of nine drinks for \$35.75. If cappuccinos cost \$3.75 each and the lattes cost \$4.25 each, how many of each drink did she buy?

2.29 Graphing and Solving Linear Inequalities

Objective

Identify solutions to and solve linear inequalities.

Review Queue

1. Solve the inequalities:

a) $8-3x \le 7$

b) $\frac{3}{4}x > -9$

c) $|5x-2| < 8$

2. Graph the linear inequality: $y \ge 3x-5$

3. Graph the linear inequality: $2x+5y < 10$

Checking for Solutions to a System of Linear Inequalities

Objective

Determine whether or not a given point is a solution to a linear system of inequalities.

Watch This

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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/60097

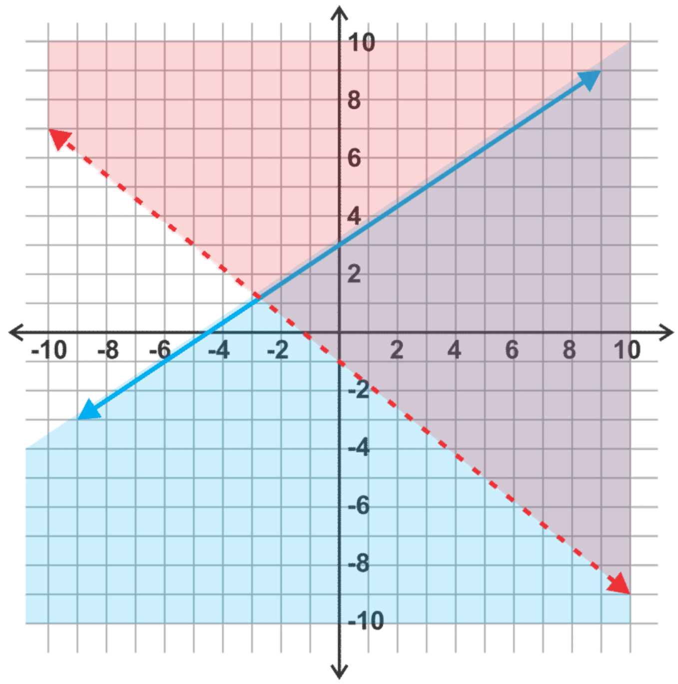
Khan Academy: Testing Solutions for a System of Inequalities

Guidance

A linear system of inequalities has an infinite number of solutions. Recall that when graphing a linear inequality the solution is a shaded region of the graph which contains all the possible solutions to the inequality. In a system, there are two linear inequalities. The solution to the system is all the points that satisfy both inequalities or the region in which the shading overlaps.

Example A

Given the system of linear inequalities shown in the graph, determine which points are solutions to the system.



a) (0, -1)

b) (2, 3)

c) (-2, -1)

d) (3, 5)

Solution:

a) The point (0, -1) is not a solution to the system of linear inequalities. It is a solution to $y \le \frac{2}{3}x+3$ (graphed in **blue**), but it lies on the line $y = -\frac{4}{5}x-1$ which is not included in the solution to $y > -\frac{4}{5}x-1$ (shown in **red**). The point must satisfy both inequalities to be a solution to the system.

b) The point (2, 3) lies in the overlapping shaded region and therefore is a solution to the system.

c) The point (-2, -1) lies outside the overlapping shaded region and therefore is not a solution the system.

d) The point (3, 5) lies on the line $y = \frac{2}{3}x+3$, which is included in the solution to $y \le \frac{2}{3}x+3$. Since this part of the line is included in the solution to $y > -\frac{4}{5}x-1$, it is a solution to the system.

Example B

Determine whether the following points are solutions to the system of linear inequalities:

$$\begin{aligned} 3x+2y &\ge 4 \\ x+5y &< 11 \end{aligned}$$

a) (3, 1)

b) (1, 2)

c) (5, 2)

d) (-3, 1)

Solution: This time we do not have a graph with which to work. Instead, we will plug the points into the equations to determine whether or not they satisfy the linear inequalities. A point must satisfy both linear inequalities to be a solution to the system.

a) Yes, $3(3)+2(1) \geq 4$ $\boxed{\checkmark}$ and $(3)+5(1) < 11$ $\boxed{\checkmark}$. Therefore, (3, 1) is a solution to the system.

b) No, $3(1)+2(2) \geq 4$ $\boxed{\checkmark}$, but $(1)+5(2) = 11$, so the point fails the second inequality.

c) No, $3(5)+2(2) \geq 4$ $\boxed{\checkmark}$, but $(5)+5(2) > 11$, so the point fails the second inequality.

d) No, $3(-3)+2(1) = -9+2 = -7 < 4$, so the point fails the first inequality. There is no need to check the point in the second inequality since it must satisfy both to be a solution.

Guided Practice

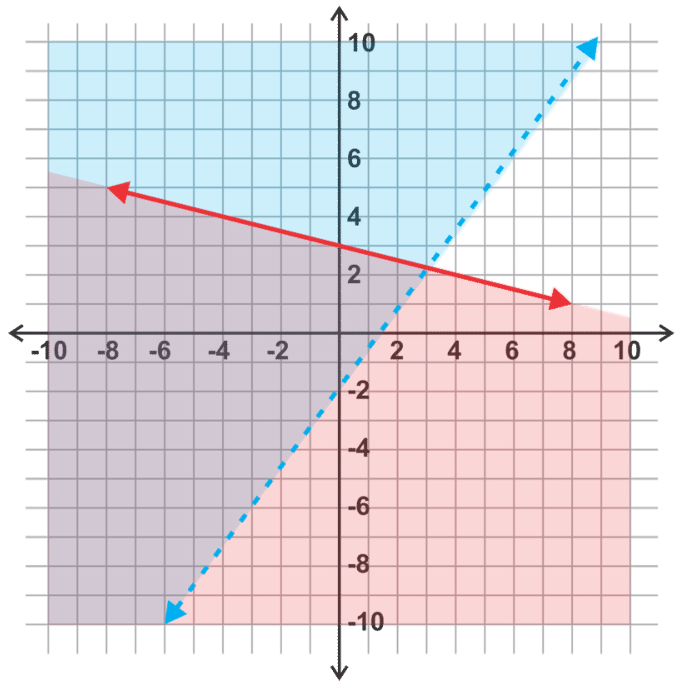
1. Determine whether the given points are solutions to the systems shown in the graph:

a) (-3, 3)

b) (4, 2)

c) (3, 2)

d) (-4, 4)



2. Determine whether the following points are solutions to the system:

$$y < 11x - 5$$
$$7x - 4y \geq 1$$

a) (4, 0)

b) (0, -5)

c) (7, 12)

d) (-1, -3)

Answers

1. a) (-3, 3) is a solution to the system because it lies in the overlapping shaded region.

b) (4, 2) is not a solution to the system. It is a solution to the red inequality only.

c) (3, 2) is not a solution to the system because it lies on the dashed blue line and therefore does not satisfy that inequality.

d) (-4, 4) is a solution to the system since it lies on the solid red line that borders the overlapping shaded region.

2. a) Yes, $0 < 11(4) - 5 \boxed{\checkmark}$, and $7(4) - 4(0) \geq 1 \boxed{\checkmark}$.

b) No, $-5 = 11(0) - 5$ so the first inequality is not satisfied.

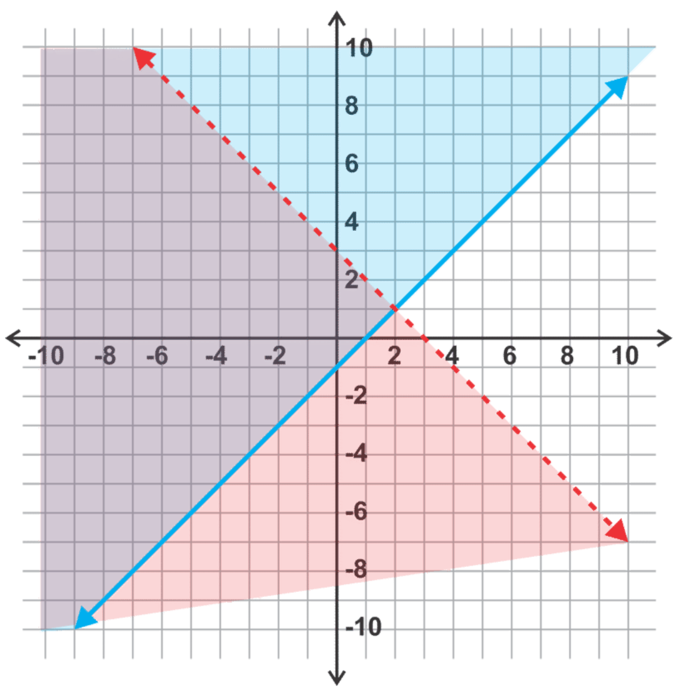
c) Yes, $12 < 11(7) - 5 \boxed{\checkmark}$, and $7(7) - 4(12) \geq 1 \boxed{\checkmark}$.

d) No, $-3 > 11(-1) - 5$ so the first inequality is not satisfied.

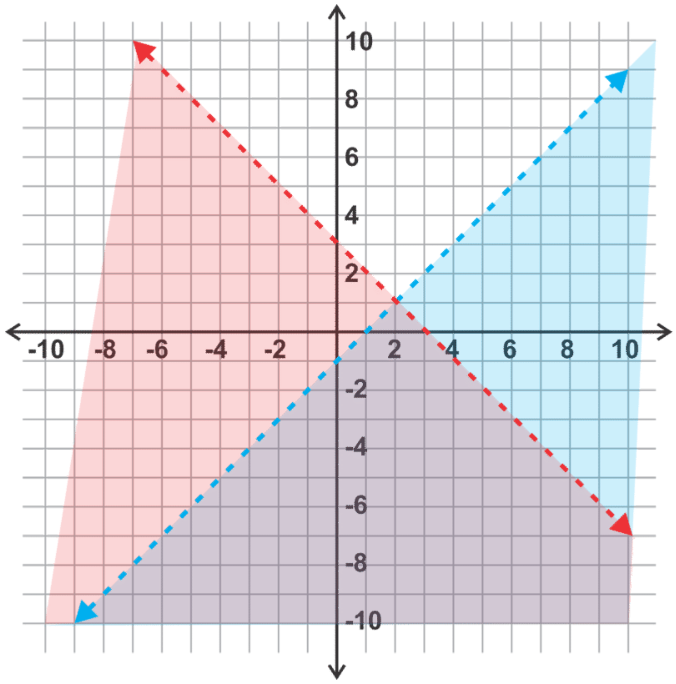
Problem Set

Given the four linear systems graphed below, match the point with the system(s) for which it is a solution.

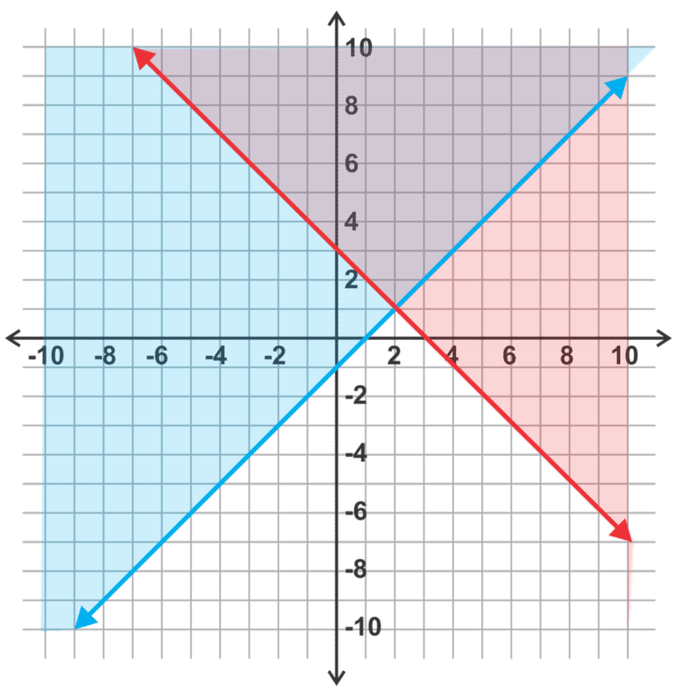
A.



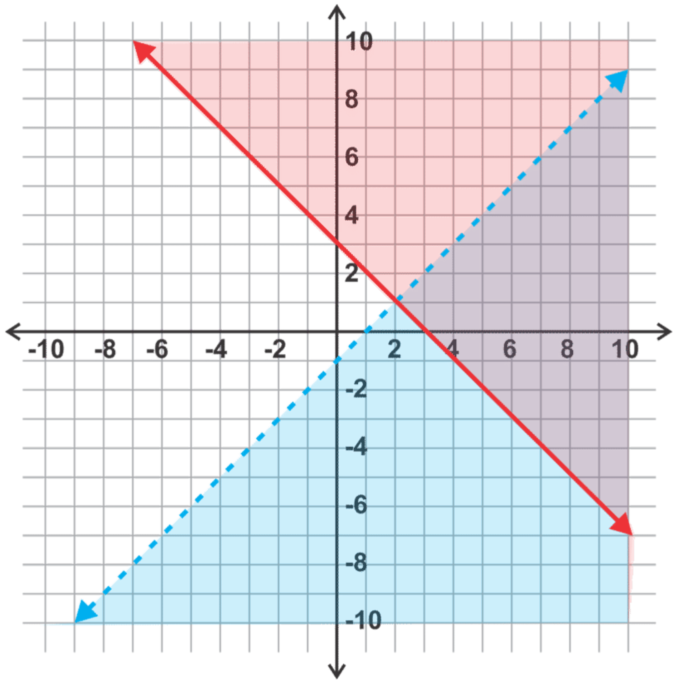
B.



C.



D.



1. (0, 3)
2. (2, -1)
3. (4, 3)
4. (2, -1)
5. (-3, 0)
6. (4, -1)
7. (-1, -2)
8. (2, 1)
9. (2, 5)
10. (0, 0)

Given the four linear systems below, match the point with the system(s) for which it is a solution.

A.

$$5x+2y \le 10$$
$$3x-4y > -12$$

B.

$$5x+2y < 10$$
$$3x-4y \le -12$$

C.

$$5x+2y > 10$$
$$3x-4y < -12$$

D.

$$5x+2y \ge 10$$
$$3x-4y \ge -12$$

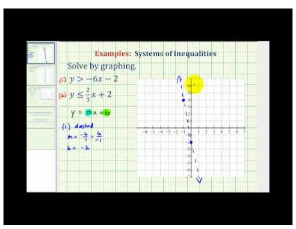
11. (0, 0)
12. (4, 6)
13. (0, 5)
14. (-3, 4)
15. (4, 3)
16. (0, 3)
17. (-8, -3)
18. (1, 6)
19. (4, -5)
20. (4, -2)

Graphing Systems of Linear Inequalities

Objective

Graph linear systems of inequalities with two or three equations and identify the region representing the solution set.

Watch This



MEDIA

Click image to the left or use the URL below.

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James Sousa: Ex 1: Graph a System of Linear Inequalities

Guidance

In this section we will be graphing two and three linear inequalities on the same grid and identifying where the shaded regions overlap. This overlapping region is the solution to the system. Note: If the shaded regions do not overlap, there is no solution as shown in Example B.

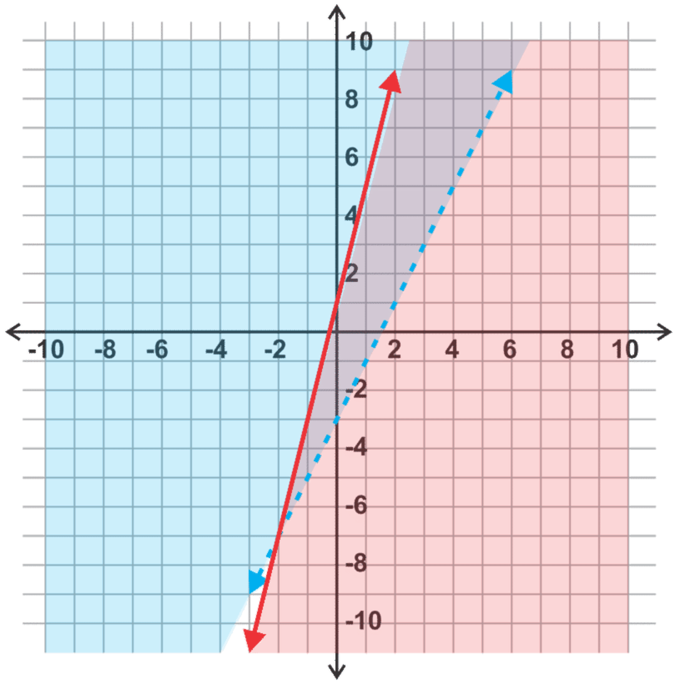
Example A

Graph and identify the solution to the system:

$$y > 2x - 3$$
$$y \leq 4x + 1$$

Solution:

Since both of these inequalities are given in slope intercept form, we can use the $y-$intercept and the slope to graph the lines. Since inequality 1 has "$y >$", we will make a dashed line to indicate that the line is not included in the solution and shade above the line where y is "greater" (where the $y-$axis is above) the line. Since inequality 2 has "$y \leq$" we will make a solid line to indicate that the line is included in the solution set and shade below the line where y is "less than" (where the $y-$axis is below) the line. Inequality 1 is graphed in **blue** and inequality 2 is graphed in **red**. The overlap of the shaded regions (**purple** shading) represents the solution.



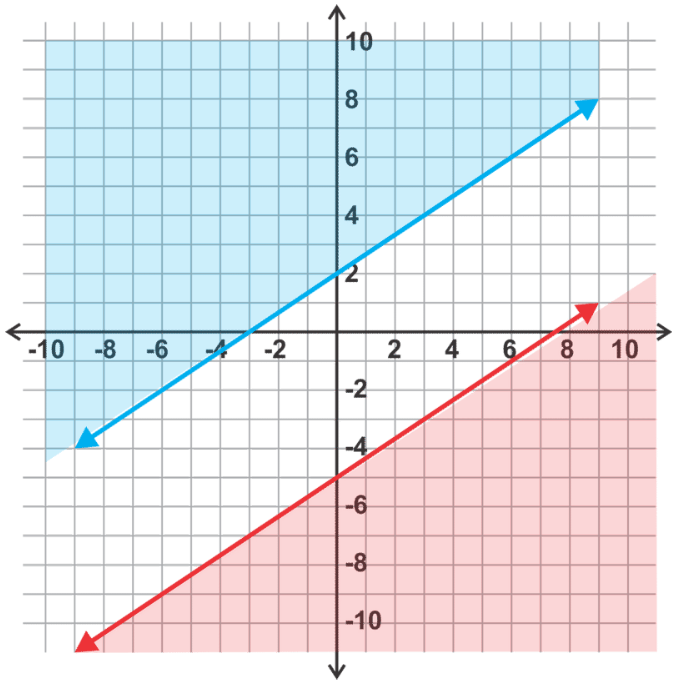
Example B

Graph and identify the solution to the system:

$$y \geq -\frac{2}{3}x+2$$
$$y \leq -\frac{2}{3}x-5$$

Solution:

Since inequality 1 has "$y \geq$", we will make a solid line to indicate that the line is included in the solution and shade above the line where y is "greater" (where the $y-$axis is above) the line. Since inequality 2 has "$y \leq$" we will make a dashed line to indicate that the line is not included in the solution set and shade below the line where y is "less than" (where the $y-$axis is below) the line. Inequality 1 is graphed in **blue** and inequality 2 is graphed in **red**. In this case the regions do not overlap. This indicates that there is no solution to the system.



Example C

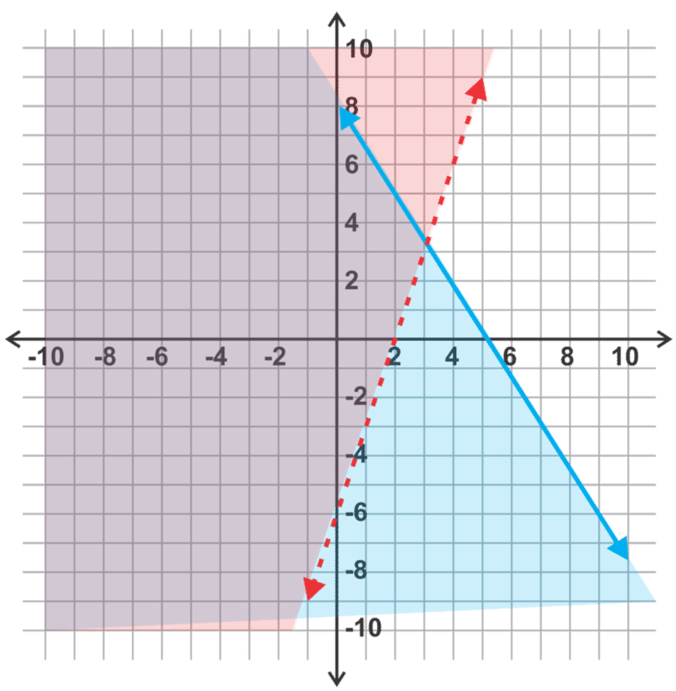
Graph and identify the solution to the system:

$$3x - y < 6$$
$$8x + 5y \leq 40$$

Solution: This time, let's use a different graphing technique. We can identify the intercepts for each equation and graph the lines using these points:

For $3x - y < 6$, the intercepts are (2, 0) and (0, -6).

For $8x + 5y \leq 40$, the intercepts are (5, 0) and (0, 8).



For the first inequality, the symbol is <so the line is dashed. Now, use a test point to determine which way to shade. (0, 0) is an easy point to test. $\Rightarrow 3(0)-(0)<6 \boxed{\checkmark}$ Since this is a true statement, (0, 0) is a solution to the inequality and we can shade on the side of the line with (0, 0).

For the second inequality, the symbol is $\leq$ so the line is solid. Using the same test point, $(0,0), \Rightarrow 8(0)+5(0) \leq 40 \boxed{\checkmark}$ This is a true statement so (0, 0) is a solution to the inequality and we can shade on the side of the line with (0, 0).

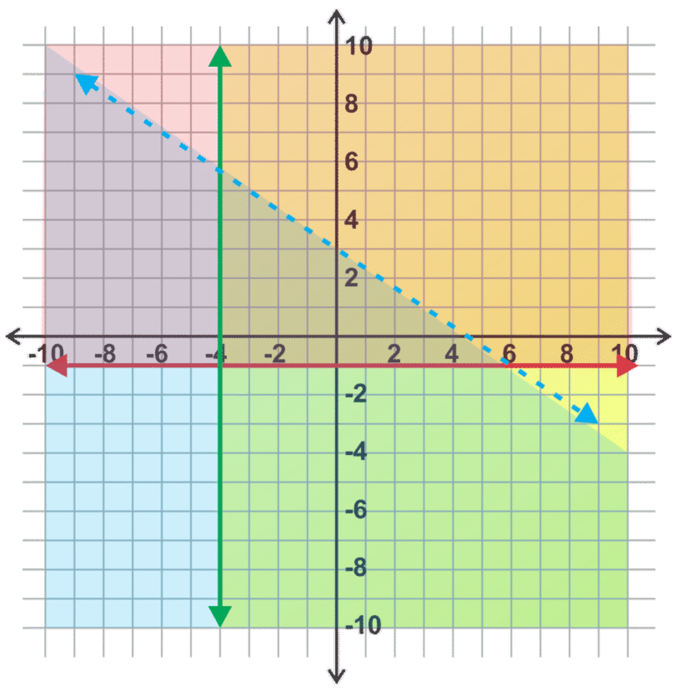
Again, Inequality 1 is graphed in **blue** and inequality 2 is graphed in **red**. The overlap of the shaded regions (**purple** shading) represents the solution.

Example D

Graph the system of linear inequalities:

$$
\begin{aligned}
& y<-\frac{2}{3} x+3 \\
& y \geq 1 \\
& x \geq-4
\end{aligned}
$$

Solution:



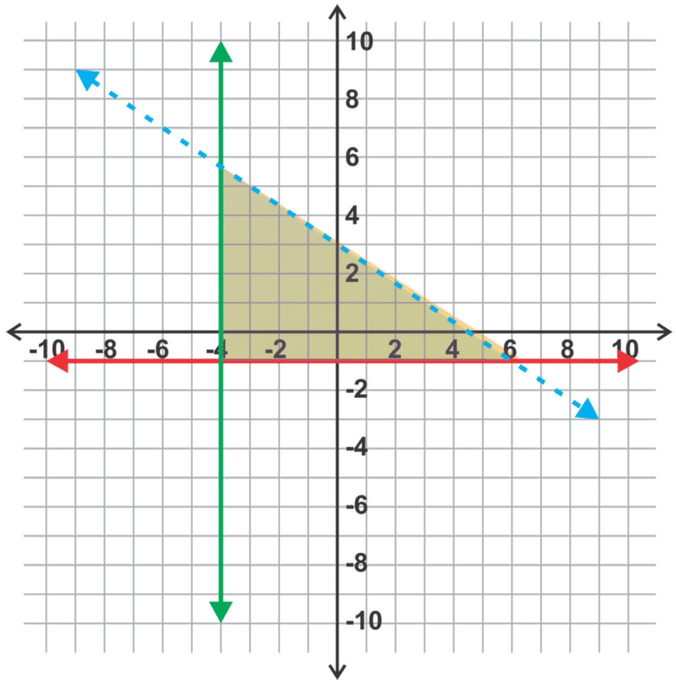
As in the previous concept, we will graph the lines and determine whether each line should be dashed or solid and which way to shade.

$y<-\frac{2}{3} x+3 \Rightarrow$ This inequality has a $y-$intercept of 3 and slope of $-\frac{2}{3}$. Since the inequality is <, we will shade below the dashed **blue** line.

$y \geq 1 \Rightarrow$ This is a horizontal line through (0, 1). The line will be solid and we shade above the **red** line.

$x \geq -4 \Rightarrow$ This is a vertical line through (-4, 0). The line will be solid and we will shade (yellow) to the right of the **green** line.

The solution to this system is the shaded region (triangular) in the center where all three shaded regions overlap. This region can be difficult to see in a graph so it is common practice to erase the shading that is not a part of the solution to make the solution region is more obvious.



Guided Practice

Graph and identify the solutions to the systems.

1.

$$y \leq \frac{1}{3}x+5$$
$$y > \frac{5}{4}x-2$$

2.

$$4x+y > 8$$
$$3x-5y \leq 15$$

3.

$$7x+2y \leq 14$$
$$3x-9y \geq 18$$

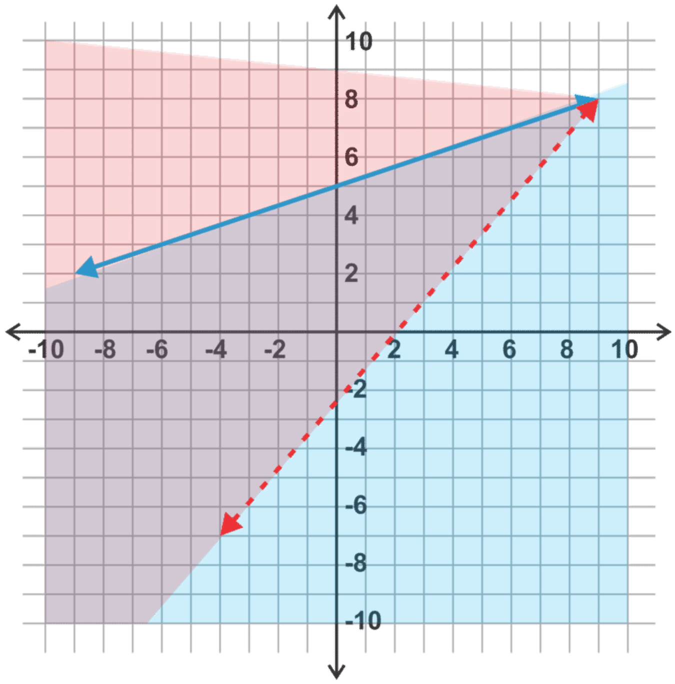
4.

$$y \geq 2x-3$$
$$2x+y > -8$$
$$y > -3$$

Answers

In each of the solutions below, the first inequality in the system is shown in **blue** and the second inequality is shown in **red**. The solution set is the overlapping shaded region in **purple**. When there are three inequalities, only the solution region is shown to eliminate confusion.

1.

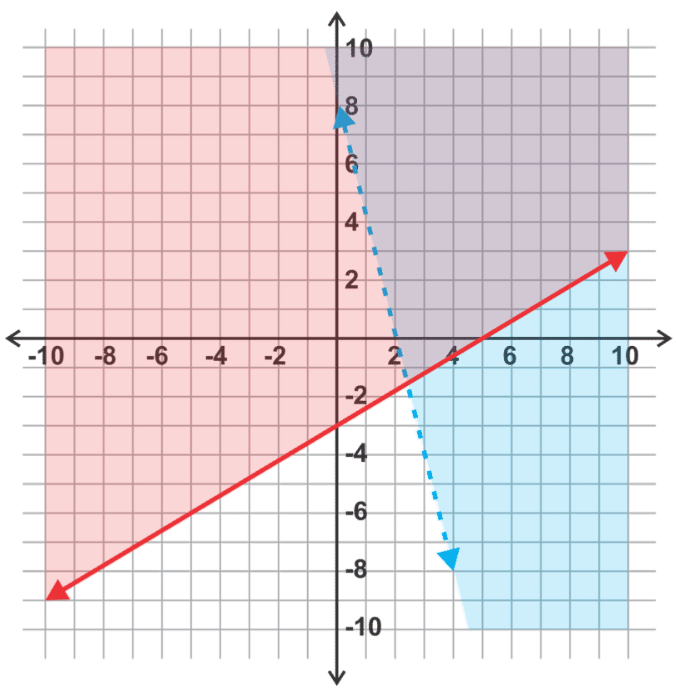


The inequalities in this system are both already in slope intercept form so we can graph them using the slope and $y-$intercept of each line and shade as shown below.

$y \leq \frac{1}{3}x + 5 \Rightarrow$ solid line and shade below

$y > \frac{5}{4}x - 2 \Rightarrow$ dashed line and shade above

2.

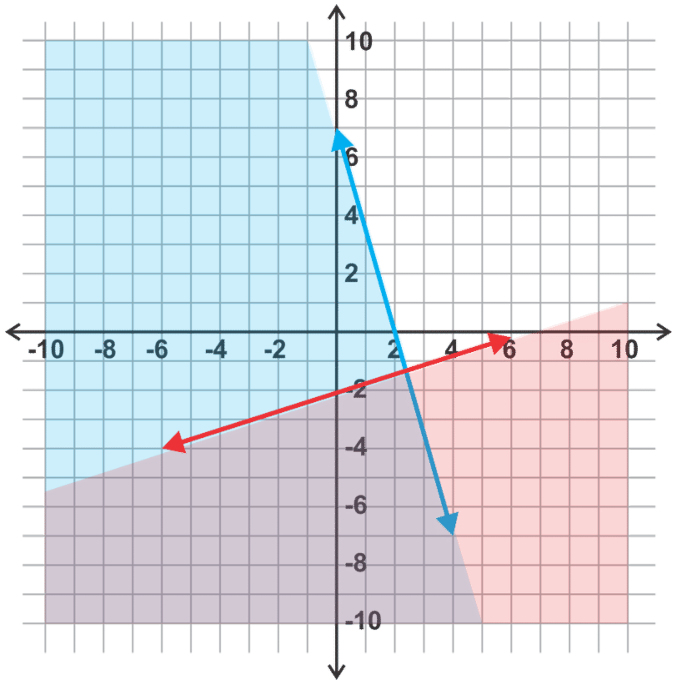


In these inequalities it is easiest to graph using the x and y intercepts. Once we have graphed the lines we can use a test point to determine which side should be shaded.

$4x + y > 8 \Rightarrow$ The intercepts are (2, 0) and (0, 8) and the line will be dashed. If we test the point (0, 0), the inequality is not true so we shade on the side of the line that does not contain (0, 0).

$3x-5y \leq 15 \Rightarrow$ The intercepts are (5, 0) and (0, -3) and the line will be solid. The test point (0, 0) satisfies the inequality so we shade on the side of the line that includes (0, 0).

3.

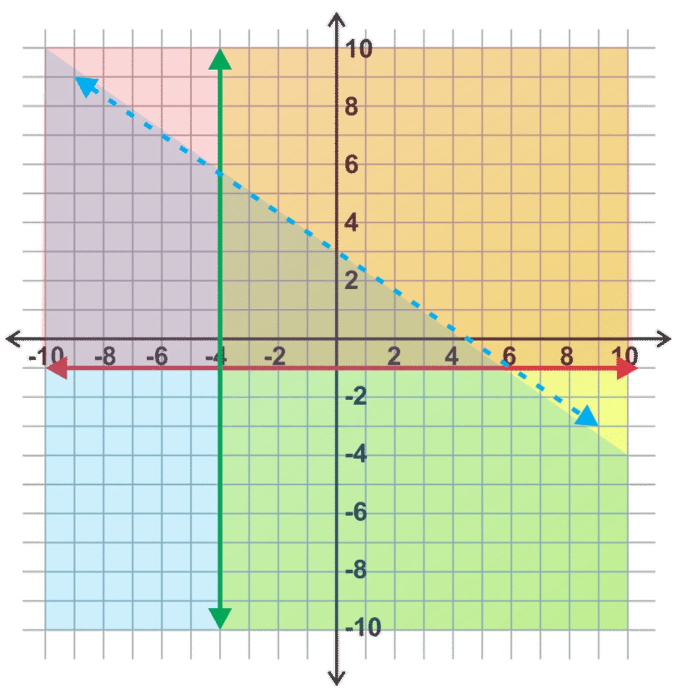


Again, it is easiest here to graph using the x and y intercepts. Once we have graphed the lines we can use a test point to determine which side should be shaded.

$7x+2y \leq 14 \Rightarrow$ The intercepts are (2, 0) and (0, 7) and the line will be solid. If we test the point (0, 0), the inequality is true so we shade on the side of the line that contains (0, 0).

$3x-9y \geq 18 \Rightarrow$ The intercepts are (6, 0) and (0, -2) and the line will be solid. The test point (0, 0) does not satisfy the inequality so we shade on the side of the line that does not include (0, 0).

4.



Inequality 1 can be graphed using the slope and $y-$intercept. This line will be solid and the shading will be above this line.

Inequality 2 can be graphed using intercepts. The line will be dashed and we can use a test point to determine that the shaded region will be above this line.

Inequality 3 is a horizontal line. It will be dashed and the shading is above this line.

The intersection of these three regions is shaded in **purple** on the graph.

Problem Set

Graph the following systems of linear inequalities.

1. $$\begin{aligned} y &> \frac{1}{2}x-2 \\ 4x+6y &\leq 24 \end{aligned}$$

2. $$\begin{aligned} y &> -\frac{3}{4}x-1 \\ y &> 3x+5 \end{aligned}$$

3. $$\begin{aligned} y &\leq -\frac{2}{3}x+2 \\ y &\geq -\frac{5}{3}x-1 \end{aligned}$$

4. $$\begin{aligned} y &\geq -x-3 \\ y &< \frac{1}{5}x+1 \end{aligned}$$

5. $$\begin{aligned} 5x-2y &> -10 \\ y &\leq -\frac{1}{3}x+2 \end{aligned}$$

6. $$\begin{aligned} y &> -\frac{4}{5}x-3 \\ y &> x \end{aligned}$$

7. $$\begin{aligned} y &\leq \frac{1}{2}x+4 \\ x-2y &\leq 2 \end{aligned}$$

8.

$$\begin{aligned} 7x - 3y &> -21 \\ x - 4y &< 8 \end{aligned}$$

9.

$$\begin{aligned} 6x + 5y &\leq 5 \\ 2x - 3y &\leq 12 \end{aligned}$$

10.

$$\begin{aligned} x &< 3 \\ y &\geq 2x + 1 \end{aligned}$$

11.

$$\begin{aligned} y &< 2 \\ y &\geq -2 \end{aligned}$$

12.

$$\begin{aligned} 2x - y &\leq 4 \\ 5x + 2y &> 10 \end{aligned}$$

13.

$$\begin{aligned} y &\leq -2x + 4 \\ y &\geq 5x + 4 \\ y &> \frac{1}{2}x - 1 \end{aligned}$$

14.

$$\begin{aligned} x + y &\leq 3 \\ x &\leq 3 \\ y &< 3 \end{aligned}$$

15.

$$\begin{aligned} x &> -2 \\ y &> -3 \\ 2x + y &\leq 2 \end{aligned}$$

16.

$$\begin{aligned} x &> -2 \\ x &\leq 4 \\ 3x + 5y &> 15 \end{aligned}$$

17.

$$\begin{aligned} 2x+3y &> 6 \\ 5x-2y &< -10 \\ x-3y &> 3 \end{aligned}$$

18.

$$\begin{aligned} y &\le x \\ y &\ge -x \\ x &< 5 \end{aligned}$$

2.30 Solving Linear Systems in Three Variables

Objective

Identify solutions to and solve linear systems in three variables.

Review Queue

1. Is the point (-6, 4) the solution to the system:

$$\begin{aligned} 2x+3y &= 0 \\ x+y &= -3 \end{aligned}$$

2. Solve the system using linear combinations:

$$\begin{aligned} 2x+8y &= -6 \\ x-y &= 7 \end{aligned}$$

3. Describe the situation (geometrically) for a linear system with no solution.

Solving a System in Three Variables Using Linear Combinations

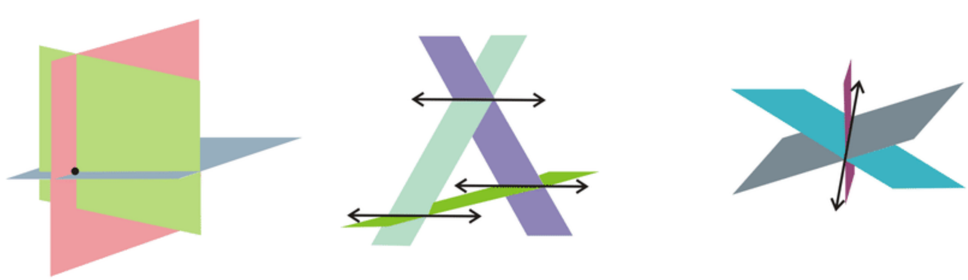
Objective

Understand the geometric situations that occur when there is one solution, infinite solutions and no solution to a system of equations in three variables. Solve systems in three variables and verify solutions to the system.

Guidance

An equation in three variables, such as $2x-3y+4z=10$, is an equation of a plane in three dimensions. In other words, this equation expresses the relationship between the three coordinates of each point on a plane. The solution to a system of three equations in three variables is a point in space which satisfies all three equations. When we add a third dimension we use the variable, z, for the third coordinate. For example, the point (3, -2, 5) would be $x=3, y=-2$ and $z=5$. A solution can be verified by substituting the x, y, and z values into the equations to see if they are valid.

A system of three equations in three variables consists of three planes in space. These planes could intersect with each other or not as shown in the diagrams below.



a unique solution exists no solution infinite solutions

- In the first diagram, the three planes intersect at a single point and thus a unique solution exists and can be found.
- The second diagram illustrates one way that three planes can exist and there is no solution to the system. It is also possible to have three parallel planes or two that are parallel and a third that intersects them. In any of these cases, there is no point that is in all three planes.
- The third diagram shows three planes intersecting in a line. Every point on this line is a solution to the system and thus there are infinite solutions.

To solve a system of three equations in three variables, we will be using the linear combination method. This time we will take two equations at a time to eliminate one variable and using the resulting equations in two variables to eliminate a second variable and solve for the third. This is just an extension of the linear combination procedure used to solve systems with two equations in two variables.

Example A

Determine whether the point, (6, -2, 5), is a solution to the system:

$$\begin{aligned} x-y+z&=13\\ 2x+5y-3z&=-13\\ 4x-y-6z&=-4 \end{aligned}$$

Solution: In order for the point to be a solution to the system, it must satisfy each of the three equations.

First equation: $(6)-(-2)+(5)=6+2+5=13\ \boxed{\checkmark}$

Second equation: $2(6)+5(-2)-3(5)=12-10-15=-13\ \boxed{\checkmark}$

Third equation: $4(6)-(-2)-6(5)=24+2-30=-4\ \boxed{\checkmark}$

The point, (6, -2, 5), satisfies all three equations. Therefore, it is a solution to the system.

Example B

Solve the system using linear combinations:

$$\begin{aligned} 2x+4y-3z&=-7\\ 3x-y+z&=20\\ x+2y-z&=-2 \end{aligned}$$

Solution: We can start by taking two equations at a time and eliminating the same variable. We can take the first two equations and eliminate z, then take the second and third equations and also eliminate z.

$$\begin{array}{lcr} 2x+4y-3z=-7 & \Rightarrow & 2x+4y-\cancel{3z}=-7\\ 3(3x-y+z=20) & & \underline{9x-3y+\cancel{3z}=60}\\ & & 11x+y=53 \end{array}$$

Result from equations 1 and 2: $11x+y=53$

$$\begin{array}{r} 3x-y+\cancel{z}=20\\ \underline{x+2y-\cancel{z}=-2}\\ 4x+y=18 \end{array}$$

Result from equations 2 and 3: $4x+y=18$

Now we have reduced our system to two equations in two variables. We can eliminate y most easily next and solve for x.

$$\begin{aligned} 11x+y&=53 \quad \Rightarrow & 11x+\cancel{y}&=53 \\ -1(4x+y&=18) & \underline{-4x-\cancel{y}}&\underline{=-18} \\ && 7x&=35 \\ && x&=5 \end{aligned}$$

Now use this value to find y:

$$\begin{aligned} 4(5)+y&=18 \\ 20+y&=18 \\ y&=-2 \end{aligned}$$

Finally, we can go back to one of the original three equations and use our x and y values to find z.

$$\begin{aligned} 2(5)+4(-2)-3z&=-7 \\ 10-8-3z&=-7 \\ 2-3z&=-7 \\ -3z&=-9 \\ z&=3 \end{aligned}$$

Therefore the solution is (5, -2, 3).

Don't forget to check your answer by substituting the point into each equation.

Equation 1: $2(5)+4(-2)-3(3)=10-8-9=-7\ \boxed{\checkmark}$

Equation 2: $3(5)-(-2)+(3)=15+2+3=20\ \boxed{\checkmark}$

Equation 3: $(5)+2(-2)-(3)=5-4-3=-2\ \boxed{\checkmark}$

Example C

Solve the system using linear combinations:

$$\begin{aligned} x-3y+4z&=14 \\ -x+2y-5z&=-13 \\ 2x+5y-3z&=-5 \end{aligned}$$

Solution: In this case, it is easiest to eliminate x first by combining the first two equations and then combining the second and third equations.

$$\begin{aligned} \cancel{x}-3y+4z&=14 \\ \underline{-\cancel{x}+2y-5z}&\underline{=-13} \\ -y-z&=1 \end{aligned}$$

Result from equations 1 and 2: $-y-z=1$

$$\begin{aligned} 2(-x+2y-5z=-13) \quad &\Rightarrow \quad -\cancel{2x}+4y-10z=-26 \\ 2x+5y-3z=-5 \quad & \quad\quad \underline{\cancel{2x}+5y-3z=-5} \\ & \quad\quad\quad 9y-13z=-31 \end{aligned}$$

Result from equations 2 and 3: $9y-13z=-31$

Now we have reduced our system to two equations in two variables. We can eliminate y most easily next and solve for z.

$$\begin{aligned} 9(-y-z=1) \quad &\Rightarrow \quad -\cancel{9y}-9z=9 \\ 9y-13z=-31 \quad & \quad\quad \underline{\cancel{9y}-13z=-31} \\ & \quad\quad -22z=-22 \\ & \quad\quad\quad z=1 \end{aligned}$$

Now use this value to find y:

$$\begin{aligned} -y-(1)&=1 \\ -y-1&=1 \\ -y&=2 \\ y&=-2 \end{aligned}$$

Finally, we can go back to one of the original three equations and use our y and z values to find x.

$$\begin{aligned} x-3(-2)+4(1)&=14 \\ x+6+4&=14 \\ x+10&=14 \\ x&=4 \end{aligned}$$

Therefore the solution is (4, -2, 1).

Don’t forget to check your answer by substituting the point into each equation.

Equation 1: $(4)-3(-2)+4(1)=4+6+4=14 \boxed{\checkmark}$

Equation 2: $-(4)+2(-2)-5(1)=-4-4-5=-13 \boxed{\checkmark}$

Equation 3: $2(4)+5(-2)-3(1)=8-10-3=-5 \boxed{\checkmark}$

Example D

Solve the system using linear combinations:

$$\begin{aligned} x+y+z&=5 \\ 5x+5y+5z&=20 \\ 2x+3y-z&=8 \end{aligned}$$

Solution: We can start by combining equations 1 and 2 together by multiplying the first equation by -5.

$$\begin{array}{rcr} -5(x+y+z=5) & \Rightarrow & -5x-5y-5z=-25 \\ 5x+5y+5z=20 & & \underline{5x+5y+5z=20} \\ & & 0=-5 \end{array}$$

Since the result is a false equation, there is no solution to the system.

Example E

Solve the system using linear combinations:

$$\begin{aligned} x+y+z&=3 \\ x+y-z&=3 \\ 2x+2y+z&=6 \end{aligned}$$

Solution: We can start by combining the first two equations and then combine equations 2 and 3.

$$\begin{array}{c} x+y+\cancel{z}=3 \\ \underline{x+y-\cancel{z}=3} \\ 2x+2y=6 \end{array}$$

Result from equations 1 and 2: $2x+2y=6$

$$\begin{array}{r} x+y-\cancel{z}=3 \\ \underline{2x+2y+\cancel{z}=6} \\ 3x+3y=9 \end{array}$$

Result from equations 2 and 3: $3x+3y=9$

Now we can combine these two equations and attempt to eliminate x or y.

$$\begin{array}{rcr} 3(2x+2y=6) & \Rightarrow & 6x+6y=18 \\ -2(3x+3y=9) & & \underline{-6x-6y=-18} \\ & & 0=0 \end{array}$$

This is a true statement. Therefore, there are infinite solutions to this system.

Guided Practice

1. Is the point, (-3, 2, 1), a solution to the system:

$$\begin{aligned} x+y+z&=0 \\ 4x+5y+z&=-1? \\ 3x+2y-4z&=-8 \end{aligned}$$

2. Solve the following system using linear combinations:

$$\begin{aligned} 5x - 3y + z &= -1 \\ x + 6y - 4z &= -17 \\ 8x - y + 5z &= 12 \end{aligned}$$

3. Solve the following system using linear combinations:

$$\begin{aligned} 2x + y - z &= 3 \\ x - 2y + z &= 5 \\ 6x + 3y - 3z &= 6 \end{aligned}$$

Answers

1. Check to see if the point satisfies all three equations.

Equation 1: $(-3) + (2) + (1) = -3 + 2 + 1 = 0 \boxed{\checkmark}$

Equation 2: $4(-3) + 5(2) + (1) = -12 + 10 + 1 = -1 \boxed{\checkmark}$

Equation 3: $3(-3) + 2(2) - 4(1) = -9 + 4 - 4 = -9 \neq -8 \boxed{\checkmark}$

Since the third equation is not satisfied by the point, the point is not a solution to the system.

2. Combine the first and second equations to eliminate z. Then combine the first and third equations to eliminate z.

$$\begin{aligned} 4(5x - 3y + z = -1) \quad &\Rightarrow \quad 20x - 12y + \cancel{4z} = -4 \\ x + 6y - 4z = -17 \quad &\qquad \underline{x + 6y - \cancel{4z} = -17} \\ &\qquad 21x - 6y = -21 \end{aligned}$$

Result from equations 1 and 2: $21x - 6y = -21$

$$\begin{aligned} -5(5x - 3y + z = -1) \quad &\Rightarrow \quad -25x + 15y - \cancel{5z} = 5 \\ 8x - y + 5z = 12 \quad &\qquad \underline{8x - y + \cancel{5z} = 12} \\ &\qquad -17x + 14y = 17 \end{aligned}$$

Result from equations 1 and 3: $-17x + 14y = 17$

Now we have reduced our system to two equations in two variables. We can eliminate y most easily next and solve for x.

$$\begin{aligned} 7(21x - 6y = -21) \quad &\Rightarrow \quad 147x - \cancel{42y} = -147 \\ 3(-17x + 14y = 17) \quad &\qquad \underline{-51x + \cancel{42y} = 51} \\ &\qquad 96x = -96 \\ &\qquad x = -1 \end{aligned}$$

Now find y:

$$\begin{aligned} 21(-1)-6y&=-21\\ -21-6y&=-21\\ -6y&=0\\ y&=0 \end{aligned}$$

Finally, we can go back to one of the original three equations and use our y and z values to find x.

$$\begin{aligned} 5(-1)-3(0)+z&=-1\\ -5+z&=-1\\ z&=4 \end{aligned}$$

Therefore the solution is (-1, 0, 4).

Don't forget to check your answer by substituting the point into each equation.

Equation 1: $5(-1)-3(0)+(4)=-5+4=-1\ \boxed{\checkmark}$

Equation 2: $(-1)+6(0)-4(4)=-1-16=-17\ \boxed{\checkmark}$

Equation 3: $8(-1)-(0)+5(4)=-8+20=12\ \boxed{\checkmark}$

3. Combine equations 1 and two to eliminate z. Then combine equations 2 and 3 to eliminate z.

$$\begin{aligned} 2x+y-\cancel{z}&=3\\ \underline{x-2y+\cancel{z}}&\underline{=5}\\ 3x-y&=8 \end{aligned}$$

Result from equations 1 and 2: $3x-y=8$

$$\begin{aligned} 3(x-2y+z=5) \quad &\Rightarrow \quad 3x-6y+\cancel{3z}=15\\ 6x+3y-3z=6 \quad & \quad\quad \underline{6x+3y-\cancel{3z}=6}\\ & \quad\quad 9x-3y=21 \end{aligned}$$

Result from equations 2 and 3: $9x-3y=21$

Now we have reduced our system to two equations in two variables. Now we can combine these two equations and attempt to eliminate another variable.

$$\begin{aligned} -3(3x-y=8) \quad &\Rightarrow \quad -9x+3y=-24\\ 9x-3y=21 \quad & \quad\quad \underline{9x-2y=21}\\ & \quad\quad 0=-3 \end{aligned}$$

Since the result is a false equation, there is no solution to the system.

Problem Set

1. Is the point, (2, -3, 5), the solution to the system:

$$\begin{aligned} 2x+5y-z &= -16 \\ 5x-y-3z &= -2? \\ 3x+2y+4z &= 20 \end{aligned}$$

2. Is the point, (-1, 3, 8), the solution to the system:

$$\begin{aligned} 8x+10y-z &= 14 \\ 11x+4y-3z &= -23? \\ 2x+3y+z &= 10 \end{aligned}$$

3. Is the point, (0, 3, 5), the solution to the system:

$$\begin{aligned} 5x-3y+2z &= 1 \\ 7x+2y-z &= 1? \\ x+4y-3z &= -3 \end{aligned}$$

Solve the following systems in three variables using linear combinations.

4.

$$\begin{aligned} 3x-2y+z &= 0 \\ 4x+y-3z &= -9 \\ 9x-2y+2z &= 20 \end{aligned}$$

5.

$$\begin{aligned} 11x+15y+5z &= 1 \\ 3x+4y+z &= -2 \\ 7x+13y+3z &= 3 \end{aligned}$$

6.

$$\begin{aligned} 2x+y+7z &= 5 \\ 3x-2y-z &= -1 \\ 4x-y+3z &= 5 \end{aligned}$$

7.

$$\begin{aligned} x+3y-4z &= -3 \\ 2x+5y-3z &= 3 \\ -x-3y+z &= -3 \end{aligned}$$

8.

$$\begin{aligned}3x+2y-5z&=-8\\3x+2y+5z&=-8\\6x+4y-10z&=-16\end{aligned}$$

9.

$$\begin{aligned}x+2y-z&=-1\\2x+4y+z&=10\\3x-y+8z&=6\end{aligned}$$

10.

$$\begin{aligned}x+y+z&=-3\\2x-y-z&=6\\4x+y+z&=0\end{aligned}$$

11.

$$\begin{aligned}4x+y+3z&=8\\8x+2y+6z&=15\\3x-3y-z&=5\end{aligned}$$

12.

$$\begin{aligned}2x+3y-z&=-1\\x-2y+3z&=-4\\-x+y-2z&=3\end{aligned}$$

2.31 Function Operations and the Inverse of a Function

Objective

To manipulate functions by taking the inverse and composing them with other functions.

Review Queue

$f(x) = x+5$ and $g(x) = x^2 - 4x + 8$. Find:

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $g(x) - f(x)$
4. $f(x) \cdot g(x)$

Function Operations

Objective

To add, subtract, multiply, divide and compose two or more functions.

Guidance

As you saw in the Review Queue, we have already dealt with adding, subtracting, and multiplying functions. To add and subtract, you combine like terms (see the *Adding and Subtracting Polynomials* concept). When multiplying, you either FOIL or use the "box" method (see the *Multiplying Polynomials* concept). When you add, subtract, or multiply functions, it is exactly the same as what you would do with polynomials, except for the notation. Notice, in the Review Queue, we didn't write out the entire function, just $f(x) - g(x)$, for example. Let's continue:

$$\begin{aligned} f(x) - g(x) &= (x+5) - (x^2 - 4x + 8) \\ &= x + 5 - x^2 + 4x - 8 \\ &= -x^2 + 5x - 3 \end{aligned}$$

Distribute the negative sign to the second function and combine like terms. Be careful! $f(x) - g(x) \neq g(x) - f(x)$. Also, this new function, $f(x) - g(x)$ has a different domain and range that either $f(x)$ or $g(x)$.

Example A

If $f(x) = \sqrt{x-8}$ and $g(x) = \frac{1}{2}x^2$, find fg and $\frac{f}{g}$. Determine any restrictions for $\frac{f}{g}$.

Solution: First, even though the x is not written along with the $f(x)$ and $g(x)$, it can be implied that f and g represent $f(x)$ and $g(x)$.

$fg = \sqrt{x-8} \cdot \frac{1}{2}x^2 = \frac{1}{2}x^2\sqrt{x-8}$

To divide the two functions, we will place f over g in a fraction.

$\frac{f}{g} = \frac{\sqrt{x-8}}{\frac{1}{2}x^2} = \frac{2\sqrt{x-8}}{x^2}$

To find the **restriction(s)** on this function, we need to determine what value(s) of x make the denominator zero because we cannot divide by zero. In this case $x \neq 0$. Also, the domain of $f(x)$ is only $x \geq 8$, because we cannot take the square root of a negative number. The portion of the domain where $f(x)$ is not defined is also considered part of the restriction. Whenever there is a restriction on a function, list it next to the function, separated by a semi-colon. We will not write $x \neq 0$ separately because it is included in $x \nless 8$.

$\frac{f}{g} = \frac{2\sqrt{x-8}}{x^2}$; $x \nless 8$

Now we will introduce a new way to manipulate functions; composing them. When you **compose** two functions, we put one function into the other, where ever there is an x. The notation can look like $f(g(x))$ or $f \circ g$, and is read "f of g of x". Let's do an example.

Example B

Using $f(x)$ and $g(x)$ from Example A find $f(g(x))$ and $g(f(x))$ and any restrictions on the domains.

Solution: For $f(g(x))$, we are going to put $g(x)$ into $f(x)$ everywhere there is an x-value.

$f(g(x)) = \sqrt{g(x) - 8}$

Now, substitute in the actual function for $g(x)$.

$$\begin{aligned} f(g(x)) &= \sqrt{g(x) - 8} \\ &= \sqrt{\frac{1}{2}x^2 - 8} \end{aligned}$$

To find the domain of $f(g(x))$, let's determine where x is defined. The radicand is equal to zero when $x = 4$ or $x = -4$. Between 4 and -4, the function is not defined because the square root would be negative. Therefore, the domain is all real numbers; $-4 \nless x \nless 4$.

Now, to find $g(f(x))$, we would put $f(x)$ into $g(x)$ everywhere there is an x-value.

$$\begin{aligned} g(f(x)) &= \frac{1}{2}[f(x)]^2 \\ &= \frac{1}{2}\left[\sqrt{x-8}\right]^2 \\ &= \frac{1}{2}(x-8) \\ &= \frac{1}{2}x - 4 \end{aligned}$$

Notice that $f(g(x)) \neq g(f(x))$. It is possible that $f \circ g = g \circ f$ and is a special case, addressed in the next concept. To find the domain of $g(f(x))$, we will determine where x is defined. $g(f(x))$ is a line, so we would think that the domain is all real numbers. However, while simplifying the composition, the square and square root canceled out. Therefore, any restriction on $f(x)$ or $g(x)$ would still exist. The domain would be all real numbers such that $x \geq 8$ from the domain of $f(x)$. *Whenever operations cancel, the original restrictions from the inner function still exist.* As with the case of $f(g(x))$, no simplifying occurred, so the domain was unique to that function.

Example C

If $f(x) = x^4 - 1$ and $g(x) = 2\sqrt[4]{x+1}$, find $g \circ f$ and the restrictions on the domain.

Solution: Recall that $g \circ f$ is another way of writing $g(f(x))$. Let's plug f into g.

$$\begin{aligned} g \circ f &= 2\sqrt[4]{f(x)+1} \\ &= 2\sqrt[4]{(x^4-1)+1} \\ &= 2\sqrt[4]{x^4} \\ &= 2|x| \end{aligned}$$

The final function, $g \circ f \neq 2x$ because x is being raised to the 4^{th} power, which will always yield a positive answer. Therefore, even when x is negative, the answer will be positive. For example, if $x=-2$, then $g \circ f = 2\sqrt[4]{(-2)^4} = 2 \cdot 2 = 4$.. An absolute value function has no restrictions on the domain. *This will always happen when even roots and powers cancel.* The range of this function is going to be all positive real numbers because the absolute value is never negative.

Recall, the previous example, however. The restrictions, if there are any, from the inner function, $f(x)$, still exist. Because there are no restrictions on $f(x)$, the domain of $g \circ f$ remains all real numbers.

Guided Practice

$f(x) = 5x^{-1}$ and $g(x) = 4x+7$. Find:

1. fg

2. $g-f$

3. $\frac{f}{g}$

4. $g(f(x))$ and the domain

5. $f \circ f$

Answers

1. fg is the product of $f(x)$ and $g(x)$.

$$\begin{aligned} fg &= 5x^{-1}(4x+7) \\ &= 20x^0 + 35x^{-1} \\ &= 20 + 35x^{-1} \ or \ \frac{20x+35}{x} \end{aligned}$$

Both representations are correct. Discuss with your teacher how s/he would like you to leave your answer.

2. Subtract $f(x)$ from $g(x)$ and simplify, if possible.

$$\begin{aligned} g-f &= (4x+7) - 5x^{-1} \\ &= 4x+7-5x^{-1} \ or \ \frac{4x^2+7x-5}{x} \end{aligned}$$

3. Divide $f(x)$ by $g(x)$. Don't forget to include the restriction(s).

$$\begin{aligned} \frac{f}{g} &= \frac{5x^{-1}}{4x+7} \\ &= \frac{5}{x(4x+7)};\ x \neq 0, -\frac{7}{4} \end{aligned}$$

Recall the properties of exponents. Anytime there is a negative exponent, it should be moved into the denominator. We set each factor in the denominator equal to zero to find the restrictions.

4. $g(f(x))$ is a composition function. Let's plug $f(x)$ into $g(x)$ everywhere there is an x.

$$\begin{aligned} g(f(x)) &= 4f(x)+7 \\ &= 4(5x^{-1})+7 \\ &= 20x^{-1}+7 \text{ or } \frac{20+7x}{x} \end{aligned}$$

The domain of $f(x)$ is all real numbers except $x \neq 0$, because we cannot divide by zero. Therefore, the domain of $g(f(x))$ is all real numbers except $x \neq 0$.

5. $f \circ f$ is a composite function on itself. We will plug $f(x)$ into $f(x)$ everywhere there is an x.

$$\begin{aligned} f(f(x)) &= 5(f(x))^{-1} \\ &= 5(5x^{-1})^{-1} \\ &= 5 \cdot 5^{-1}x^1 \\ &= x \end{aligned}$$

Vocabulary

Restriction
A value of the domain where x cannot be defined.

Composite Function
A function, $h(x)$, such that $h(x) = f(g(x))$, also written $h = f \circ g$. When $f(x)$ and $g(x)$ are composed, we plug $g(x)$ into $f(x)$ everywhere there is an x-value, resulting in a new function, $h(x)$. The domain of $h(x)$ is the set of all x-values that are in the domain of $f(x)$ and $g(x)$.

Problem Set

For problems 1-8, use the following functions to form the indicated compositions and clearly indicate any restrictions to the domain of the composite function.

$$f(x) = x^2+5 \qquad g(x) = 3\sqrt{x-5} \qquad h(x) = 5x+1$$

1. $f+h$
2. $h-g$
3. $\frac{f}{g}$
4. fh
5. $f \circ g$
6. $h(f(x))$
7. $g \circ f$
8. $f \circ g \circ h$

For problems 9-16, use the following functions to form the indicated compositions and clearly indicate any restrictions to the domain of the composite function.

$$p(x) = \frac{5}{x} \qquad q(x) = 5\sqrt{x} \qquad r(x) = \frac{\sqrt{x}}{5} \qquad s(x) = \frac{1}{5}x^2$$

9. ps
10. $\frac{q}{r}$
11. $q+r$
12. $p(q(x))$
13. $s(q(x))$
14. $q \circ s$
15. $q \circ p \circ s$
16. $p \circ r$

Inverse Functions

Objective

To find the inverse of a relation and function.

Guidance

By now, you are probably familiar with the term "inverse". Multiplication and division are inverses of each other. More examples are addition and subtraction and the square and square root. We are going to extend this idea to functions. An **inverse relation** maps the output values to the input values to create another relation. In other words, we switch the x and y values. The domain of the original relation becomes the range of the inverse relation and the range of the original relation becomes the domain of the inverse relation.

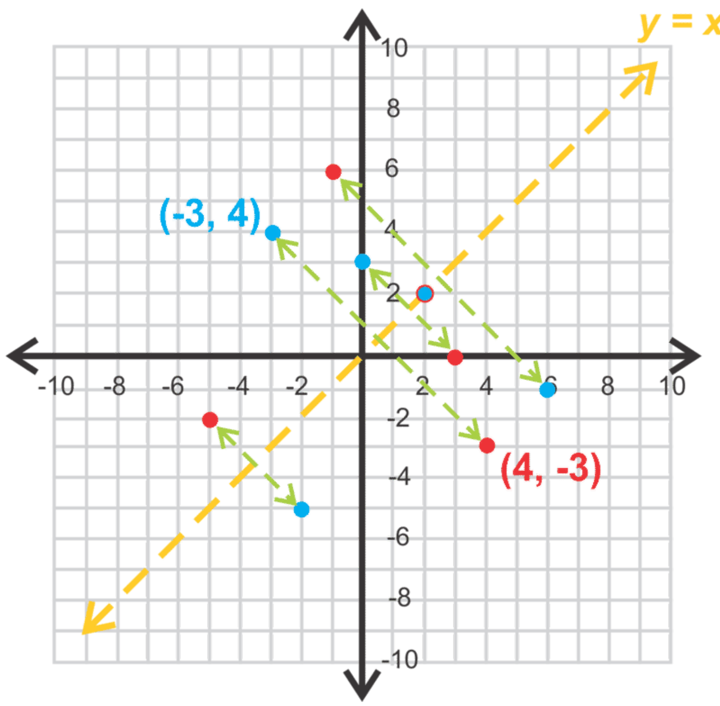
Example A

Find the inverse mapping of $S = \{(6,-1),(-2,-5),(-3,4),(0,3),(2,2)\}$.

Solution: Here, we will find the inverse of this relation by mapping it over the line $y = x$. As was stated above in the definition, the inverse relation switched the domain and range of the original function. So, the inverse of this relation, S, is S^{-1} (said "s inverse") and will flip all the x and y values.

$S^{-1} = \{(-1,6),(-5,-2),(4,-3),(3,0),(2,2)\}$

If we plot the two relations on the $x-y$ plane, we have:



The blue points are all the points in S and the red points are all the points in S^{-1}. Notice that the points in S^{-1} are a reflection of the points in S over the line, $y = x$. All inverses have this property.

If we were to fold the graph on $y = x$, each inverse point S^{-1} should lie on the original point from S. The point $(2, 2)$ lies on this line, so it has no reflection. Any value on this line will remain the same.

Domain of S: $x \in \{6, -2, -3, 0, 2\}$

Range of S: $y \in \{-1, -5, 4, 3, 2\}$

Domain of S': $x \in \{-1, -5, 4, 3, 2\}$

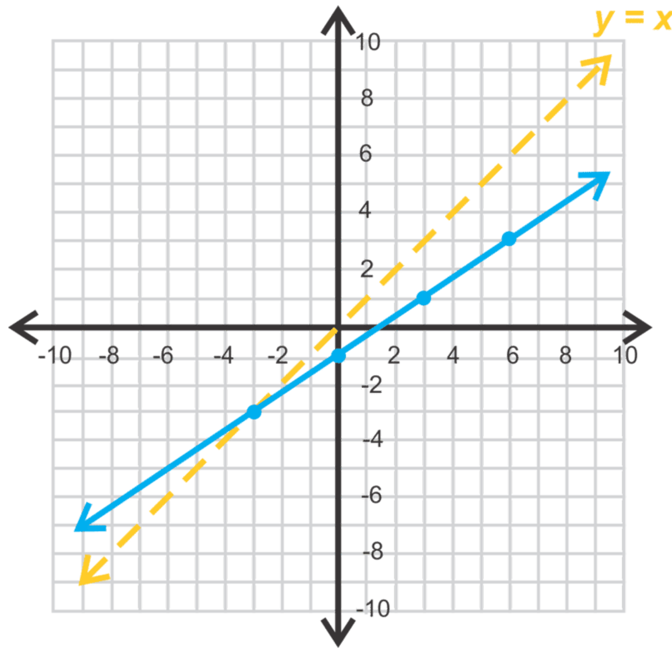
Range of S': $y \in \{6, -2, -3, 0, 2\}$

By looking at the domains and ranges of S and S^{-1}, we see that they are both functions (no x-values repeat). When the inverse of a function is also a function, we say that the original function is a **one-to-one function**. Each value maps one unique value onto another unique value.

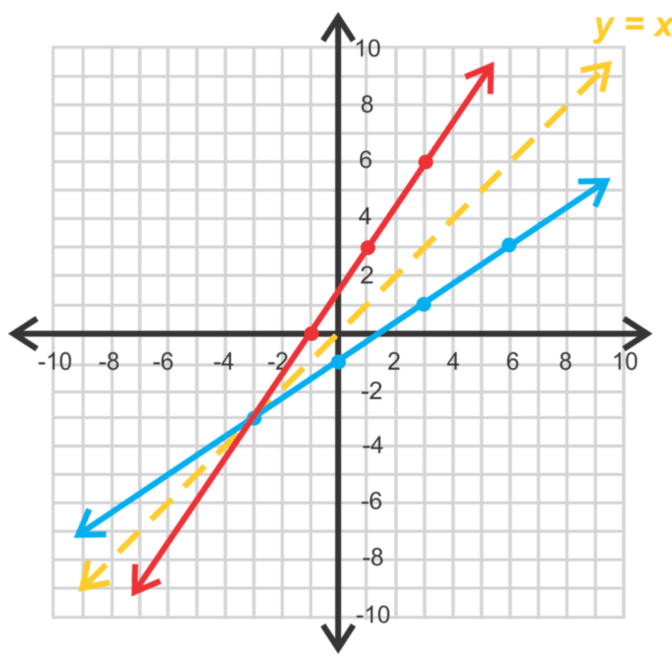
Example B

Find the inverse of $f(x) = \frac{2}{3}x - 1$.

Solution: This is a linear function. Let's solve by doing a little investigation. First, draw the line along with $y = x$ on the same set of axes.



Notice the points on the function (blue line). Map these points over $y = x$ by switching their x and y values. You could also fold the graph along $y = x$ and trace the reflection.



The red line in the graph to the right is the inverse of $f(x) = \frac{2}{3}x - 1$. Using slope triangles between (-1, 0) and (1, 3), we see that the slope is $\frac{3}{2}$. Use (-1, 0) to find the y-intercept.

$$f^{-1}(x) = \frac{3}{2}x + b$$
$$0 = \frac{3}{2}(-1) + b$$
$$\frac{3}{2} = b$$

The equation of the inverse, read "f inverse", is $f^{-1}(x) = \frac{3}{2}x + \frac{3}{2}$.

You may have noticed that the slopes of f and f^{-1} are reciprocals of each other. This will always be the case for linear functions. Also, the x-intercept of f becomes the y-intercept of f^{-1} and vise versa.

Alternate Method: There is also an algebraic approach to finding the inverse of any function. Let's repeat this example using algebra.

1. Change $f(x)$ to y.

$y = \frac{2}{3}x - 1$

2. Switch the x and y. Change y to y^{-1} for the inverse.

$x = \frac{2}{3}y^{-1} - 1$

3. Solve for y'.

$$\begin{aligned}
x &= \frac{2}{3}y^{-1} - 1 \\
\frac{3}{2}(x+1) &= \frac{3}{2} \cdot \left(\frac{2}{3}y^{-1}\right) \\
\frac{3}{2}x + \frac{3}{2} &= y^{-1}
\end{aligned}$$

The algebraic method will work for any type of function.

Example C

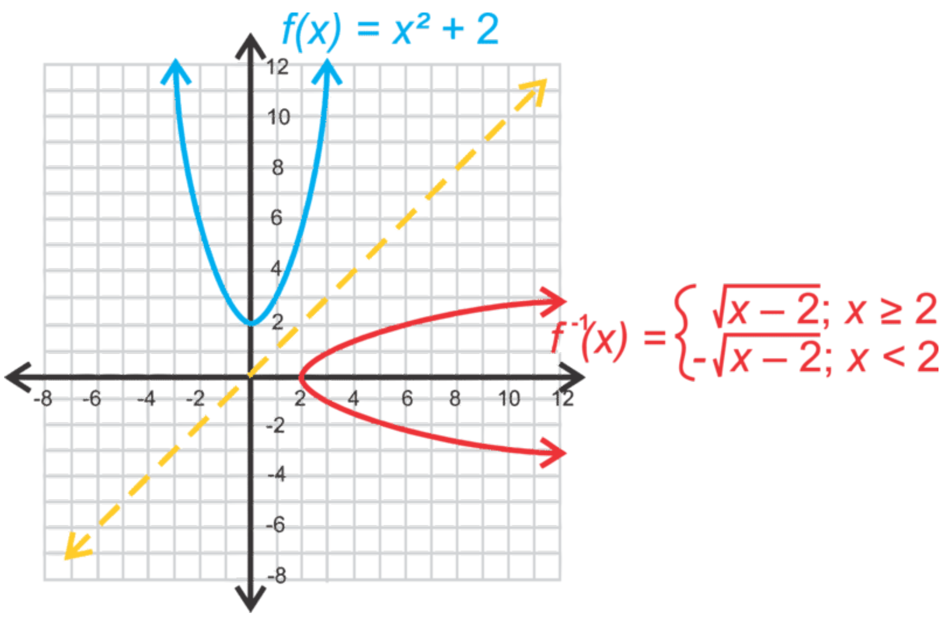
Determine if $g(x) = \sqrt{x-2}$ and $f(x) = x^2 + 2$ are inverses of each other.

Solution: There are two different ways to determine if two functions are inverses of each other. The first, is to find f^{-1} and g^{-1} and see if $f^{-1} = g$ and $g^{-1} = f$.

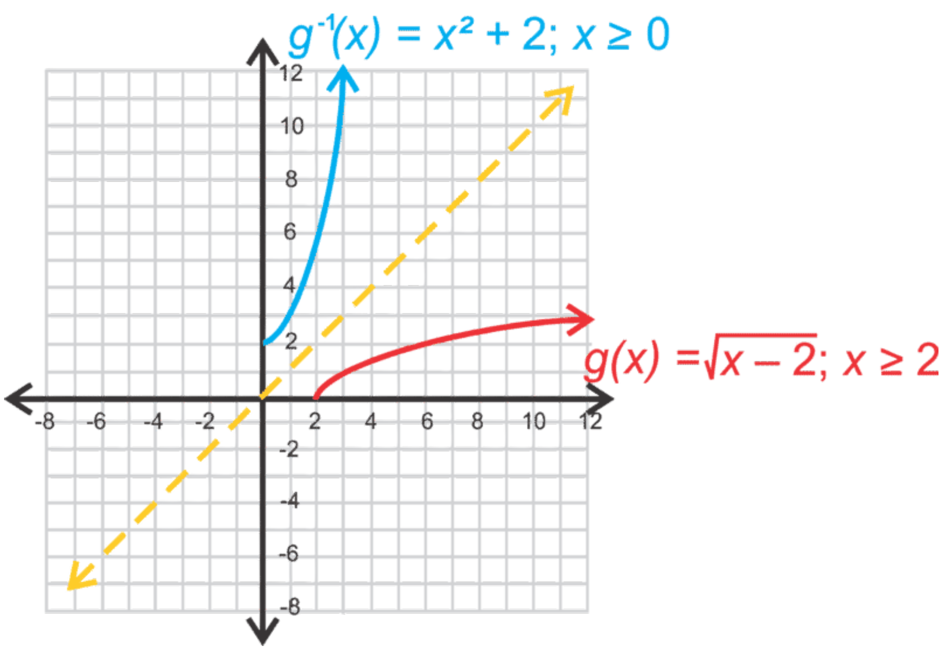
$$\begin{aligned}
x &= \sqrt{y^{-1} - 2} \\
x^2 &= y^{-1} - 2 \\
x^2 + 2 &= y^{-1} = g^{-1}(x)
\end{aligned} \qquad \textit{and} \qquad \begin{aligned}
x &= (y^{-1})^2 + 2 \\
x - 2 &= (y^{-1})^2 \\
\pm\sqrt{x-2} &= y^{-1} = f^{-1}(x)
\end{aligned}$$

Notice the $\pm$ sign in front of the square root for f^{-1}. That means that g^{-1} is $\sqrt{x-2}$ and $-\sqrt{x-2}$.

Therefore, f^{-1} is not really a function because it fails the vertical line test. However, if you were to take each part separately, individually, they are functions. You can also think about reflecting $f(x)$ over $y = x$. It would be a parabola on its side, which is not a function.



The inverse of g would then be only half of the parabola, see below. Despite the restrictions on the domains, f and g are inverses of each other.



Alternate Method: The second, and easier, way to determine if two functions are inverses of each other is to use composition. If $f \circ g = g \circ f = x$, then f and g are inverses of each other. Think about it; if everything cancels out and all that remains is x, each operation within the functions are opposites, making the functions "opposites" or inverses of each other.

$$\begin{aligned} f \circ g &= \sqrt{(x^2+2)-2} \\ &= \sqrt{x^2} \\ &= x \end{aligned} \qquad \textit{and} \qquad \begin{aligned} g \circ f &= \sqrt{x-2}^2+2 \\ &= x-2+2 \\ &= x \end{aligned}$$

Because $f\circ g = g\circ f = x$, f and g are inverses of each other. Both $f\circ g = x$ and $g\circ f = x$ in order for f and g to be inverses of each other.

Guided Practice

1. Find the inverse of $g(x) = -\frac{3}{4}x + 12$ algebraically.

2. Find the inverse of $f(x) = 2x^3 + 5$ algebraically. Is the inverse a function?

3. Determine if $h(x) = 4x^4 - 7$ and $j(x) = \frac{1}{4}\sqrt[4]{x-7}$ are inverses of each other using compositions.

Answers

1. Use the steps given in the Alternate Method for Example B.

$$\begin{aligned} y &= -\frac{3}{4}x + 12 \\ x &= -\frac{3}{4}y^{-1} + 12 \\ x - 12 &= -\frac{3}{4}y^{-1} \\ -\frac{4}{3}(x-12) &= y^{-1} \\ g^{-1}(x) &= -\frac{4}{3}x + 16 \end{aligned}$$

2. Again, use the steps from Example B.

$$\begin{aligned} y &= 2x^3 + 5 \\ x &= 2(y^{-1})^3 + 5 \\ x - 5 &= 2(y^{-1})^3 \\ \frac{x-5}{2} &= (y^{-1})^3 \\ f^{-1}(x) &= \sqrt[3]{\frac{x-5}{2}} \end{aligned}$$

Yes, f^{-1} is a function. Plot in your graphing calculator if you are unsure and see if it passes the vertical line test.

3. First, find $h(j(x))$.

$$\begin{aligned} h(j(x)) &= 4\left(\frac{1}{4}\sqrt[4]{x+7}\right)^4 - 7 \\ &= 4\cdot\left(\frac{1}{4}\right)^4 x + 7 - 7 \\ &= \frac{1}{64}x \end{aligned}$$

Because $h(j(x)) \neq x$, we know that h and j are not inverses of each other. Therefore, there is no point to find $j(h(x))$.

Vocabulary

Inverse Relation/Function

When a relation or function's output values are mapped to create input values for a new relation (or function). The input values of the original function would become the output values for the new relation (or function).

One-to-one Function
When the inverse of a function is also a function.

Problem Set

Write the inverses of the following functions. State whether or not the inverse is a function.

1. $(2,3),(-4,8),(-5,9),(1,1)$
2. $(9,-6),(8,-5),(7,3),(4,3)$

Find the inverses of the following functions algebraically. Note any restrictions to the domain of the inverse functions.

3. $f(x) = 6x-9$
4. $f(x) = \frac{1}{4x+3}$
5. $f(x) = \sqrt{x+7}$
6. $f(x) = x^2+5$
7. $f(x) = x^3-11$
8. $f(x) = \sqrt[5]{x+16}$

Determine whether f and g are inverses of each other by checking to see whether finding $f \circ g = x$ or $g \circ f = x$. You do not need to show both.

9. $f(x) = \frac{2}{3}x-14$ and $g(x) = \frac{3}{2}x+21$
10. $f(x) = \frac{x+5}{8}$ and $g(x) = 8x+5$
11. $f(x) = \sqrt[3]{3x-7}$ and $g(x) = \frac{x^3}{3}-7$
12. $f(x) = \frac{x}{x-9}, x \neq 9$ and $g(x) = \frac{9x}{x-1}$

Find the inverses of the following functions algebraically. Note any restrictions to the domain of the inverse functions. These problems are a little trickier as you will need to factor out the y variable to solve. Use the example below as a guide.

$f(x) = \frac{3x+13}{2x-11}$

Example:

1. $= \frac{3y+13}{2y-11}$ First, switch x and y
2. $2xy-11x = 3y+13$ Multiply both sides by $2y-11$ to eliminate the fraction
3. $2xy-3y = 11x+13$ Now rearrange the terms to get both terms with y in them on one side and everything else on the other side
4. $y(2x-3) = 11x+13$ Factor out the y
5. $y = \frac{11x+13}{2x-3}$ Finally, Divide both sides by $2x-3$ to isolate y.

So, the inverse of $f(x) = \frac{3x+13}{2x-11}, x \neq \frac{11}{2}$ is $f^{-1}(x) = \frac{11x+13}{2x-3}, x \neq \frac{3}{2}$.

13. $f(x) = \frac{x+7}{x}, x \neq 0$
14. $f(x) = \frac{x}{x-8}, x \neq 8$

Multi-step problem.

15. In many countries, the temperature is measured in degrees Celsius. In the US we typically use degrees Fahrenheit. For travelers, it is helpful to be able to convert from one unit of measure to another. The following problem will help you learn to do this using an inverse function.

a. The temperature at which water freezes will give us one point on a line in which x represents the degrees in Celsius and y represents the degrees in Fahrenheit. Water freezes at 0 degrees Celsius and 32 degrees Fahrenheit so the first point is (0, 32). The temperature at which water boils gives us the second point (100, 212), because water boils at 100 degrees Celsius or 212 degrees Fahrenheit. Use this information to show that the equation to convert from Celsius to Fahrenheit is $y = \frac{9}{5}x + 32$ or $F = \frac{9}{5}C + 32$.
b. Find the inverse of the equation above by solving for C to derive a formula that will allow us to convert from Fahrenheit to Celsius.
c. Show that your inverse is correct by showing that the composition of the two functions simplifies to either F or C (depending on which one you put into the other.)

2.32 Exponential Growth and Decay

Objective

To analyze and use exponential growth and decay functions.

Review Queue

Simplify the following expressions. Your final answer should have only positive exponents.

1. $2x^2 \cdot 6x^4$
2. $\frac{5xy^{-1}}{15x^3y^3}$
3. $(3x^2y^5)^3$
4. Solve for x: $2^x = 32$

Exponential Growth Function

Objective

To analyze an exponential growth function and its graph.

Guidance

An **exponential function** has the variable in the exponent of the expression. All exponential functions have the form: $f(x) = a \cdot b^{x-h} + k$, where h and k move the function in the x and y directions respectively, much like the other functions we have seen in this text. b is the base and a changes how quickly or slowly the function grows. Let's take a look at the parent graph, $y = 2^x$.

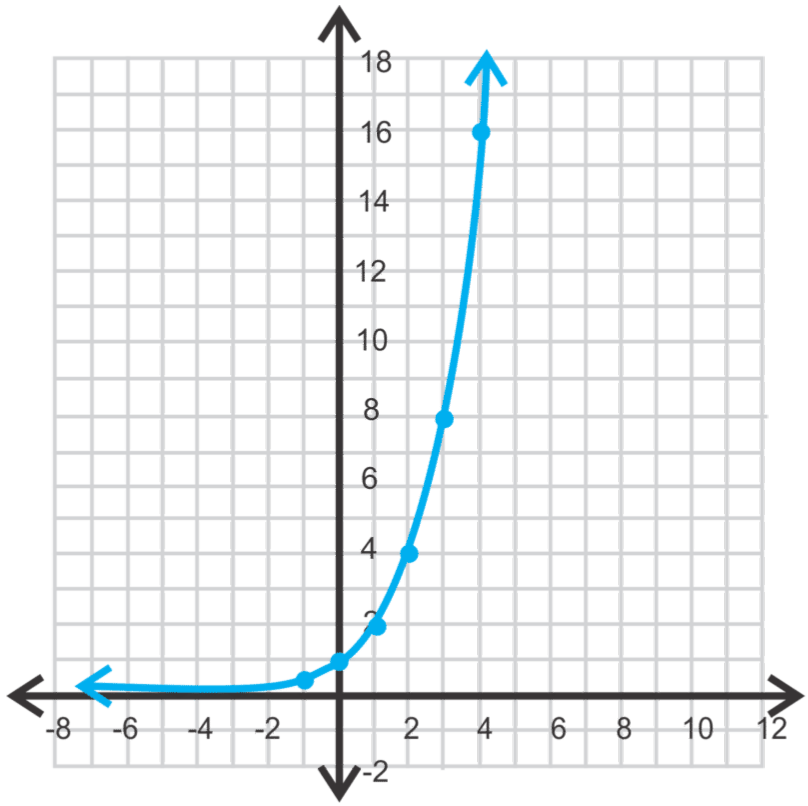
Example A

Graph $y = 2^x$. Find the y-intercept.

Solution: Let's start by making a table. Include some positive and negative values for x and zero.

TABLE 2.9:

x	2^x	y
3	2^3	8
2	2^2	4
1	2^1	2
0	2^0	1
-1	2^{-1}	$\frac{1}{2}$
-2	2^{-2}	$\frac{1}{4}$
-3	2^{-3}	$\frac{1}{8}$



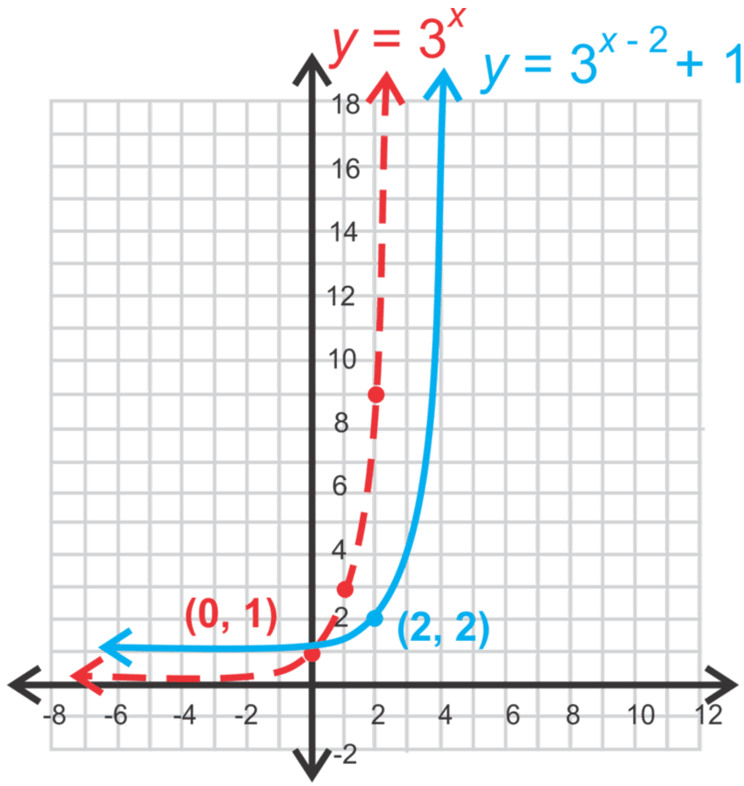
This is the typical shape of an **exponential growth function**. The function grows "exponentially fast". Meaning, in this case, the function grows in powers of 2. For an exponential function to be a growth function, $a > 0$ and $b > 1$ and h and k are both zero $(y = ab^x)$. From the table, we see that the y-intercept is (0, 1).

Notice that the function gets very, very close to the x-axis, but never touches or passes through it. Even if we chose $x = -50$, y would be $2^{-50} = \frac{1}{2^{50}}$, which is still not zero, but very close. In fact, the function will never reach zero, even though it will get smaller and smaller. Therefore, this function approaches the line $y = 0$, but will never touch or pass through it. This type of boundary line is called an **asymptote**. In the case with all exponential functions, there will be a horizontal asymptote. If $k = 0$, then the asymptote will be $y = 0$.

Example B

Graph $y = 3^{x-2} + 1$. Find the y-intercept, asymptote, domain and range.

Solution: This is not considered a growth function because h and k are not zero. To graph something like this (without a calculator), start by graphing $y = 3^x$ and then shift it h units in the x-direction and k units in the y-direction.



Notice that the point (0, 1) from $y = 3^x$ gets shifted to the right 2 units and up one unit and is (2, 2) in the translated function, $y = 3^{x-2} + 1$. Therefore, the asymptote is $y = 1$. To find the y-intercept, plug in $x = 0$.

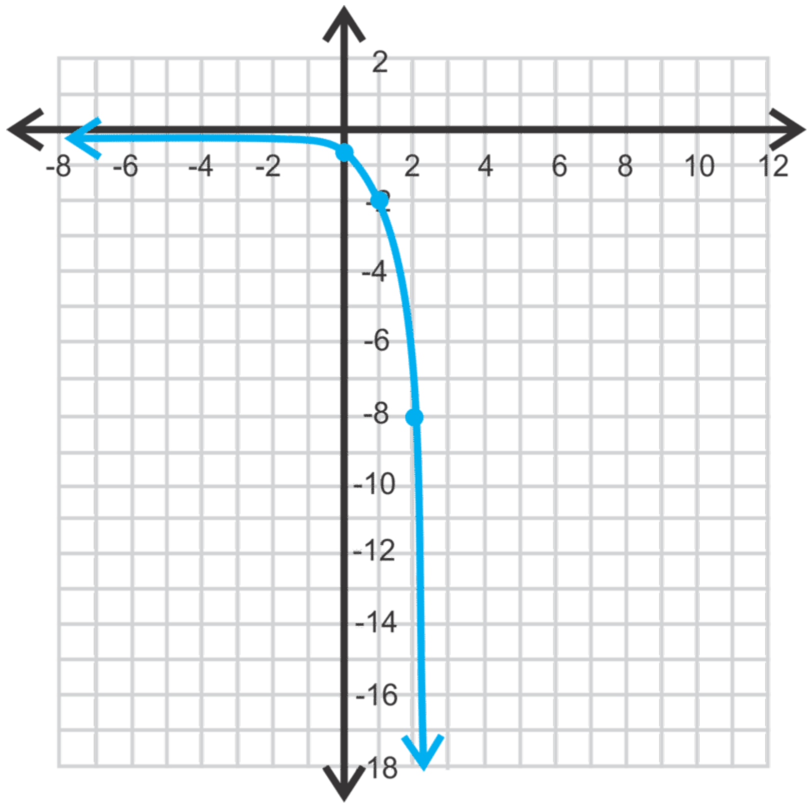
$$y = 3^{0-2} + 1 = 3^{-2} + 1 = 1\frac{1}{9} = 1.\overline{1}$$

The domain of all exponential functions is all real numbers. The range will be everything greater than the asymptote. In this example, the range is $y > 1$.

Example C

Graph the function $y = -\frac{1}{2} \cdot 4^x$. Determine if it is an exponential growth function.

Solution: In this example, we will outline how to use the graphing calculator to graph an exponential function. First, clear out anything in $Y =$. Next, input the function into $Y1, -\left(\frac{1}{2}\right) 4^\wedge X$ and press GRAPH. Adjust your window accordingly.



This is not an exponential growth function, because it does not grow in a positive direction. By looking at the definition of a growth function, $a > 0$, and it is not here.

Guided Practice

Graph the following exponential functions. Determine if they are growth functions. Then, find the y-intercept, asymptote, domain and range. Use an appropriate window.

1. $y = 3^{x-4} - 2$

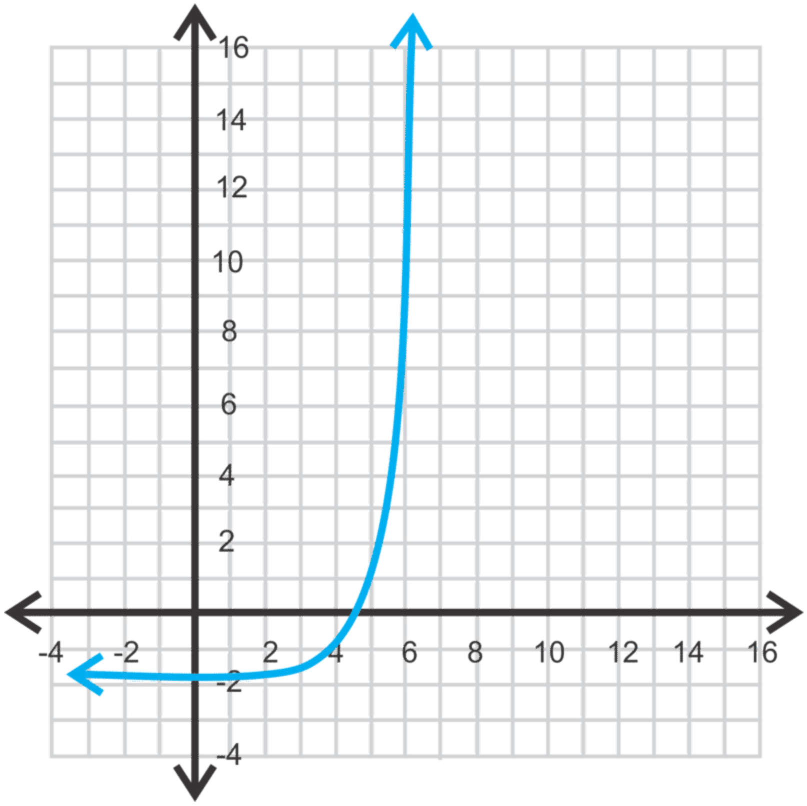
2. $f(x) = (-2)^{x+5}$

3. $f(x) = 5^x$

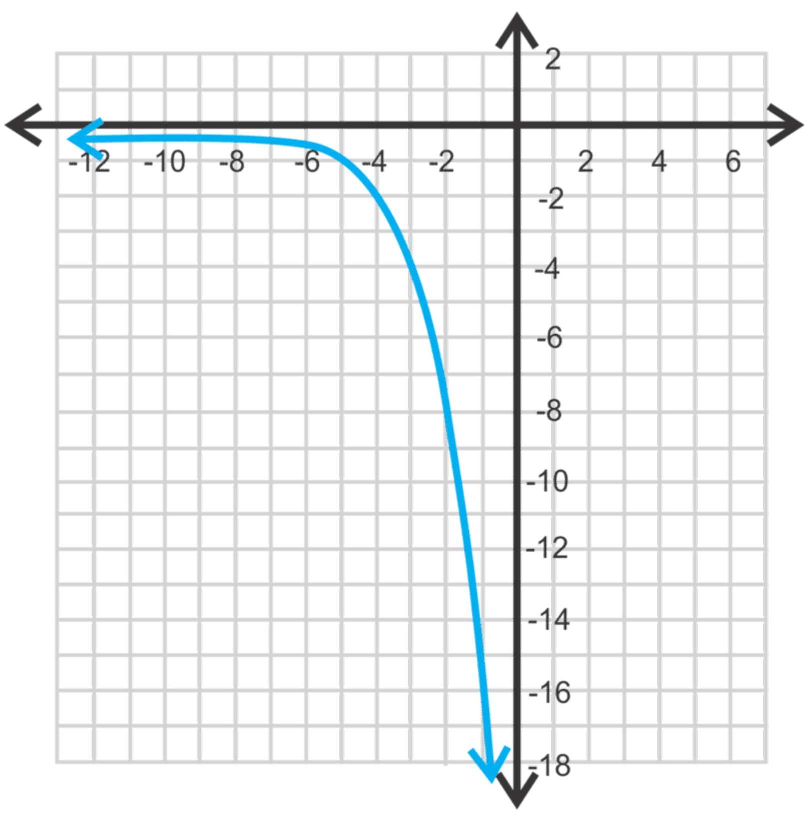
4. Abigail is in a singles tennis tournament. She finds out that there are eight rounds until the final match. If the tournament is single elimination, how many games will be played? How many competitors are in the tournament?

Answers

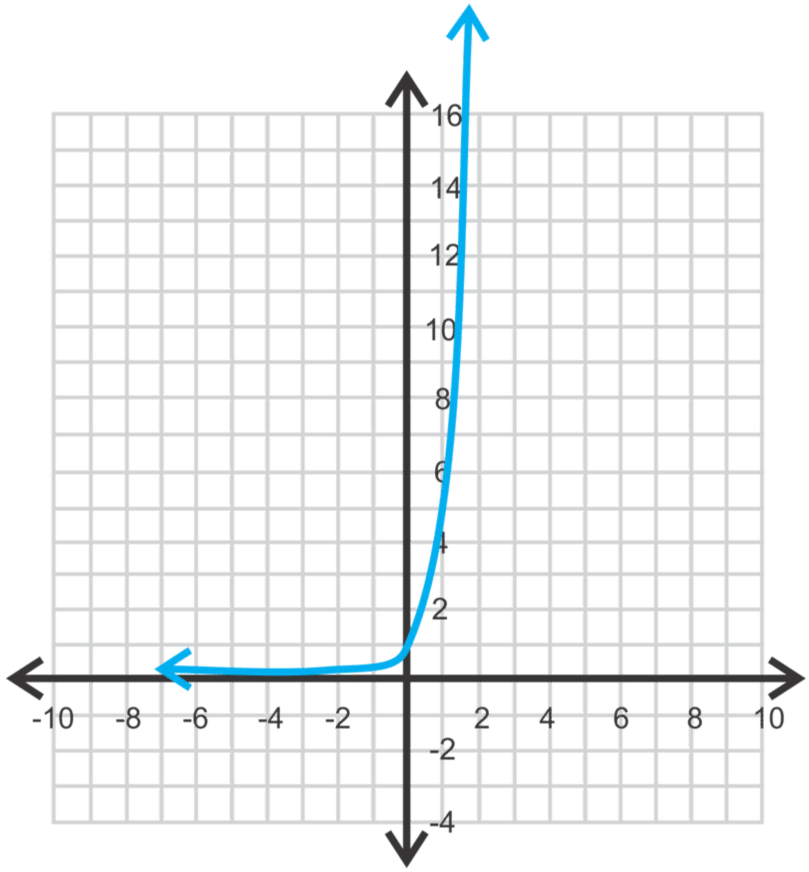
1. This is not a growth function because h and k are not zero. The y-intercept is $y = 3^{0-4} - 2 = \frac{1}{81} - 2 = -1\frac{80}{81}$, the asymptote is at $y = -2$, the domain is all real numbers and the range is $y > -2$.



2. This is not a growth function because h is not zero. The y-intercept is $y = (-2)^{0+5} = (-2)^5 = -32$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



3. This is a growth function. The y-intercept is $y = 5^{\circ} = 1$, the asymptote is at $y = 0$, the domain is all real numbers and the range is $y > 0$.



4. If there are eight rounds to single's games, there are will be $2^8 = 256$ competitors. In the first round, there will be 128 matches, then 64 matches, followed by 32 matches, then 16 matches, 8, 4, 2, and finally the championship game. Adding all these all together, there will be $128+64+32+16+8+4+2+1$ or 255 total matches.

Vocabulary

Exponential Function
A function whose variable is in the exponent. The general form is $y = a \cdot b^{x-h} + k$.

Exponential Growth Function
A specific type of exponential function where $h = k = 0, a > 0$, and $b > 1$. The general form is $y = ab^x$.

Asymptote
A boundary line that restricts the domain or range. This line is not part of the graph.

Problem Set

Graph the following exponential functions. Give the y-intercept, the equation of the asymptote and the domain and range for each function.

1. $y = 4^x$
2. $y = (-1)(5)^x$
3. $y = 3^x - 2$
4. $y = 2^x + 1$
5. $y = 6^{x+3}$
6. $y = -\frac{1}{4}(2)^x + 3$

7. $y = 7^{x+3} - 5$
8. $y = -(3)^{x-4} + 2$
9. $y = 3(2)^{x+1} - 5$
10. An investment grows according the function $A = P(1.05)^t$ where P represents the initial investment, A represents the value of the investment and t represents the number of years of investment. If \$10,000 was the initial investment, how much would the value of the investment be after 10 years to the nearest dollar?

Exponential Decay Function

Objective

To graph and analyze an exponential decay function.

Guidance

In the last concept, we only addressed functions where $|b| > 1$. So, what happens when b is less than 1? Let's analyze $y = \left(\frac{1}{2}\right)^x$.

Example A

Graph $y = \left(\frac{1}{2}\right)^x$ and compare it to $y = 2^x$.

Solution: Let's make a table of both functions and then graph.

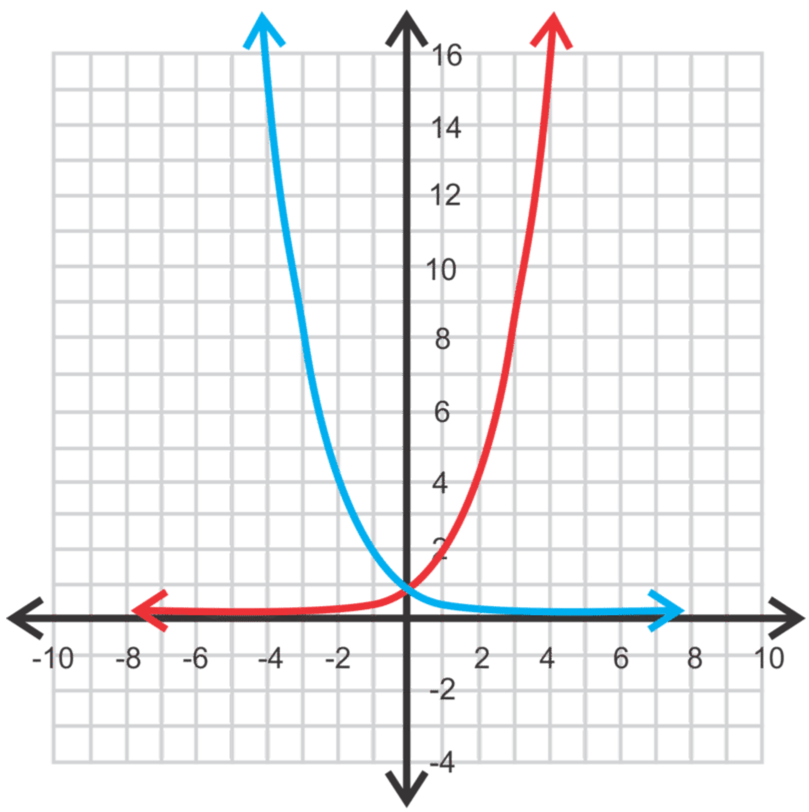


TABLE 2.10:

x	$\left(\frac{1}{2}\right)^x$	2^x
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$2^3 = 8$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$2^2 = 4$

TABLE 2.10: (continued)

x	$\left(\frac{1}{2}\right)^x$	2^x
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$2^1 = 2$
0	$\left(\frac{1}{2}\right)^0 = 1$	$2^0 = 1$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$	$2^{-1} = \frac{1}{2}$
-2	$\left(\frac{1}{2}\right)^{-2} = 4$	$2^{-2} = \frac{1}{4}$
-3	$\left(\frac{1}{2}\right)^{3} = 8$	$2^{-3} = \frac{1}{8}$

Notice that $y = \left(\frac{1}{2}\right)^x$ is a reflection over the y-axis of $y = 2^x$. Therefore, instead of exponential growth, the function $y = \left(\frac{1}{2}\right)^x$ *decreases exponentially,* or *exponentially decays*. Anytime b is a fraction or decimal between zero and one, the exponential function will decay. And, just like an exponential growth function, and **exponential decay function** has the form $y = ab^x$ and $a > 0$. However, to be a decay function, $0 < b < 1$. The exponential decay function also has an asymptote at $y = 0$.

Example B

Determine which of the following functions are exponential decay functions, exponential growth functions, or neither. Briefly explain your answer.

a) $y = 4(1.3)^x$

b) $f(x) = 3\left(\frac{6}{5}\right)^x$

c) $y = \left(\frac{3}{10}\right)^x$

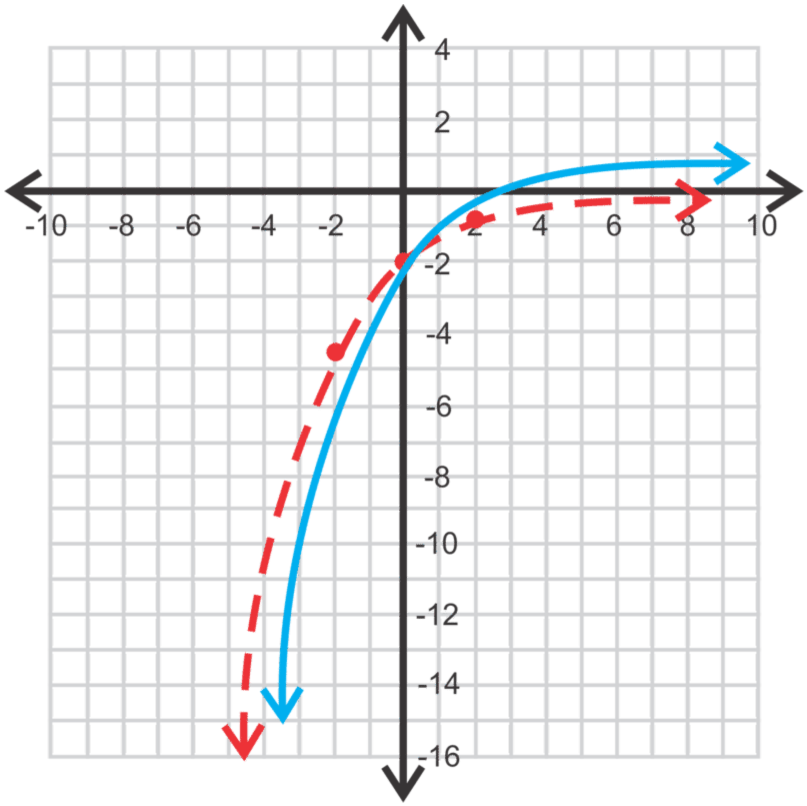
d) $g(x) = -2(0.65)^x$

Solution: a) and b) are exponential growth functions because $b > 1$. c) is an exponential decay function because b is between zero and one. d) is neither growth or decay because a is negative.

Example C

Graph $g(x) = -2\left(\frac{2}{3}\right)^{x-1} + 1$. Find the y-intercept, asymptote, domain, and range.

Solution: To graph this function, you can either plug it into your calculator (entered $Y = -2\left(\frac{2}{3}\right)^{\wedge} X - 1 + 1$) or graph $y = -2\left(\frac{2}{3}\right)^x$ and shift it to the right one unit and up one unit. We will use the second method; final answer is the blue function below.



The y-intercept is:

$y=-2\left(\frac{2}{3}\right)^{0-1}+1=-2\cdot\frac{3}{2}+1=-3+1=-2$

The horizontal asymptote is $y=1$, the domain is all real numbers and the range is $y<1$.

Guided Practice

Graph the following exponential functions. Find the y-intercept, asymptote, domain, and range.

1. $f(x)=4\left(\frac{1}{3}\right)^x$

2. $y=-2\left(\frac{2}{3}\right)^{x+3}$

3. $g(x)=\left(\frac{3}{5}\right)^x-6$

4. Determine if the following functions are exponential growth, exponential decay, or neither.

a) $y=2.3^x$

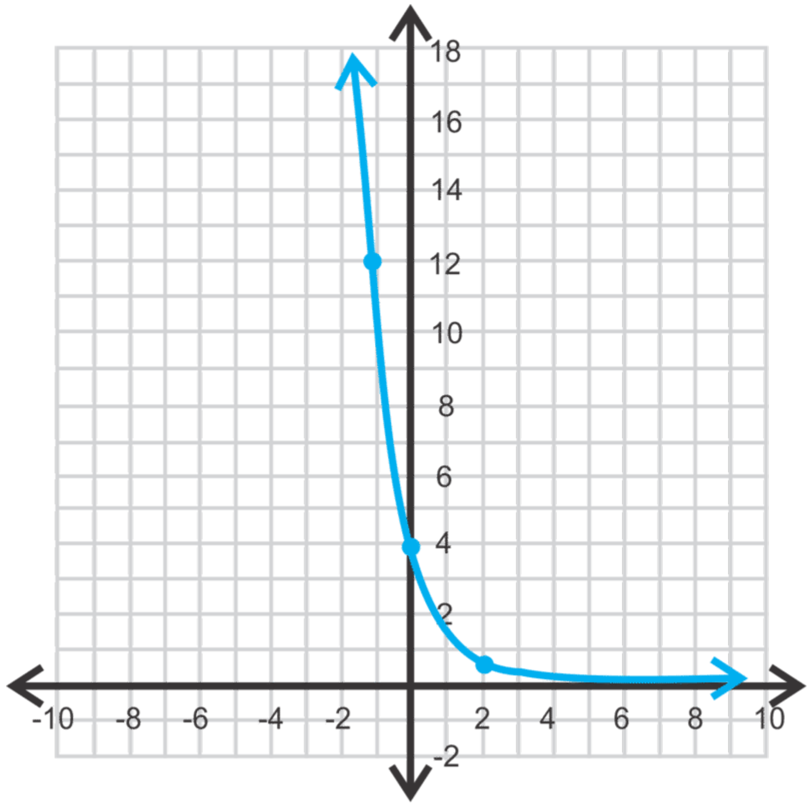
b) $y=2\left(\frac{4}{3}\right)^{-x}$

c) $y=3\cdot 0.9^x$

d) $y=\frac{1}{2}\left(\frac{4}{5}\right)^x$

Answers

1.



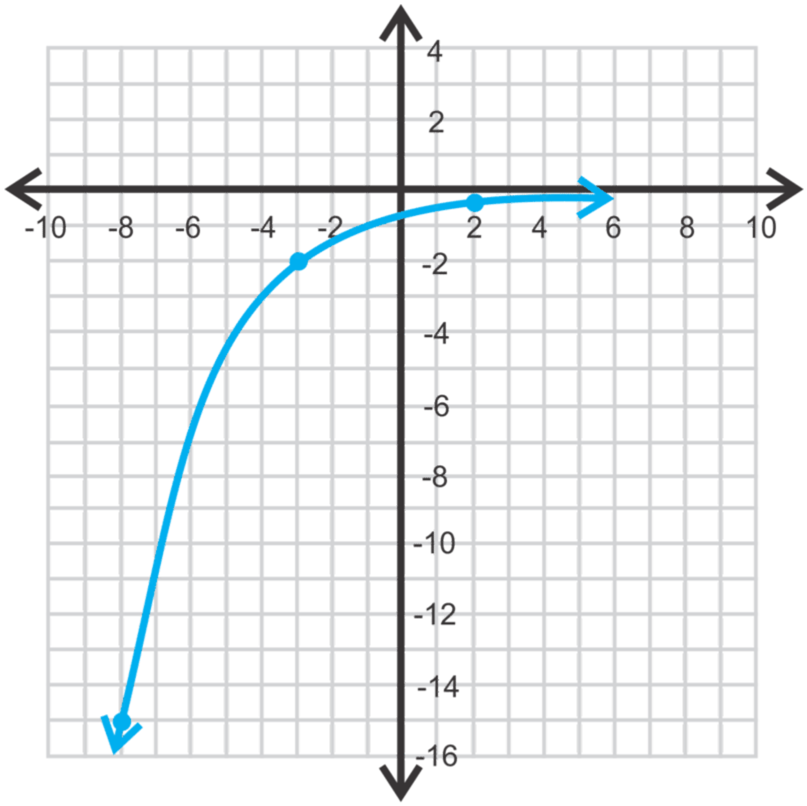
y-intercept: $(4, 0)$

asymptote: $y = 0$

domain: all reals

range: $y < 0$

2.



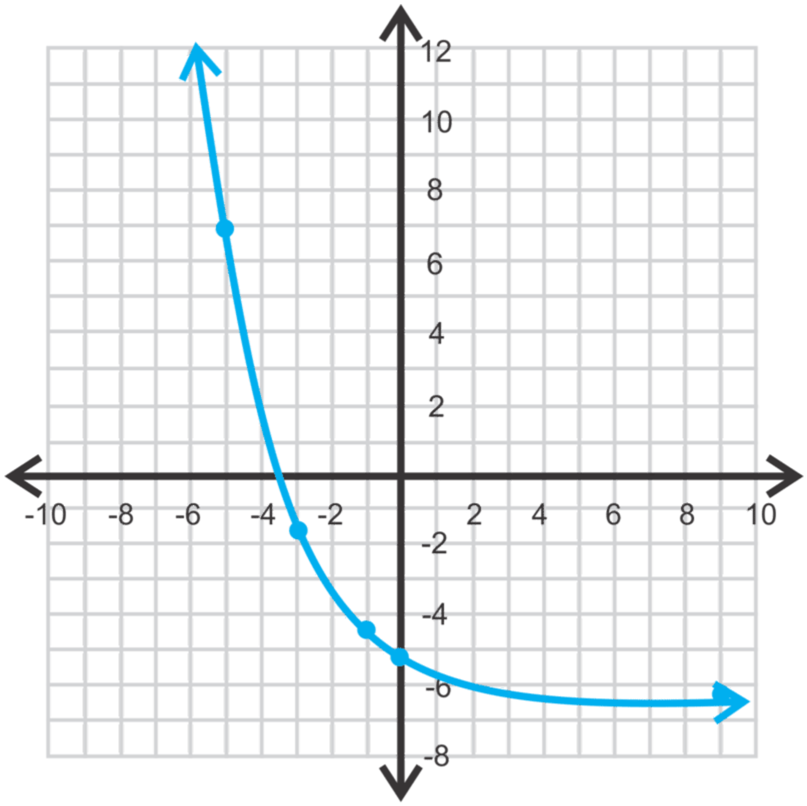
y-intercept: $\left(0, -\frac{16}{27}\right)$

asymptote: $y = 0$

domain: all reals

range: $y < 0$

3.



y-intercept: $(-5,0)$

asymptote: $y=-6$

domain: all reals

range: $y>-6$

4. a) exponential growth

b) exponential decay; recall that a negative exponent flips whatever is in the base. $y=2\left(\frac{4}{3}\right)^{-x}$ is the same as $y=2\left(\frac{3}{4}\right)^{x}$, which looks like our definition of a decay function.

c) exponential decay

d) neither; $a<0$

Vocabulary

Exponential Decay Function
An exponential function that has the form $y=ab^x$ where $a>0$ and $0<b<1$.

Problem Set

Determine which of the following functions are exponential growth, exponential decay or neither.

1. $y=-\left(\frac{2}{3}\right)^{x}+5$
2. $y=\left(\frac{4}{3}\right)^{x-4}$
3. $y=5^x-2$

Graph the following exponential functions. Give the y-intercept, the equation of the asymptote and the domain and range for each function.

4. $y=\left(\frac{1}{2}\right)^{x}$

5. $y = (0.8)^{x+2}$
6. $y = 4\left(\frac{2}{3}\right)^{x-1} - 5$
7. $y = -\left(\frac{5}{7}\right)^{x} + 3$
8. $y = \left(\frac{8}{9}\right)^{x+5} - 2$
9. $y = (0.75)^{x-2} + 4$
10. A discount retailer advertises that items will be marked down at a rate of 10% per week until sold. The initial price of one item is \$50.
    a. Write an exponential decay function to model the price of the item x weeks after it is first put on the rack.
    b. What will the price be after the item has been on display for 5 weeks?
    c. After how many weeks will the item be half its original price?

Using Exponential Growth and Decay Models

Objective

To use different exponential functions in real-life situations.

Guidance

When a real-life quantity increases by a percentage over a period of time, the final amount can be modeled by the equation: $A = P(1+r)^t$, where A is the final amount, P is the initial amount, r is the rate (or percentage), and t is the time (in years). $1+r$ is known as the **growth factor**.

Conversely, a real-life quantity can decrease by a percentage over a period of time. The final amount can be modeled by the equation: $A = P(1-r)^t$, where $1-r$ is the **decay factor**.

Example A

The population of Coleman, Texas grows at a 2% rate annually. If the population in 2000 was 5981, what was the population is 2010? Round up to the nearest person.

Solution: First, set up an equation using the growth factor. $r = 0.02, t = 10$, and $P = 5981$.

$$\begin{aligned} A &= 5981(1+0.02)^{10} \\ &= 5981(1.02)^{10} \\ &= 7291 \text{ people} \end{aligned}$$

Example B

You deposit \$1000 into a savings account that pays 2.5% annual interest. Find the balance after 3 years if the interest rate is compounded a) annually, b) monthly, c) daily.

Solution: For part a, we will use $A = 1000(1.025)^3 = 1008.18$, as we would expect from Example A.

But, to determine the amount if it is compounded in amounts other than yearly, we need to alter the equation. For compound interest, the equation is $A = P\left(1+\frac{r}{n}\right)^{nt}$, where n is the number of times the interest is compounded within a year. For part b, $n = 12$.

$$\begin{aligned} A &= 1000\left(1+\frac{0.025}{12}\right)^{12\cdot 3} \\ &= 1000(1.002)^{36} \\ &= 1077.80 \end{aligned}$$

In part c, $n = 365$.

$$\begin{aligned} A &= 1000\left(1+\frac{0.025}{365}\right)^{365\cdot 3} \\ &= 1000(1.000068)^{1095} \\ &= 1077.88 \end{aligned}$$

Example C

You buy a new car for $35,000. If the value of the car decreases by 12% each year, what will the value of the car be in 5 years?

Solution: This is a decay function because the value *decreases*.

$$\begin{aligned} A &= 35000(1-0.12)^5 \\ &= 35000(0.88)^5 \\ &= 18470.62 \end{aligned}$$

The car would be worth $18,470.62 after five years.

Guided Practice

1. Tommy bought a truck 7 years ago that is now worth $12,348. If the value of his truck decreased 14% each year, how much did he buy it for? Round to the nearest dollar.

2. The Wetakayomoola credit card company charges an Annual Percentage Rate (APR) of 21.99%, compounded monthly. If you have a balance of $2000 on the card, what would the balance be after 4 years (assuming you do not make any payments)? If you pay $200 a month to the card, how long would it take you to pay it off? You may need to make a table to help you with the second question.

3. As the altitude increases, the atmospheric pressure (the pressure of the air around you) decreases. For every 1000 feet up, the atmospheric pressure decreases about 4%. The atmospheric pressure at sea level is 101.3. If you are on top of Hevenly Mountain at Lake Tahoe (elevation about 10,000 feet) what is the atmospheric pressure?

Answers

1. Tommy needs to use the formula $A = P(1-r)^t$ and solve for P.

$$\begin{aligned} 12348 &= P(1-0.14)^7 \\ 12348 &= P(0.86)^7 \qquad \text{Tommy's truck was originally } \$35,490. \\ \frac{12348}{(0.86)^7} &= P \approx 35490 \end{aligned}$$

2. you need to use the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$, where $n = 12$ because the interest is compounded monthly.

$$\begin{aligned} A &= 2000\left(1+\frac{0.2199}{12}\right)^{12\cdot 4} \\ &= 2000(1018325)^{48} \\ &= 4781.65 \end{aligned}$$

To determine how long it will take you to pay off the balance, you need to find how much interest is compounded in one month, subtract \$200, and repeat. A table might be helpful. For each month after the first, we will use the equation, $B = R\left(1+\frac{0.2199}{12}\right)^{12\cdot\left(\frac{1}{12}\right)} = R(1.018325)$, where B is the current balance and R is the remaining balance from the previous month. For example, in month 2, the balance (including interest) would be $B = 1800\left(1+\frac{0.2199}{12}\right)^{12\cdot\left(\frac{1}{12}\right)} = 1800 \cdot 1.08325 = 1832.99$.

TABLE 2.11:

Month	**1**	**2**	**3**	**4**	**5**	**6**
Balance	2000	1832.99	1662.91	1489.72	1313.35	930.09
Payment	200	200.00	200.00	200.00	200.00	200.00
Remainder	\$1800	1632.99	1462.91	1289.72	913.35	730.09

TABLE 2.12:

Month	**7**	**8**	**9**	**10**	**11**
Balance	790.87	640.06	476.69	299.73	108.03
Payment	200.00	200.00	200.00	200.00	108.03
Remainder	590.87	440.06	276.69	99.73	0

It is going to take you 11 months to pay off the balance and you are going to pay 108.03 in interest, making your total payment \$2108.03.

3. The equation will be $A = 101,325(1-0.04)^{100} = 1709.39$. The decay factor is only raised to the power of 100 because for every 1000 feet the pressure decreased. Therefore, $10,000 \div 1000 = 100$. Atmospheric pressure is what you don't feel when you are at a higher altitude and can make you feel light-headed. The picture below demonstrates the atmospheric pressure on a plastic bottle. The bottle was sealed at 14,000 feet elevation (1), and then the resulting pressure at 9,000 feet (2) and 1,000 feet (3). The lower the elevation, the higher the atmospheric pressure, thus the bottle was crushed at 1,000 feet.

Vocabulary

Growth Factor

The amount, $1+r$, an exponential function grows by. Populations and interest commonly use growth factors.

Decay Factor

The amount, $1-r$, an exponential function decreases by. Populations, depreciated values, and radioactivity commonly use decay factors.

Compounded Interest

When an amount of money is charges a particular interest rate and that rate is collected yearly, monthly, quarterly, or even daily. It is compounded because after the first "collection" interest is taken on interest.

Problem Set

Use an exponential growth or exponential decay function to model the following scenarios and answer the questions.

1. Sonya's salary increases at a rate of 4% per year. Her starting salary is \$45,000. What is her annual salary, to the nearest \$100, after 8 years of service?
2. The value of Sam's car depreciates at a rate of 8% per year. The initial value was \$22,000. What will his car be worth after 12 years to the nearest dollar?
3. Rebecca is training for a marathon. Her weekly long run is currently 5 miles. If she increase her mileage each week by 10%, will she complete a 20 mile training run within 15 weeks?
4. An investment grows at a rate of 6% per year. How much, to the nearest \$100, should Noel invest if he wants to have \$100,000 at the end of 20 years?
5. Charlie purchases a 7 year old used RV for \$54,000. If the rate of depreciation was 13% per year during those 7 years, how much was the RV worth when it was new? Give your answer to the nearest one thousand dollars.
6. The value of homes in a neighborhood increase in value an average of 3% per year. What will a home purchased for \$180,000 be worth in 25 years to the nearest one thousand dollars?
7. The population of school age children in a community is decreasing at a rate of 2% per year. The current population is \$152,000. How many children were there 5 years ago?
8. The value of a particular piece of land worth \$40,000 is increasing at a rate of 1.5% per year. Assuming the rate of appreciation continues, how long will the owner need to wait to sell the land if he hopes to get \$50,000 for it? Give your answer to the nearest year.

For problems 9-12, use the formula for compound interest: $A = P\left(1+\frac{r}{n}\right)^{nt}$.

9. If \$12,000 is invested at 4% annual interest compounded monthly, how much will the investment be worth in 10 years? Give your answer to the nearest dollar.
10. If \$8,000 is invested at 5% annual interest compounded semiannually, how much will the investment be worth in 6 years? Give your answer to the nearest dollar.
11. If \$20,000 is invested at 6% annual interested compounded quarterly, how much will the investment be worth in 12 years. Give your answer to the nearest dollar.
12. How much of an initial investment is required to insure an accumulated amount of at least \$25,000 at the end of 8 years at an annual interest rate of 3.75% compounded monthly. Give your answer to the nearest one hundred dollars.

The Number e

Objective

To use the natural number, e, in exponential functions and real-life situations.

Guidance

There are many special numbers in mathematics: π, zero, $\sqrt{2}$, among others. In this concept, we will introduce another special number that is known only by a letter, e. It is called the **natural number** (or base), or the **Euler**

number, named after the Swiss mathematician Leonhard Euler who popularized the use of the letter e for the constant. Credit for discovery of the constant itself goes to another important Swiss mathematician, Jacob Bernoulli, and his study of sequences in compound interest.

From the previous concept, we learned that the formula for compound interest is $A = P\left(1+\frac{r}{n}\right)^{nt}$. Let's set P, r and t equal to one and see what happens, $A = \left(1+\frac{1}{n}\right)^{n}$.

Investigation: Finding the values of $\left(1+\frac{1}{n}\right)^{n}$ as n gets larger

1. Copy the table below and fill in the blanks. Round each entry to the nearest 4 decimal places.

TABLE 2.13:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\left(1+\frac{1}{n}\right)^{n}$	$\left(1+\frac{1}{1}\right)^{1} =$ 2	$\left(1+\frac{1}{2}\right)^{2} =$ 2.25						

2. Does it seem like the numbers in the table are approaching a certain value? What do you think the number is?

3. Find $\left(1+\frac{1}{100}\right)^{100}$ and $\left(1+\frac{1}{1000}\right)^{1000}$. Does this change your answer from #2?

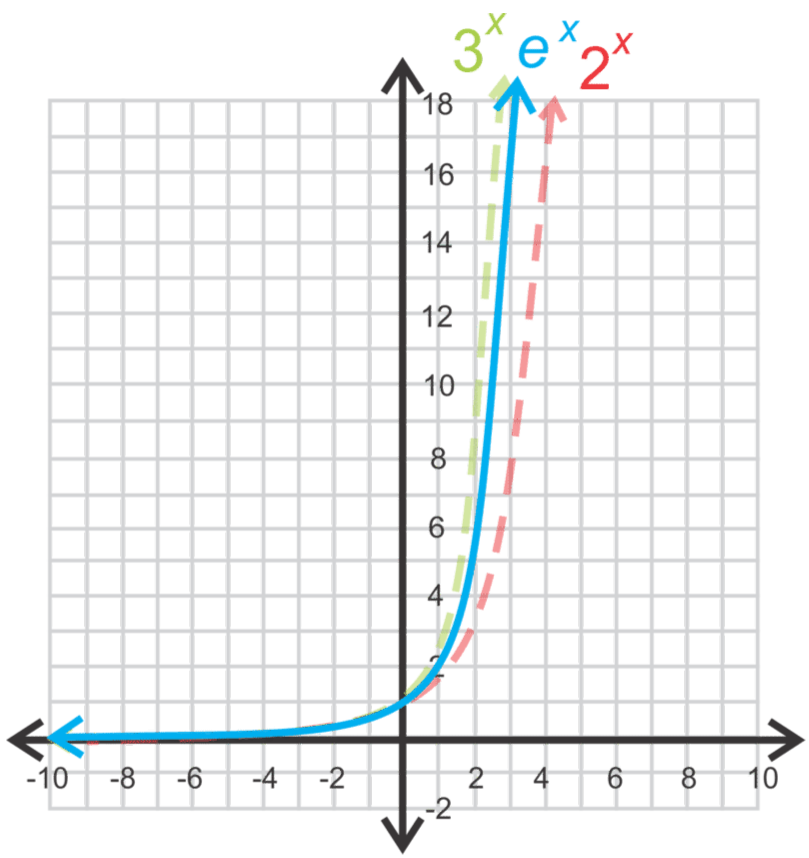
4. Fill in the blanks: As n approaches ___________, ___________ approaches $e \approx 2.718281828459\ldots$

We define e as the number that $\left(1+\frac{1}{n}\right)^{n}$ approaches as $n \to \infty$ (n approaches infinity). e is an irrational number with the first 12 decimal places above.

Example A

Graph $y = e^x$. Identify the asymptote, y-intercept, domain and range.

Solution: As you would expect, the graph of e^x will curve between 2^x and 3^x.



The asymptote is $y = 0$ and the y-intercept is (0, 1) because anything to the zero power is one. The domain is all real numbers and the range is all positive real numbers; $y > 0$.

Example B

Simplify $e^2 \cdot e^4$.

Solution: The bases are the same, so you can just add the exponents. The answer is e^6.

Example C

Gianna opens a savings account with \$1000 and it accrues interest *continuously* at a rate of 5%. What is the balance in the account after 6 years?

Solution: In the previous concept, the word problems dealt with interest that compounded monthly, quarterly, annually, etc. In this example, the interest compounds continuously. The equation changes slightly, from $A = P\left(1+\frac{r}{n}\right)^{nt}$ to $A = Pe^{rt}$, without n, because there is no longer any interval. Therefore, the equation for this problem is $A = 1000e^{0.05(6)}$ and the account will have \$1349.86 in it. Compare this to daily accrued interest, which would be $A = 1000\left(1+\frac{0.05}{365}\right)^{365(6)} = 1349.83$.

Guided Practice

1. Determine if the following functions are exponential growth, decay, or neither.

a) $y = \frac{1}{2}e^x$

b) $y = -4e^x$

c) $y = e^{-x}$

d) $y = 2\left(\frac{1}{e}\right)^{-x}$

2. Simplify the following expressions with e.

a) $2e^{-3} \cdot e^2$

b) $\frac{4e^6}{16e^2}$

3. The rate of radioactive decay of radium is modeled by $R = Pe^{-0.00043t}$, where R is the amount (in grams) of radium present after t years and P is the initial amount (also in grams). If there is 698.9 grams of radium present after 5,000 years, what was the initial amount?

Answers

1. Recall to be exponential growth, the base must be greater than one. To be exponential decay, the base must be between zero and one.

a) Exponential growth; $e > 1$

b) Neither; $a < 0$

c) Exponential decay; $e^{-x} = \left(\frac{1}{e}\right)^x$ and $0 < \frac{1}{e} < 1$

d) Exponential growth; $\left(\frac{1}{e}\right)^{-x} = e^x$

2. a) $2e^{-3} \cdot e^2 = 2e^{-1}$ or $\frac{2}{e}$

b) $\frac{4e^6}{16e^2} = \frac{e^4}{4}$

3. Use the formula given in the problem and solve for what you don't know.

$$\begin{aligned} R &= Pe^{-0.00043t} \\ 698.9 &= Pe^{-0.00043(5000)} \\ 698.9 &= P(0.11648) \\ 6000 &= P \end{aligned}$$

There was about 6000 grams of radium to start with.

Vocabulary

Natural Number (Euler Number)
The number e, such that as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$. $e \approx 2.71828$.

Problem Set

Determine if the following functions are exponential growth, decay or neither. Give a reason for your answer.

1. $y = \frac{4}{3}e^x$
2. $y = -e^{-x} + 3$
3. $y = \left(\frac{1}{e}\right)^x + 2$
4. $y = \left(\frac{3}{e}\right)^{-x} - 5$

Simplify the following expressions with e.

5. $\frac{5e^{-4}}{e^3}$
6. $6e^5e^{-4}$
7. $\left(\frac{4e^4}{3e^{-2}e^3}\right)^{-2}$

Solve the following word problems.

8. The population of Springfield is growing exponentially. The growth can be modeled by the function $P = Ie^{0.055t}$, where P represents the projected population, I represents the current population of 100,000 in 2012 and $t$represents the number of years after 2012.
   a. To the nearest person, what will the population be in 2022?
   b. In what year will the population double in size if this growth rate continues?
9. The value of Steve's car decreases in value according to the exponential decay function: $V = Pe^{-0.12t}$, where V is the current value of the vehicle, t is the number of years Steve has owned the car and $P$is the purchase price of the car, \$25,000.
   a. To the nearest dollar, what will the value of Steve's car be in 2 years?
   b. To the nearest dollar, what will the value be in 10 years?
10. Naya invests \$7500 in an account which accrues interest continuously at a rate of 4.5%.
   a. Write an exponential growth function to model the value of her investment after t years.
   b. How much interest does Naya earn in the first six months to the nearest dollar?
   c. How much money, to the nearest dollar, is in the account after 8 years?

2.33 Logarithmic Functions

Objective

To learn about the inverse of an exponential function, the logarithm.

Review Queue

Find the inverse of the following functions.

1. $f(x) = \frac{1}{2}x - 5$

2. $g(x) = \sqrt{x+5}$

3. $h(x) = 6x^2 - 1$

Solve the equations below.

4. $3^x = 27$

5. $2^x = \frac{1}{8}$

6. $5^x = 1$

Defining Logarithms

Objective

To define and use logarithms.

Guidance

You can probably guess that $x = 3$ in $2^x = 8$ and $x = 4$ in $2^x = 16$. But, what is x if $2^x = 12$? Until now, we did not have an inverse to an exponential function. But, because we have variables in the exponent, we need a way to get them out of the exponent. Introduce the logarithm. A **logarithm** is defined as the inverse of an exponential function. It is written $\log_b a = x$ such that $b^x = a$. Therefore, if $5^2 = 25$ (**exponential form**), then $\log_5 25 = 2$ (**logarithmic form**).

There are two special logarithms, or logs. One has base 10, and rather that writing $\log_{10}$, we just write log. The other is the **natural log**, the inverse of the natural number. The natural log has base e and is written ln This is the only log that is not written using log.

Example A

Rewrite $\log_3 27 = 3$ in exponential form.

Solution: Use the definition above, also called the "key".

$$\log_b a = x \leftrightarrow b^x = a$$
$$\log_3 27 = 3 \leftrightarrow 3^3 = 27$$

Example B

Find:

a) $\log 1000$

b) $\log_7 \frac{1}{49}$

c) $\log_{\frac{1}{2}}(-8)$

Solution: Using the key, we can rearrange all of these in terms of exponents.

a) $\log 1000 = x \Rightarrow 10^x = 1000, x = 3$.

b) $\log_7 \frac{1}{49} = x \Rightarrow 7^x = \frac{1}{49}, x = -2$.

c) $\log_{\frac{1}{2}}(-8) = x \Rightarrow \left(\frac{1}{2}\right)^x = -8$. There is no solution. A positive number when raised to any power will never be negative.

There are two special logarithms that you may encounter while writing them into exponential form.

The first is $\log_b 1 = 0$, because $b^0 = 1$. The second is $\log_b b = 1$ because $b^1 = b \cdot b$ can be any number except 1.

Example C

Use your calculator to find the following logarithms. Round your answer to the nearest hundredth.

a) $\ln 7$

b) $\log 35$

c) $\log_5 226$

Solution:

a) Locate the LN button on your calculator. Depending on the brand, you may have to input the number first. For a TI-83 or 84, press the LN, followed by the 7 and ENTER. The answer is 1.95.

b) The LOG button on the calculator is base 10. Press LOG, 35, ENTER. The answer is 1.54.

c) To use the calculator for a base other than 10 or the natural log, you need to use the change of base formula.

Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$, such that x, a, and $b > 0$ and a and $b \neq 1$.

So, to use this for a calculator, you can use either LN or LOG.

$\log_5 226 = \frac{\log 226}{\log 5}$ or $\frac{\ln 226}{\ln 5} \approx 3.37$

In the TI-83 or 84, the keystrokes would be $\frac{\text{LOG(226)}}{\text{LOG(5)}}$, ENTER.

Guided Practice

1. Write $6^2 = 36$ in logarithmic form.

2. Evaluate the following expressions without a calculator.

a) $\log_{\frac{1}{2}} 16$

b) $\log 100$

c) $\log_{64} \frac{1}{8}$

3. Use a calculator to evaluate each expression. Round your answers to the hundredths place.

a) $\ln 32$

b) $\log_7 94$

c) $\log 65$

4. Use the change of base formula to evaluate $\log_8 \frac{7}{9}$ in a calculator.

Answers

1. Using the key, we have: $6^2 = 36 \rightarrow \log_6 36 = 2$.

2. Change each logarithm into exponential form and solve for x.

a) $\log_{\frac{1}{2}} 16 \rightarrow \left(\frac{1}{2}\right)^x = 16$. x must be negative because the answer is not a fraction, like the base.

$2^4 = 16$, so $\left(\frac{1}{2}\right)^{-4} = 16$. Therefore, $\log_{\frac{1}{2}} 16 = -4$.

b) $\log 100 \rightarrow 10^x = 100$. $x = 2$, therefore, $\log 100 = 2$.

c) $\log_{64} \frac{1}{8} \rightarrow 64^x = \frac{1}{8}$. First, $\sqrt{64} = 8$, so $64^{\frac{1}{2}} = 8$. To make this a fraction, we need to make the power negative. $64^{-\frac{1}{2}} = \frac{1}{8}$, therefore $\log_{64} \frac{1}{8} = -\frac{1}{2}$.

3. Using a calculator, we have:

a) 3.47

b) 2.33

c) 1.81

4. Rewriting $\log_8 \frac{7}{9}$ using the change of base formula, we have: $\frac{\log \frac{7}{9}}{\log 8}$. Plugging it into a calculator, we get $\frac{\log\left(\frac{7}{9}\right)}{\log 8} \approx -0.12$.

Vocabulary

Logarithm
The inverse of an exponential function and is written $\log_b a = x$ such that $b^x = a$.

Exponential Form
$b^x = a$, such that b is the base and x is the exponent.

Logarithmic Form
$\log_b a = x$, such that b is the base.

Natural Log
The inverse of the natural number, e, written ln.

Change of Base Formula
Let b, x, and y be positive numbers, $b \neq 1$ and $y \neq 1$. Then, $\log_y x = \frac{\log_b x}{\log_b y}$. More specifically, $\log_y x = \frac{\log x}{\log y}$ and $\log_y x = \frac{\ln x}{\ln y}$, so that expressions can be evaluated using a calculator.

Problem Set

Convert the following exponential equations to logarithmic equations.

1. $3^x = 5$
2. $a^x = b$
3. $4(5^x) = 10$

Convert the following logarithmic equations to exponential equations.

4. $\log_2 32 = x$
5. $\log_{\frac{1}{3}} x = -2$
6. $\log_a y = b$

convert the following logarithmic expressions without a calculator.

7. $\log_5 25$
8. $\log_{\frac{1}{3}} 27$
9. $\log \frac{1}{10}$
10. $\log_2 64$

Evaluate the following logarithmic expressions using a calculator. You may need to use the Change of Base Formula for some problems.

11. $\log 72$
12. $\ln 8$
13. $\log_2 12$
14. $\log_3 9$

Inverse Properties of Logarithmic Functions

Objective

To understand the inverse properties of a logarithmic function.

Guidance

By the definition of a logarithm, it is the inverse of an exponent. Therefore, a logarithmic function is the inverse of an exponential function. Recall what it means to be an inverse of a function. When two inverses are composed (see the *Inverse of a Function* concept), they equal x. Therefore, if $f(x) = b^x$ and $g(x) = \log_b x$, then:

$f \circ g = b^{\log_b x} = x$ and $g \circ f = \log_b b^x = x$

These are called the Inverse Properties of Logarithms.

Example A

Find:

a) $10^{\log 56}$

b) $e^{\ln 6} \cdot e^{\ln 2}$

Solution: For each of these examples, we will use the Inverse Properties.

a) Using the first property, we see that the bases cancel each other out. $10^{\log 56} = 56$

b) Here, e and the natural log cancel out and we are left with $6 \cdot 2 = 12$.

Example B

Find $\log_4 16^x$

Solution: We will use the second property here. Also, rewrite 16 as 4^2.

$\log_4 16^x = \log_4 (4^2)^x = \log_4 4^{2x} = 2x$

Example C

Find the inverse of $f(x) = 2e^{x-1}$.

Solution: See the *Finding the Inverse* concept for the steps on how to find the inverse.

Change $f(x)$ to y. Then, switch x and y.

$$y = 2e^{x-1}$$
$$x = 2e^{y-1}$$

Now, we need to isolate the exponent and take the logarithm of both sides. First divide by 2.

$$\frac{x}{2} = e^{y-1}$$
$$\ln\left(\frac{x}{2}\right) = \ln e^{y-1}$$

Recall the Inverse Properties from earlier in this concept. $\log_b b^x = x$; applying this to the right side of our equation, we have $\ln e^{y-1} = y - 1$. Solve for y.

$$\ln\left(\frac{x}{2}\right) = y - 1$$
$$\ln\left(\frac{x}{2}\right) + 1 = y$$

Therefore, $\ln\left(\frac{x}{2}\right) + 1$ is the inverse of $2e^{y-1}$.

Guided Practice

1. Simplify $5^{\log_5 6x}$.

2. Simplify $\log_9 81^{x+2}$.

3. Find the inverse of $f(x) = 4^{x+2} - 5$.

Answers

1. Using the first inverse property, the log and the base cancel out, leaving $6x$ as the answer.

$$5^{\log_5 6x} = 6x$$

2. Using the second inverse property and changing 81 into 9^2 we have:

$$\begin{aligned}\log_9 81^{x+2} &= \log_9 9^{2(x+2)}\\ &= 2(x+2)\\ &= 2x+4\end{aligned}$$

3. Follow the steps from Example C to find the inverse.

$$\begin{aligned}f(x) &= 4^{x+2} - 5\\ y &= 4^{x+2} - 5\\ x &= 4^{y+2} - 5\\ x+5 &= 4^{y+2}\\ \log_4(x+5) &= y+2\\ \log_4(x+5) - 2 &= y\end{aligned}$$

Vocabulary

Inverse Properties of Logarithms

$\log_b b^x = x$ and $b^{\log_b x} = x, b \neq 1$

Problem Set

Use the Inverse Properties of Logarithms to simplify the following expressions.

1. $\log_3 27^x$
2. $\log_5 \left(\frac{1}{5}\right)^x$
3. $\log_2 \left(\frac{1}{32}\right)^x$
4. $10^{\log(x+3)}$
5. $\log_6 36^{(x-1)}$
6. $9^{\log_9(3x)}$
7. $e^{\ln(x-7)}$
8. $\log \left(\frac{1}{100}\right)^{3x}$
9. $\ln e^{(5x-3)}$

Find the inverse of each of the following exponential functions.

10. $y = 3e^{x+2}$
11. $f(x) = \frac{1}{5}e^{\frac{x}{7}}$
12. $y = 2 + e^{2x-3}$
13. $f(x) = 7^{\frac{3}{x}+1-5}$
14. $y = 2(6)^{\frac{x-5}{2}}$
15. $f(x) = \frac{1}{3}(8)^{\frac{x}{2}-5}$

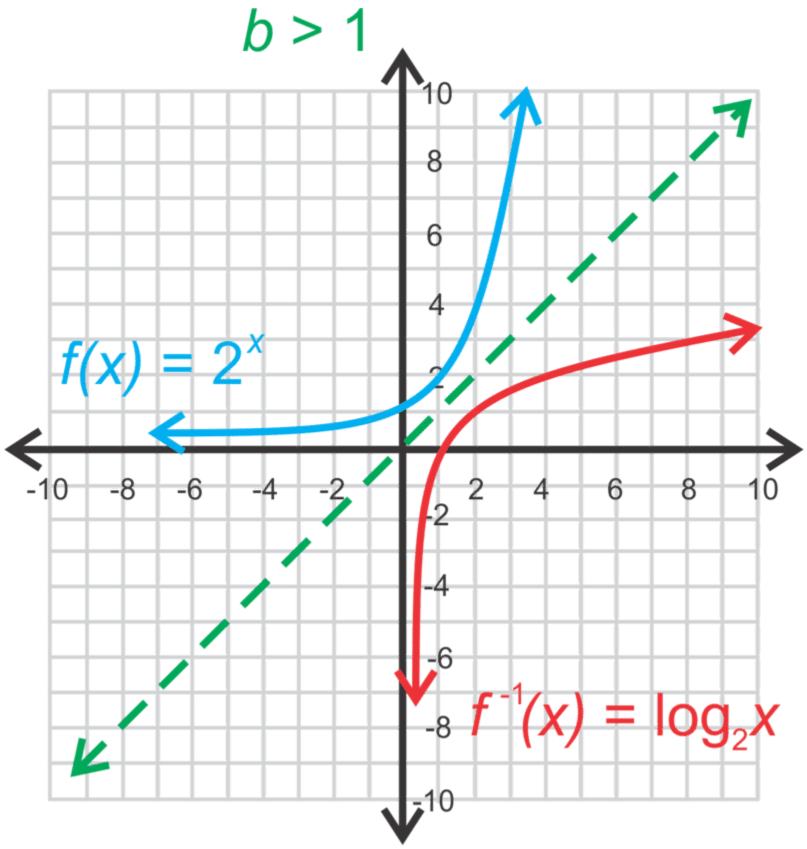
Graphing Logarithmic Functions

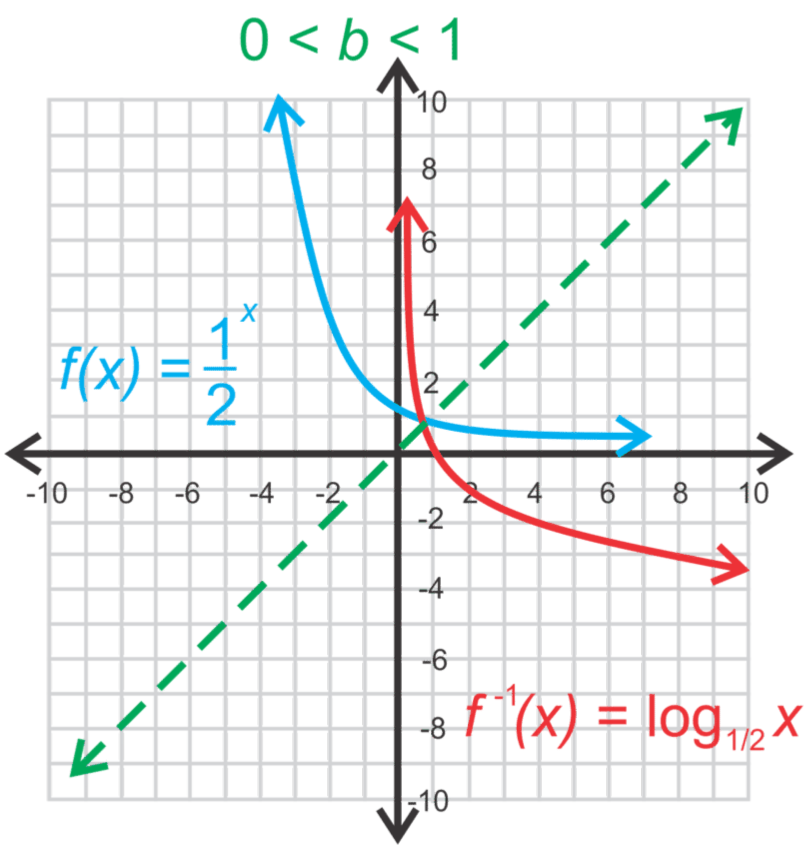
Objective

To graph a logarithmic function by hand and using a calculator.

Guidance

Now that we are more comfortable with using these functions as inverses, let's use this idea to graph a logarithmic function. Recall that functions are inverses of each other when they are mirror images over the line $y = x$. Therefore, if we reflect $y = b^x$ over $y = x$, then we will get the graph of $y = \log_b x$.



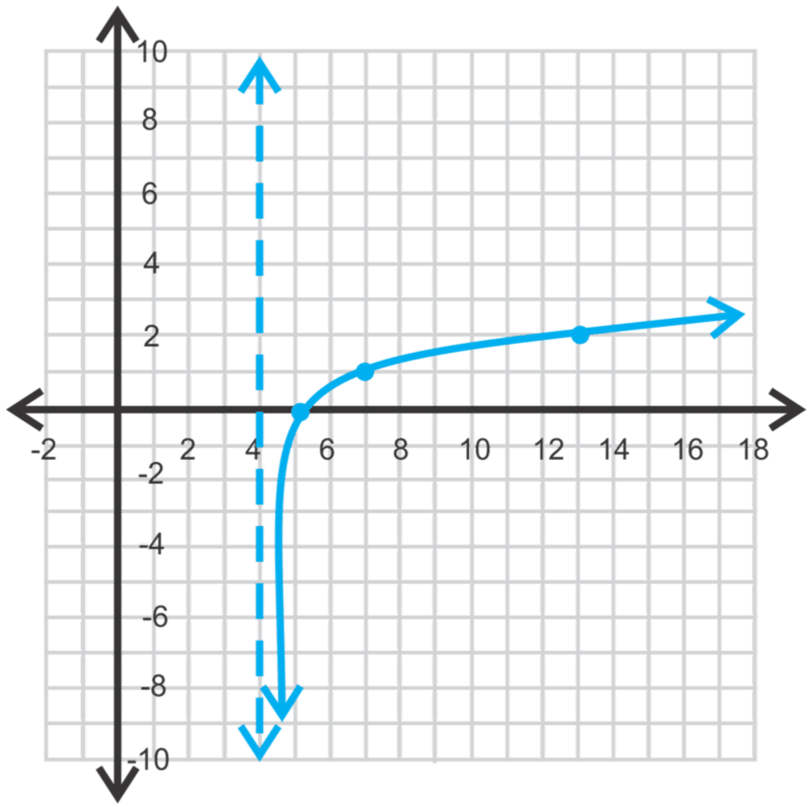


Recall that an exponential function has a horizontal asymptote. Because the logarithm is its inverse, it will have a *vertical* asymptote. The general form of a logarithmic function is $f(x) = \log_b(x-h)+k$ and the vertical asymptote is $x = h$. The domain is $x > h$ and the range is all real numbers. Lastly, if $b > 1$, the graph moves *up* to the right. If $0 < b < 1$, the graph moves *down* to the right.

Example A

Graph $y = \log_3(x-4)$. State the domain and range.

Solution:



To graph a logarithmic function without a calculator, start by drawing the vertical asymptote, at $x = 4$. We know the graph is going to have the general shape of the first function above. Plot a few "easy" points, such as (5, 0), (7, 1), and (13, 2) and connect.

The domain is $x > 4$ and the range is all real numbers.

Example B

Is (16, 1) on $y = \log(x-6)$?

Solution: Plug in the point to the equation to see if it holds true.

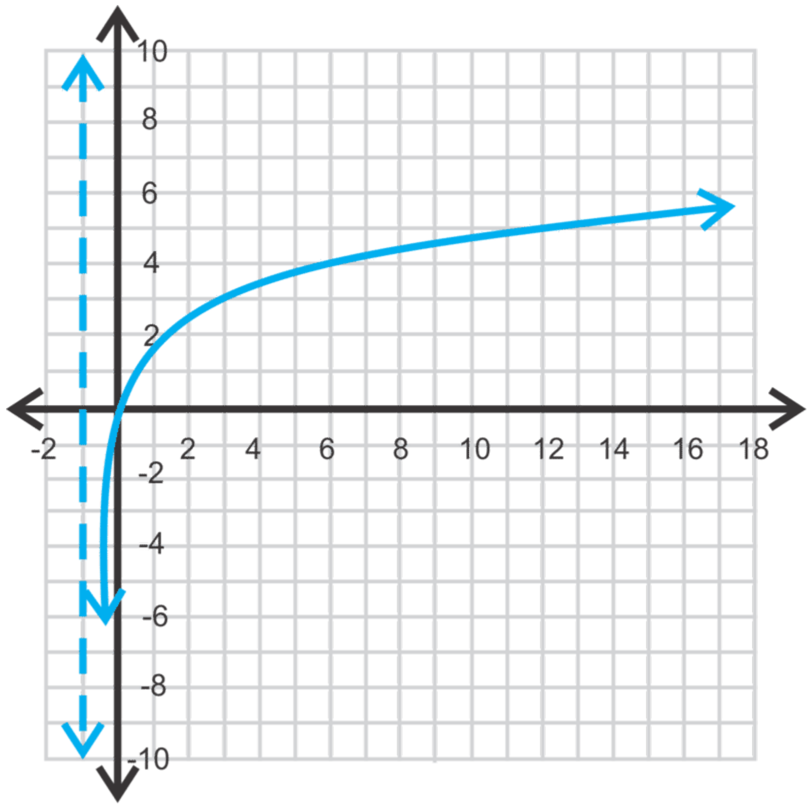
$$1 = \log(16-6)$$
$$1 = \log 10$$
$$1 = 1$$

Yes, this is true, so (16, 1) is on the graph.

Example C

Graph $f(x) = 2\ln(x+1)$.

Solution: To graph a natural log, we need to use a graphing calculator. Press $Y =$ and enter in the function, $Y = 2\ln(x+1)$, GRAPH.



Guided Practice

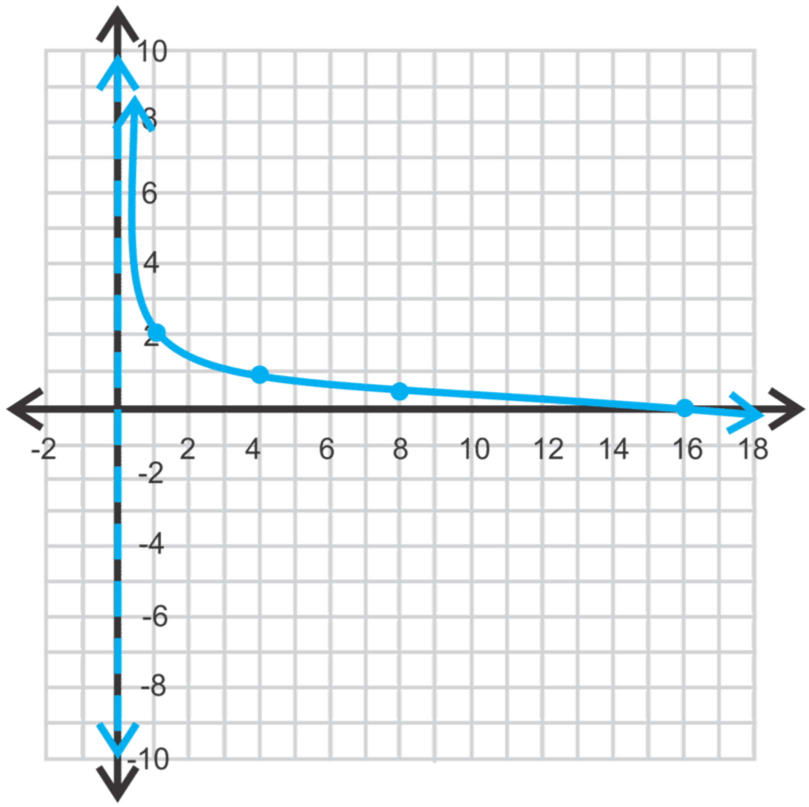
1. Graph $y = \log_{\frac{1}{4}} x + 2$ in an appropriate window.

2. Graph $y = -\log x$ using a graphing calculator. Find the domain and range.

3. Is (-2, 1) on the graph of $f(x) = \log_{\frac{1}{2}}(x+4)$?

Answers

1. First, there is a vertical asymptote at $x = 0$. Now, determine a few easy points, points where the log is easy to find; such as (1, 2), (4, 1), (8, 0.5), and (16, 0).

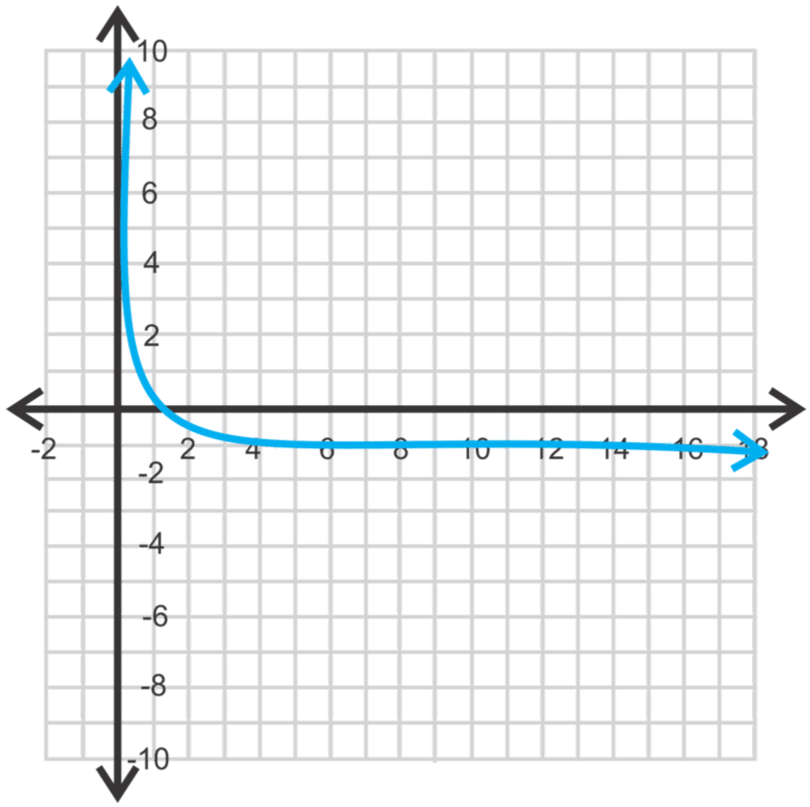


To graph a logarithmic function using a TI-83/84, enter the function into $Y =$ and use the *Change of Base Formula*. The keystrokes would be:

$Y = \frac{\log(x)}{\log\left(\frac{1}{4}\right)} + 2$, GRAPH

To see a table of values, press $2^{nd} \rightarrow$ GRAPH.

2. The keystrokes are $Y = -\log(x)$, GRAPH.



The domain is $x > 0$ and the range is all real numbers.

3. Plug (-2, 1) into $f(x) = \log_{\frac{1}{2}}(x+4)$ to see if the equation holds true.

$$1 = \log_{\frac{1}{2}}(-2+4)$$
$$1 = \log_{\frac{1}{2}} 2 \rightarrow \frac{1}{2}^x = 2$$
$$1 \neq -1$$

Therefore, (-2, 1) is not on the graph. However, (-2, -1) is.

Problem Set

Graph the following logarithmic functions without using a calculator. State the equation of the asymptote, the domain and the range of each function.

1. $y = \log_5 x$
2. $y = \log_2(x+1)$
3. $y = \log(x) - 4$
4. $y = \log_{\frac{1}{3}}(x-1) + 3$
5. $y = -\log_{\frac{1}{2}}(x+3) - 5$
6. $y = \log_4(2-x) + 2$

Graph the following logarithmic functions using your graphing calculator.

7. $y = \ln(x+6) - 1$
8. $y = -\ln(x-1) + 2$
9. $y = \ln(1-x) + 3$

10. Is (3, 8) on the graph of $y = \log_3(2x-3)+7$?
11. Is (9, -2) on the graph of $y = \log_{\frac{1}{4}}(x-5)$?
12. Is (4, 5) on the graph of $y = 5\log_2(8-x)$?

2.34 Properties of Logarithms

Objective

To simplify expressions involving logarithms.

Review Queue

Simplify the following exponential expressions.

1. $(3x^{\frac{1}{2}})^4$
2. $5x^2y \cdot 8x^{-1}y^6$
3. $\left(\frac{2x^{-1}y^2z^8}{5x^0y^{12}z}\right)^{-1}$

Product and Quotient Properties

Objective

To use and apply the product and quotient properties of logarithms.

Guidance

Just like exponents, logarithms have special properties, or shortcuts, that can be applied when simplifying expressions. In this lesson, we will address two of these properties.

Example A

Simplify $\log_b x + \log_b y$.

Solution: First, notice that these logs have the same base. If they do not, then the properties do not apply.

$\log_b x = m$ and $\log_b y = n$, then $b^m = x$ and $b^n = y$.

Now, multiply the latter two equations together.

$$\begin{aligned} b^m \cdot b^n &= xy \\ b^{m+n} &= xy \end{aligned}$$

Recall, that when two exponents with the same base are multiplied, we can add the exponents. Now, reapply the logarithm to this equation.

$b^{m+n} = xy \rightarrow \log_b xy = m+n$

Recall that $m = \log_b x$ and $n = \log_b y$, therefore $\log_b xy = \log_b x + \log_b y$.

This is the **Product Property of Logarithms**.

Example B

Expand $\log_{12} 4y$.

Solution: Applying the Product Property from Example A, we have:

$\log_{12} 4y = \log_{12} 4 + \log_{12} y$

Example C

Simplify $\log_3 15 - \log_3 5$.

Solution: As you might expect, the **Quotient Property of Logarithms** is $\log_b \frac{x}{y} = \log_b x - \log_b y$ (proof in the Problem Set). Therefore, the answer is:

$$\begin{aligned}\log_3 15 - \log_3 5 &= \log_3 \frac{15}{5}\\ &= \log_3 3\\ &= 1\end{aligned}$$

Guided Practice

Simplify the following expressions.

1. $\log_7 8 + \log_7 x^2 + \log_7 3y$
2. $\log y - \log 20 + \log 8x$
3. $\log_2 32 - \log_2 z$
4. $\log_8 \frac{16x}{y^2}$

Answers

1. Combine all the logs together using the Product Property.

$$\begin{aligned}\log_7 8 + \log_7 x^2 + \log_7 3y &= \log_7 8x^2 3y\\ &= \log_7 24x^2 y\end{aligned}$$

2. Use both the Product and Quotient Property to condense.

$$\begin{aligned}\log y - \log 20 + \log 8x &= \log \frac{y}{20} \cdot 8x\\ &= \log \frac{2xy}{5}\end{aligned}$$

3. Be careful; you do not have to use either rule here, just the definition of a logarithm.

$$\log_2 32 - \log_2 z = 5 - \log_2 z$$

4. When expanding a log, do the division first and then break the numerator apart further.

$$\begin{aligned}\log_8 \frac{16x}{y^2} &= \log_8 16x - \log_8 y^2\\ &= \log_8 16 + \log_8 x - \log_8 y^2\\ &= \frac{4}{3} + \log_8 x - \log_8 y^2\end{aligned}$$

To determine $\log_8 16$, use the definition and powers of 2: $8^n = 16 \rightarrow 2^{3n} = 2^4 \rightarrow 3n = 4 \rightarrow n = \frac{4}{3}$.

Vocabulary

Product Property of Logarithms

As long as $b \neq 1$, then $\log_b xy = \log_b x + \log_b y$

Quotient Property of Logarithms

As long as $b \neq 1$, then $\log_b \frac{x}{y} = \log_b x - \log_b y$

Problem Set

Simplify the following logarithmic expressions.

1. $\log_3 6 + \log_3 y - \log_3 4$
2. $\log 12 - \log x + \log y^2$
3. $\log_6 x^2 - \log_6 x - \log_6 y$
4. $\ln 8 + \ln 6 - \ln 12$
5. $\ln 7 - \ln 14 + \ln 10$
6. $\log_{11} 22 + \log_{11} 5 - \log_{11} 55$

Expand the following logarithmic functions.

7. $\log_3(abc)$
8. $\log\left(\frac{a^2}{b}\right)$
9. $\log_9\left(\frac{xy}{5}\right)$
10. $\log\left(\frac{2x}{y}\right)$
11. $\log\left(\frac{8x^2}{15}\right)$
12. $\log_4\left(\frac{5}{9y}\right)$
13. Write an algebraic proof of the quotient property. Start with the expression $\log_a x - \log_a y$ and the equations $\log_a x = m$ and $\log_a y = n$ in your proof. Refer to the proof of the product property in Example A as a guide for your proof.

Power Property of Logarithms

Objective

To use the Power Property of logarithms.

Guidance

The last property of logs is the **Power Property**.

$\log_b x = y$

Using the definition of a log, we have $b^y = x$. Now, raise both sides to the n power.

$$(b^y)^n = x^n$$
$$b^{ny} = x^n$$

Let's convert this back to a log with base b, $\log_b x^n = ny$. Substituting for y, we have $\log_b x^n = n \log_b x$.

Therefore, the Power Property says that if there is an exponent within a logarithm, we can pull it out in front of the logarithm.

Example A

Expand $\log_6 17x^5$.

Solution: To expand this log, we need to use the Product Property and the Power Property.

$$\begin{aligned}\log_6 17x^5 &= \log_6 17 + \log_6 x^5 \\ &= \log_6 17 + 5\log_6 x\end{aligned}$$

Example B

Expand $\ln\left(\frac{2x}{y^3}\right)^4$.

Solution: We will need to use all three properties to expand this example. Because the expression within the natural log is in parenthesis, start with moving the 4^{th} power to the front of the log.

$$\begin{aligned}\ln\left(\frac{2x}{y^3}\right)^4 &= 4\ln\frac{2x}{y^3} \\ &= 4(\ln 2x - \ln y^3) \\ &= 4(\ln 2 + \ln x - 3\ln y) \\ &= 4\ln 2 + 4\ln x - 12\ln y\end{aligned}$$

Depending on how your teacher would like your answer, you can evaluate $4\ln 2 \approx 2.77$, making the final answer $2.77 + 4\ln x - 12\ln y$.

Example C

Condense $\log 9 - 4\log 5 - 4\log x + 2\log 7 + 2\log y$.

Solution: This is the opposite of the previous two examples. Start with the Power Property.

$$\begin{aligned}&\log 9 - 4\log 5 - 4\log x + 2\log 7 + 2\log y \\ &\log 9 - \log 5^4 - \log x^4 + \log 7^2 + \log y^2\end{aligned}$$

Now, start changing things to division and multiplication within one log.

$\log\frac{9\cdot 7^2 y^2}{5^4 x^4}$

Lastly, combine like terms.

$\log\frac{441y^2}{625x^4}$

Guided Practice

Expand the following logarithmic expressions.

1. $\ln x^3$

2. $\log_{16}\frac{x^2 y}{32z^5}$

3. $\log(5c^4)^2$

4. Condense into one log: $\ln 5 - 7\ln x^4 + 2\ln y$.

Answers

1. The only thing to do here is apply the Power Property: $3\ln x$.

2. Let's start with using the Quotient Property.

$$\log_{16}\frac{x^2y}{32z^5} = \log_{16}x^2y - \log_{16}32z^5$$

Now, apply the Product Property, followed by the Power Property.

$$\begin{aligned} &= \log_{16}x^2 + \log_{16}y - \left(\log_{16}32 + \log_{16}z^5\right) \\ &= 2\log_{16}x + \log_{16}y - \frac{5}{4} - 5\log_{16}z \end{aligned}$$

Simplify $\log_{16}32 \rightarrow 16^n = 32 \rightarrow 2^{4n} = 2^5$ and solve for n. Also, notice that we put parenthesis around the second log once it was expanded to ensure that the z^5 would also be subtracted (because it was in the denominator of the original expression).

3. For this problem, you will need to apply the Power Property twice.

$$\begin{aligned} \log(5c^4)^2 &= 2\log 5c^4 \\ &= 2(\log 5 + \log c^4) \\ &= 2(\log 5 + 4\log c) \\ &= 2\log 5 + 8\log c \end{aligned}$$

Important Note: You can write this particular log several different ways. Equivalent logs are: $\log 25 + 8\log c, \log 25 + \log c^8$ and $\log 25c^8$. Because of these properties, there are several different ways to write one logarithm.

4. To condense this expression into one log, you will need to use all three properties.

$$\begin{aligned} \ln 5 - 7\ln x^4 + 2\ln y &= \ln 5 - \ln x^{28} + \ln y^2 \\ &= \ln\frac{5y^2}{x^{28}} \end{aligned}$$

Important Note: If the problem was $\ln 5 - (7\ln x^4 + 2\ln y)$, then the answer would have been $\ln\frac{5}{x^{28}y^2}$. But, because there are no parentheses, the y^2 is in the numerator.

Vocabulary

Power Property
As long as $b \neq 1$, then $\log_b x^n = n\log_b x$.

Problem Set

Expand the following logarithmic expressions.

1. $\log_4(9x)^3$
2. $\log\left(\frac{3x}{y}\right)^2$

3. $\log_8 \frac{x^3y^2}{z^4}$
4. $\log_5 \left(\frac{25x^4}{y}\right)^2$
5. $\ln \left(\frac{6x}{y^3}\right)^{-2}$
6. $\ln \left(\frac{e^5x^{-2}}{y^3}\right)^6$

Condense the following logarithmic expressions.

7. $2\log_6 x + 5\log_6 y$
8. $3(\log x - \log y)$
9. $\frac{1}{2}\log(x+1) - 3\log y$
10. $4\log_2 y + \frac{1}{3}\log_2 x^3$
11. $\frac{1}{5}\left[10\log_2(x-3) + \log_2 32 - \log_2 y\right]$
12. $4\left[\frac{1}{2}\log_3 y - \frac{1}{3}\log_3 x - \log_3 z\right]$

2.35 Solving Exponential and Logarithmic Equations

Objective

To solve exponential and logarithmic equations.

Review Queue

Solve the following equations.

1. $2^x = 32$
2. $x^2 - 9x + 20 = 0$
3. $\sqrt{x-5} + 3 = 11$
4. $8^x = 128$

Solving Exponential Equations

Objective

To learn how to solve exponential equations.

Guidance

Until now, we have only solved pretty basic exponential equations, like #1 in the Review Queue above. We know that $x = 5$, because $2^5 = 32$. Ones like #4 are a little more challenging, but if we put everything into a power of 2, we can set the exponents equal to each other and solve.

$$\begin{aligned} 8^x &= 128 \\ 2^{3x} &= 2^7 \\ 3x &= 7 \qquad \text{So}, 8^{\frac{7}{3}} = 128. \\ x &= \frac{7}{3} \end{aligned}$$

But, what happens when the power is not easily found? We must use logarithms, followed by the Power Property to solve for the exponent.

Example A

Solve $6^x = 49$. Round your answer to the nearest three decimal places.

Solution: To solve this exponential equation, let's take the logarithm of both sides. The easiest logs to use are either ln (the natural log), or log (log, base 10). We will use the natural log.

$$\begin{aligned}
6^x &= 49 \\
\ln 6^x &= \ln 49 \\
x \ln 6 &= \ln 49 \\
x &= \frac{\ln 49}{\ln 6} \approx 2.172
\end{aligned}$$

Example B

Solve $10^{x-3} = 100^{3x+11}$.

Solution: Change 100 into a power of 10.

$$\begin{aligned}
10^{x-3} &= 10^{2(3x+11)} \\
x-3 &= 6x+22 \\
-25 &= 5x \\
-5 &= x
\end{aligned}$$

Example C

Solve $8^{2x-3} - 4 = 5$.

Solution: Add 4 to both sides and then take the log of both sides.

$$\begin{aligned}
8^{2x-3} - 4 &= 5 \\
8^{2x-3} &= 9 \\
\log 8^{2x-3} &= \log 9 \\
(2x-3)\log 8 &= \log 9 \\
2x-3 &= \frac{\log 9}{\log 8} \\
2x &= 3 + \frac{\log 9}{\log 8} \\
x &= \frac{3}{2} + \frac{\log 9}{2\log 8} \approx 2.56
\end{aligned}$$

Notice that we did not find the numeric value of $\log 9$ or $\log 8$ until the very end. This will ensure that we have the most accurate answer.

Guided Practice

Solve the following exponential equations.

1. $4^{x-8} = 16$
2. $2(7)^{3x+1} = 48$
3. $\frac{2}{3} \cdot 5^{x+2} + 9 = 21$

Answers

1. Change 16 to 4^2 and set the exponents equal to each other.

$$\begin{aligned} 4^{x-8} &= 16 \\ 4^{x-8} &= 4^2 \\ x-8 &= 2 \\ x &= 10 \end{aligned}$$

2. Divide both sides by 2 and then take the log of both sides.

$$\begin{aligned} 2(7)^{3x+1} &= 48 \\ 7^{3x+1} &= 24 \\ \ln 7^{3x+1} &= \ln 24 \\ (3x+1)\ln 7 &= \ln 24 \\ 3x+1 &= \frac{\ln 24}{\ln 7} \\ 3x &= -1 + \frac{\ln 24}{\ln 7} \\ x &= -\frac{1}{3} + \frac{\ln 24}{3\ln 7} \approx 0.211 \end{aligned}$$

3. Subtract 9 from both sides and multiply both sides by $\frac{3}{2}$. Then, take the log of both sides.

$$\begin{aligned} \frac{2}{3}\cdot 5^{x+2} + 9 &= 21 \\ \frac{2}{3}\cdot 5^{x+2} &= 12 \\ 5^{x+2} &= 18 \\ (x+2)\log 5 &= \log 18 \\ x &= \frac{\log 18}{\log 5} - 2 \approx -0.204 \end{aligned}$$

Problem Set

Use logarithms and a calculator to solve the following equations for x. Round answers to three decimal places.

1. $5^x = 65$
2. $2^x = 90$
3. $6^{x+1} + 3 = 13$
4. $6(11^{3x-2}) = 216$
5. $8 + 13^{2x-5} = 35$
6. $\frac{1}{2}\cdot 7^{x-3} - 5 = 14$

Solve the following exponential equations without a calculator.

7. $4^x = 8$
8. $5^{2x+1} = 125$
9. $9^3 = 3^{4x-6}$
10. $7(2^{x-3}) = 56$
11. $16^x \cdot 4^{x+1} = 32^{x+1}$
12. $3^{3x+5} = 3\cdot 9^{x+3}$

Solving Logarithmic Equations

Objective

To solve a logarithmic equation with any base.

Guidance

A logarithmic equation has the variable within the log. To solve a logarithmic equation, you will need to use the inverse property, $b^{\log_b x} = x$, to cancel out the log.

Example A

Solve $\log_2(x+5) = 9$.

Solution: There are two different ways to solve this equation. The first is to use the definition of a logarithm.

$$\begin{aligned}\log_2(x+5) &= 9\\ 2^9 &= x+5\\ 512 &= x+5\\ 507 &= x\end{aligned}$$

The second way to solve this equation is to put everything into the exponent of a 2, and then use the inverse property.

$$\begin{aligned}2^{\log_2(x+5)} &= 2^9\\ x+5 &= 512\\ x &= 507\end{aligned}$$

Make sure to check your answers for logarithmic equations. There can be times when you get an extraneous solution. $\log_2(507+5) = 9 \rightarrow \log_2 512 = 9 \boxed{\checkmark}$

Example B

Solve $3\ln(-x) - 5 = 10$.

Solution: First, add 5 to both sides and then divide by 3 to isolate the natural log.

$$\begin{aligned}3\ln(-x) - 5 &= 10\\ 3\ln(-x) &= 15\\ \ln(-x) &= 5\end{aligned}$$

Recall that the inverse of the natural log is the natural number. Therefore, everything needs to be put into the exponent of e in order to get rid of the log.

$$\begin{aligned}e^{\ln(-x)} &= e^5\\ -x &= e^5\\ x &= -e^5 \approx -148.41\end{aligned}$$

Checking the answer, we have $3\ln(-(-e^5))-5=10 \rightarrow 3\ln e^5-5=10 \rightarrow 3\cdot 5-5=10 \boxed{\checkmark}$

Example C

Solve $\log 5x+\log(x-1)=2$

Solution: Condense the left-hand side using the Product Property.

$$\begin{aligned}\log 5x+\log(x-1)&=2\\ \log[5x(x-1)]&=2\\ \log(5x^2-5x)&=2\end{aligned}$$

Now, put everything in the exponent of 10 and solve for x.

$$\begin{aligned}10^{\log(5x^2-5x)}&=10^2\\ 5x^2-5x&=100\\ x^2-x-20&=0\\ (x-5)(x+4)&=0\\ x&=5,-4\end{aligned}$$

Now, check both answers.

$$\begin{aligned}\log 5(5)+\log(5-1)&=2\\ \log 25+\log 4&=2 \boxed{\checkmark}\\ \log 100&=2\end{aligned} \qquad \begin{aligned}\log 5(-4)+\log((-4)-1)&=2\\ \log(-20)+\log(-5)&=2 \boxed{\times}\end{aligned}$$

-4 is an extraneous solution. In the step $\log(-20)+\log(-5)=2$, we cannot take the log of a negative number, therefore -4 is not a solution. 5 is the only solution.

Guided Practice

Solve the following logarithmic equations.

1. $9+2\log_3 x=23$
2. $\ln(x-1)-\ln(x+1)=8$
3. $\frac{1}{2}\log_5(2x+5)=5$

Answers

1. Isolate the log and put everything in the exponent of 3.

$$\begin{aligned}9+2\log_3 x&=23\\ 2\log_3 x&=14\\ \log_3 x&=7\\ x&=3^7=2187\end{aligned} \qquad \begin{aligned}&9+2\log_3 2187=23\\ &9+2\cdot 7=23 \boxed{\checkmark}\\ &9+14=23\end{aligned}$$

2. Condense the left-hand side using the Quotient Rule and put everything in the exponent of e.

$$
\begin{aligned}
\ln(x-1)-\ln(x+1)&=8\\
\ln\left(\frac{x-1}{x+1}\right)&=8\\
\frac{x-1}{x+1}&=\ln 8\\
x-1&=(x+1)\ln 8\\
x-1&=x\ln 8+\ln 8\\
x-x\ln 8&=1+\ln 8\\
x(1-\ln 8)&=1+\ln 8\\
x&=\frac{1+\ln 8}{1-\ln 8}\approx -2.85
\end{aligned}
$$

Checking our answer, we get $\ln(-2.85-1)-\ln(2.85+1)=8$, which does not work because the first natural log is of a negative number. Therefore, there is no solution for this equation.

3. Multiply both sides by 2 and put everything in the exponent of a 5.

$$
\begin{aligned}
\frac{1}{2}\log_5(2x+5)&=2\\
\log_5(2x+5)&=4\\
2x+5&=625\\
2x&=620\\
x&=310
\end{aligned}
\qquad
\text{Check : }\quad
\begin{aligned}
\frac{1}{2}\log_5(2\cdot 310+5)&=2\\
\frac{1}{2}\log_5 625&=2\ \boxed{\checkmark}\\
\frac{1}{2}\cdot 4&=2
\end{aligned}
$$

Problem Set

Use properties of logarithms and a calculator to solve the following equations for x. Round answers to three decimal places and check for extraneous solutions.

1. $\log_7(2x+3)=3$
2. $8\ln(3-x)=5$
3. $4\log_3 3x-\log_3 x=5$
4. $\log(x+5)+\log x=\log 14$
5. $2\ln x-\ln x=0$
6. $3\log_3(x-5)=3$
7. $\frac{2}{3}\log_3 x=2$
8. $5\log\frac{x}{2}-3\log\frac{1}{x}=\log 8$
9. $2\ln x^{e+2}-\ln x=10$
10. $2\log_6 x+1=\log_6(5x+4)$
11. $2\log_{\frac{1}{2}} x+2=\log_{\frac{1}{2}}(x+10)$
12. $3\log_{\frac{2}{3}} x-\log_{\frac{2}{3}} 27=\log_{\frac{2}{3}} 8$

CHAPTER 3 Advanced Functions and Basic Trigonometry

Chapter Outline

3.1 Fundamental Theorem of Algebra

Learning Objectives

Here you will state the connection between zeroes of a polynomial and the Fundamental Theorem of Algebra and start to use complex numbers.

You have learned that a quadratic has at most two real zeroes and a cubic has at most three real zeros. You may have noticed that the number of real zeros is always less than or equal to the degree of the polynomial. By looking at a graph you can see when a parabola crosses the x axis 0, 1 or 2 times, but what does this have to do with complex numbers?

The Fundamental Theorem of Algebra

A **real number** is any rational or irrational number. When a real number is squared, it will always produce a non-negative value. **Complex numbers** include real numbers and another type of number called **imaginary numbers**. Unlike real numbers, imaginary numbers may produce a negative value when squared. The square root of negative one is defined to be the imaginary number i.

$i = \sqrt{-1}$ and $i^2 = -1$

Complex numbers are written with a real component and an imaginary component. All complex numbers can be written in the form $a+bi$. When the imaginary component is zero, the number is simply a real number. This means that real numbers are a subset of complex numbers.

The **Fundamental Theorem of Algebra** states that an n^{th} degree polynomial with real or complex coefficients has, with multiplicity, exactly n complex roots. This means a cubic will have exactly 3 roots, some of which may be complex.

Take the following polynomial.

$f(x) = x^2+9$

At first you may think that this does not have any roots but the Fundamental Theorem of Algebra states that it must have 2 roots. Both roots for this polynomial are complex.

To find the roots for this complex polynomial, set $y = 0$ and solve for x. This will give you the two zeros.

$$\begin{aligned} 0 &= x^2+9 \\ -9 &= x^2 \\ \pm 3i &= x \end{aligned}$$

Thus the linear factorization of the function is:

$f(x) = (x-3i)(x+3i)$

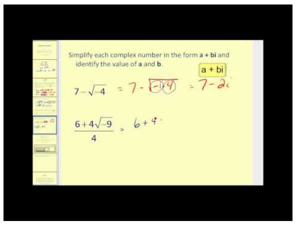
Multiplicity refers to when a root counts more than once. For example, in the following function the only root occurs at $x = 3$.

$f(x) = (x-3)^2$

The Fundamental Theorem of Algebra states that this 2^{nd} degree polynomial must have exactly 2 roots with multiplicity. This means that the root $x = 3$ has multiplicity 2. One way to determine the multiplicity is to simply look at the degree of each of the factors in the factorized polynomial.

$g(x) = (x-1)(x-3)^4(x+2)$

This function has 6 roots. The first two roots $x = 1$ and $x = -2$ have multiplicities of 1 because the power of each of their binomial factors is 1. The third root $x = 3$ has a multiplicity of 4 because the power of its binomial factor is 4. Keep in mind that all polynomials can be written in factorized form like the above polynomial, due to a theorem called the Linear Factorization Theorem.



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Examples

Example 1

Earlier, you were asked about the graphs of parabolas and their relationships to the zeroes of the parabolas. When a parabola fails to cross the x axis it still has 2 roots. These two roots happen to be imaginary numbers. The function $f(x) = x^2 + 4$ does not cross the x axis, but its roots are $x = \pm 2i$.

Example 2

Identify the polynomial that has the following five roots. $x = 0, 2, 3, \pm\sqrt{5}i$

Write the function in factorized form.

$f(x) = (x-0)(x-2)(x-3)(x-\sqrt{5}i)(x+\sqrt{5}i)$

When you multiply through, it will be helpful to do the complex conjugates first. Remember that **Complex conjugates** are pairs of complex numbers with real parts that are identical and imaginary parts that are of equal magnitude but opposite signs. The complex conjugates in this equation are $(x-\sqrt{5}i)(x+\sqrt{5}i)$.

$$
\begin{aligned}
f(x) &= x(x^2-5x+6)(x^2-5\cdot(-1)) \\
f(x) &= (x^3-5x^2+6x)(x^2+5) \\
f(x) &= x^5-5x^4+6x^3+5x^3-25x^2+30x \\
f(x) &= x^5-5x^4+11x^3-25x^2+30x
\end{aligned}
$$

Example 3

Write the polynomial that has the following roots: 4 (with multiplicity 3), 2 (with multiplicity 2) and 0.

$f(x) = (x-4)^3 \cdot (x-2)^2 \cdot x$

Example 4

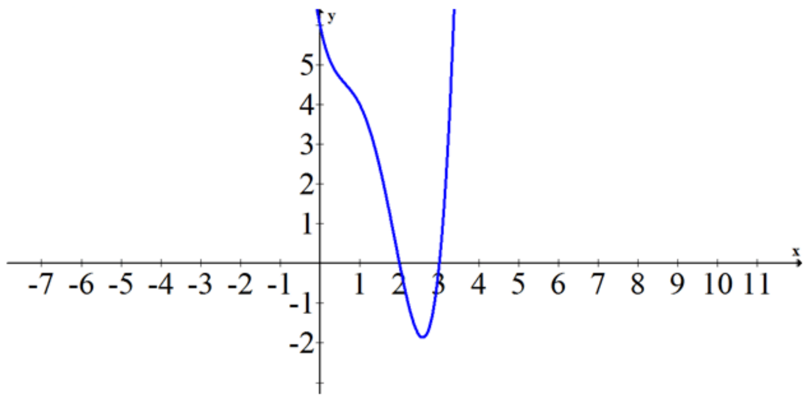
Factor the following polynomial into its linear factorization and state all of its roots.

$$f(x) = x^4 - 5x^3 + 7x^2 - 5x + 6$$

You can use polynomial long division to obtain the following factorization.

$$f(x) = (x-3)(x-2)(x-i)(x+i)$$

If you need a place to start, it is helpful to look at the graph of the polynomial and notice that the graph shows you exactly where the real roots appear.



Example 5

Can you create a polynomial with real coefficients that has one imaginary root? Why or why not?

No, if a polynomial has real coefficients then either it has no imaginary roots, or the imaginary roots come in pairs of complex conjugates (so that the imaginary portions cancel out when the factors are multiplied).

Review

For 1 - 4, find the polynomial with the given roots.

1. 2 (with multiplicity 2), 4 (with multiplicity 3), 1, $\sqrt{2}i, -\sqrt{2}i$.
2. 1, -3 (with multiplicity 3), -1, $\sqrt{3}i, -\sqrt{3}i$
3. 5 (with multiplicity 2), -1 (with multiplicity 2), $2i, -2i$
4. $i, -i$, $\sqrt{2}i, -\sqrt{2}i$

For each polynomial, factor into its linear factorization and state all of its roots.

5. $f(x) = x^5 + 4x^4 - 2x^3 - 14x^2 - 3x - 18$
6. $g(x) = x^4 - 1$
7. $h(x) = x^6 - 12x^5 + 61x^4 - 204x^3 + 532x^2 - 864x + 576$
8. $j(x) = x^7 - 11x^6 + 49x^5 - 123x^4 + 219x^3 - 297x^2 + 243x - 81$
9. $k(x) = x^5 + 3x^4 - 11x^3 - 15x^2 + 46x - 24$
10. $m(x) = x^6 - 12x^4 + 23x^2 + 36$
11. $n(x) = x^6 - 3x^5 - 10x^4 - 32x^3 - 81x^2 - 85x - 30$
12. $p(x) = x^6 + 4x^5 + 7x^4 + 12x^3 - 16x^2 - 112x - 112$

13. How can you tell the number of roots that a polynomial has from its equation?
14. Explain the meaning of the term "multiplicity".
15. A polynomial with real coefficients has one root that is $\sqrt{3}i$. What other root(s) must the polynomial have?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.1.

3.2 Arithmetic with Complex Numbers

Learning Objectives

Here you will add, subtract, multiply and divide complex numbers. You will also find the absolute value of complex numbers and plot complex numbers in the complex plane.

The idea of a complex number can be hard to comprehend, especially when you start thinking about absolute value. In the past you may have thought of the absolute value of a number as just the number itself or its positive version. How should you think about the absolute value of a complex number?

Arithmetic Operations with Complex Numbers

Complex numbers follow all the same rules as real numbers for the operations of adding, subtracting, multiplying and dividing. There are a few important ideas to remember when working with complex numbers:

1. When simplifying, you must remember to combine imaginary parts with imaginary parts and real parts with real parts. For example, $4+5i+2-3i = 6+2i$.
2. If you end up with a complex number in the denominator of a fraction, eliminate it by multiplying both the numerator and denominator by the complex conjugate of the denominator.
3. The powers of i are:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -\sqrt{-1} = -i$
- $i^4 = 1$
- $i^5 = i$
- . . . and the pattern repeats

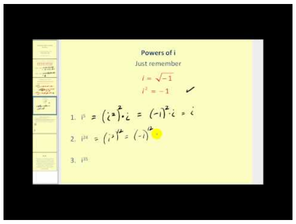
Take the following complex expression.

$(2+3i)(1-5i)-3i+8$

First, multiply the two binomials and then combine the imaginary parts with imaginary parts and real parts with real parts.

$$
\begin{aligned}
&= 2-10i+3i-15i^2-3i+8 \\
&= 10-10i+15 \\
&= 25-10i
\end{aligned}
$$

Note that a power higher than 1 of i can be simplified using the pattern above.



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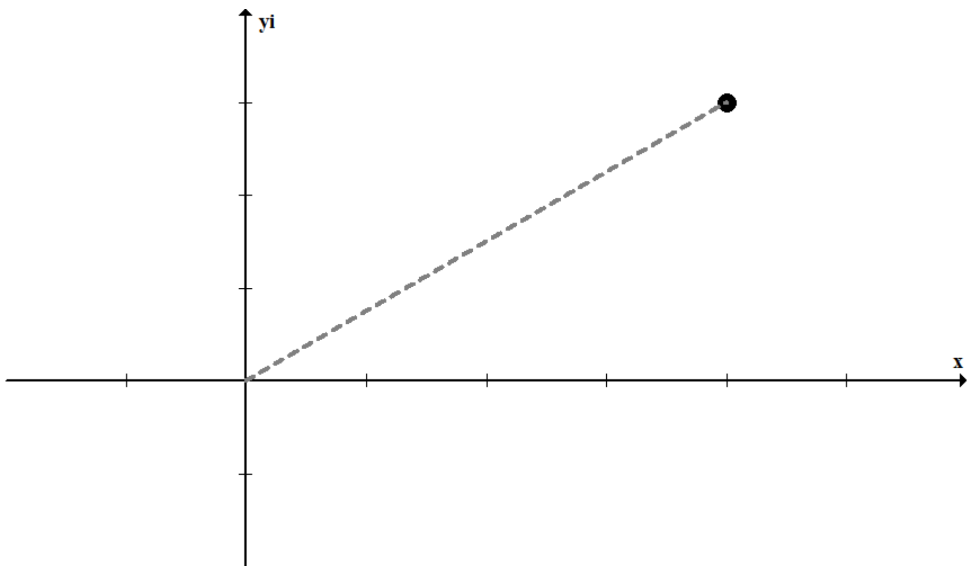
The **complex plane** is set up in the same way as the regular x, y plane, except that real numbers are counted horizontally and complex numbers are counted vertically. The following is the number $4+3i$ plotted in the complex number plane. Notice how the point is four units over and three units up.



The absolute value of a complex number like $|4+3i|$ is defined as the distance from the complex number to the origin. You can use the Pythagorean Theorem to get the absolute value. In this case, $|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5$.

Examples

Example 1

Earlier, you were asked how to think about the absolute value of a complex number. A good way to think about the absolute value for all numbers is to define it as the distance from a number to zero. In the case of complex numbers where an individual number is actually a coordinate on a plane, zero is the origin.

Example 2

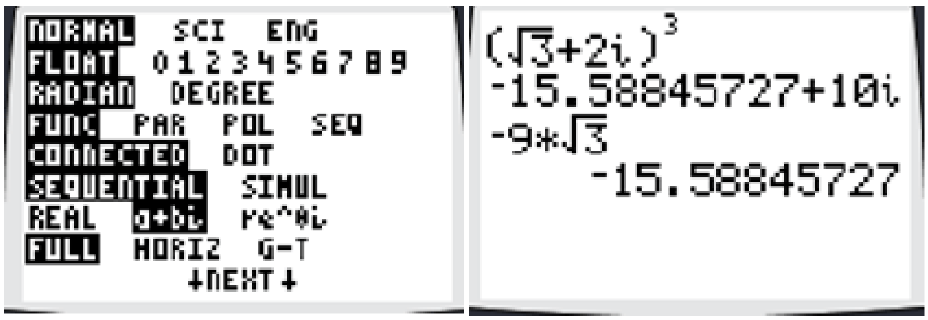
Compute the following power by hand and use your calculator to support your work.

$\left(\sqrt{3}+2i\right)^3$

$\left(\sqrt{3}+2i\right) \cdot \left(\sqrt{3}+2i\right) \cdot \left(\sqrt{3}+2i\right)$

$$\begin{aligned}
&= \left(3+4i\sqrt{3}-4\right)\left(\sqrt{3}+2i\right)\\
&= \left(-1+4i\sqrt{3}\right)\left(\sqrt{3}+2i\right)\\
&= -\sqrt{3}-2i+12i-8\sqrt{3}\\
&= -9\sqrt{3}+10i
\end{aligned}$$

A TI-84 can be switched to imaginary mode and then compute exactly what you just did. Note that the calculator will give a decimal approximation for $-9\sqrt{3}$.



Example 3

Simplify the following complex expression.

$\frac{7-9i}{4-3i}+\frac{3-5i}{2i}$

To add fractions you need to find a common denominator.

$$\begin{aligned}
&\frac{(7-9i)\cdot 2i}{(4-3i)\cdot 2i}+\frac{(3-5i)\cdot(4-3i)}{2i\cdot(4-3i)}\\
&=\frac{14i+18}{8i+6}+\frac{12-20i-9i-15}{8i+6}\\
&=\frac{15-15i}{8i+6}
\end{aligned}$$

Lastly, eliminate the imaginary component from the denominator by using the conjugate.

$$\begin{aligned}
&=\frac{(15-15i)\cdot(8i-6)}{(8i+6)\cdot(8i-6)}\\
&=\frac{120i-90+120+90i}{100}\\
&=\frac{30i+30}{100}\\
&=\frac{3i+3}{10}
\end{aligned}$$

Example 4

Simplify the following complex number.

i^{2013}

When simplifying complex numbers, i should not have a power greater than 1. The powers of i repeat in a four part cycle:

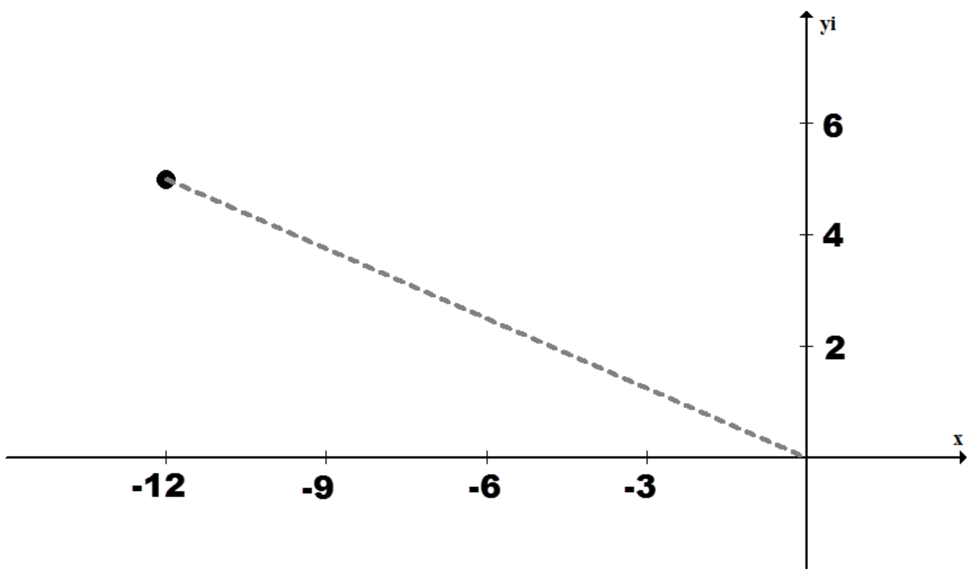
$$\begin{aligned}
i^5 &= i = \sqrt{-1} \\
i^6 &= i^2 = -1 \\
i^7 &= i^3 = -\sqrt{-1} = -i \\
i^8 &= i^4 = 1
\end{aligned}$$

Therefore, you just need to determine where 2013 is in the cycle. To do this, determine the remainder when you divide 2013 by 4. The remainder is 1 so $i^{2013} = i$.

Example 5

Plot the following complex number on the complex coordinate plane and determine its absolute value.

$-12+5i$



The sides of the right triangle are 5 and 12, which you should recognize as a Pythagorean triple with a hypotenuse of 13. $|-12+5i| = 13$.

Review

Simplify the following complex numbers.

1. i^{252}
2. i^{312}
3. i^{411}
4. i^{2345}

For each of the following, plot the complex number on the complex coordinate plane and determine its absolute value.

5. $6-8i$

6. $2+i$

7. $4-2i$

8. $-5i+1$

Let $c=2+7i$ and $d=3-5i$.

9. What is $c+d$?

10. What is $c-d$?

11. What is $c \cdot d$?

12. What is $2c-4d$?

13. What is $2c \cdot 4d$?

14. What is $\frac{c}{d}$?

15. What is c^2-d^2 ?

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.2.

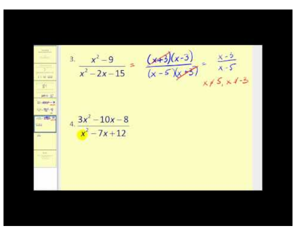
3.3 Rational Expressions

Learning Objectives

Here you will add, subtract, multiply and divide rational expressions in order to help you solve and graph rational expressions in the future. A rational expression is a ratio, just like a fraction. However, instead of a ratio between numbers, a **rational expression** is a ratio between two expressions. One driving question to ask is: Are the rules for simplifying and operating on rational expressions the same as the rules for simplifying and operating on fractions?

Working with Rational Expressions

When simplifying or operating on rational expressions, it is vital that each polynomial be fully factored. Once all expressions are factored, identical factors in the numerator and denominator may be canceled. The reason they can be "canceled" is that any expression divided by itself is equal to 1. An identical expression in the numerator and denominator is just an expression being divided by itself, and so equals 1.



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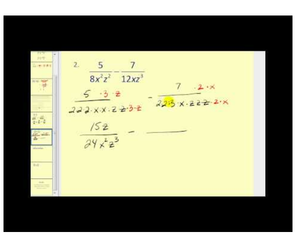
Adding and Subtracting Rational Expressions

To add or subtract rational expressions, it is essential to first find a common denominator. While any common denominator will work, using the least common denominator is a means of keeping the number of additional factors under control. Look at each rational expression you are working with and identify your desired common denominator. Multiply each expression by an appropriate form of 1, such as $\frac{x-2}{x-2}$, and then you should have your common denominator. In addition and subtraction problems, the numerator must be multiplied, combined, and re-factored to be considered simplified.



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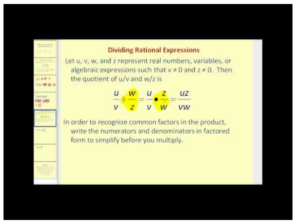
Multiplying and Dividing Rational Expressions

To multiply rational expressions, you should write the product of all the numerator factors over the product of all the denominator factors and then cancel identical factors.To divide rational expressions, you should rewrite the division problem as a multiplication problem. Multiply the first rational expression by the reciprocal of the second rational expression. Follow the steps above for multiplying. In both multiplication and division problems answers are most commonly left entirely factored to demonstrate everything has been reduced appropriately.



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Examples

Example 1

Earlier, you were asked if the rules for simplifying and operating on rational expressions are the same as the rules for simplifying and operating on fractions. Rational expressions are an extension of fractions and the operations of simplifying, adding, subtracting, multiplying and dividing work in exactly the same way.

Example 2

Simplify the following expression.

$$\frac{\frac{1}{x+1}-\frac{1}{x+2}}{\frac{1}{x-2}-\frac{1}{x+1}}$$

The expression itself does not look like a rational expression, but it can be rewritten so it is more recognizable. Also working with fractions within fractions is an important skill.

$$\begin{aligned}
&= \left(\frac{1}{x+1}-\frac{1}{x+2}\right)\div\left(\frac{1}{x-2}-\frac{1}{x+1}\right)\\
&= \left[\frac{(x+2)}{(x+1)(x+2)}-\frac{(x+1)}{(x+1)(x+2)}\right]\div\left[\frac{(x+1)}{(x+1)(x-2)}-\frac{(x-2)}{(x+1)(x-2)}\right]\\
&= \left[\frac{1}{(x+1)(x+2)}\right]\div\left[\frac{3}{(x+1)(x-2)}\right]\\
&= \frac{1}{(x+1)(x+2)}\cdot\frac{(x+1)(x-2)}{3}\\
&= \frac{(x-2)}{3(x+2)}
\end{aligned}$$

Example 3

Subtract the following rational expressions.

$$\frac{x-2}{x+3}-\frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$$

Being able to factor effectively is of paramount importance.

$$=\frac{x-2}{x+3}-\frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$$

Before subtracting, simplify where possible so you don't contribute to unnecessarily complicated denominators.

$$=\frac{(x-2)}{(x+3)}-\frac{x^2+8}{2(x+2)(x+3)}$$

The left expression lacks $2(x+2)$, so multiply both its numerator and denominator by $2(x+2)$.

Simplify the following rational expression.

$$\frac{x^2+7x+12}{x^2+4x+3}\cdot\frac{x^2+9x+8}{2x^2-128}\div\frac{x+4}{x-8}\cdot\frac{14}{1}$$

First factor everything. Second, turn division into multiplication (only one term). Third, cancel appropriately which will leave the answer.

$$=\frac{(x+3)(x+4)}{(x+3)(x+1)}\cdot\frac{(x+8)(x+1)}{2(x+8)(x-8)}\cdot\frac{(x-8)}{(x+4)}\cdot\frac{14}{1}$$

In this example, the strike through is shown. You should use this technique to match up factors in the numerator and the denominator when simplifying.

Example 5

Combine the following rational expressions.

$$\frac{1}{x^2+5x+6}-\frac{1}{x^2-4}+\frac{(x-7)(x+5)+5}{(x+2)(x-2)(x+3)(x-4)}$$

First factor everything and decide on a common denominator. While the numerators do not really need to be factored, it is sometimes helpful in simplifying individual expressions before combining them. Note that the numerator of the expression on the right hand seems factored but it really is not. Since the 5 is not connected to the rest of the numerator through multiplication, that part of the expression needs to be multiplied out and like terms need to be combined.

$$=\frac{1}{(x+2)(x+3)}-\frac{1}{(x+2)(x-2)}+\frac{x^2-2x-35+5}{(x+2)(x-2)(x+3)(x-4)}$$

Note that the right expression has 4 factors in the denominator while each of the left expressions have two that match and two that are missing from those four factors. This tells you what you need to multiply each expression by in order to have the denominators match up.

$=\frac{(x-2)(x-4)}{(x+2)(x-2)(x+3)(x-4)}-\frac{(x+3)(x-4)}{(x+2)(x-2)(x+3)(x-4)}+\frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$

Now since the rational expressions have a common denominator, the numerators may be multiplied out and combined. Sometimes instead of rewriting an expression repeatedly in mathematics you can use an abbreviation. In this case, you can replace the denominator with the letter D and then replace it at the end.

$$=\frac{(x-2)(x-4)-(x+3)(x-4)+x^2-2x-30}{D}$$

Notice how it is extremely important to use brackets to indicate that the subtraction applies to all the terms of the middle expression not just x^2. This is one of the most common mistakes.

$$=\frac{x^2-6x+8-x^2+x+12+x^2-2x-30}{D}$$

$$=\frac{x^2-7x-10}{D}$$

After the numerator has been entirely simplified try to factor the remaining expression and see if anything cancels with the denominator which you now need to replace.

$$= \frac{\cancel{(x+2)}(x-5)}{\cancel{(x+2)}(x-2)(x+3)(x-4)}$$

Bonus Example

Simplify the following expression which has an infinite number of fractions nested within other fractions.

$$\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\dots}}}}}$$

It would be an exercise in futility to attempt to try to compute this expression directly. Instead, notice that the repeating nature of the expression lends itself to an extremely nice substitution.

Let $\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\dots}}}}} = x$

Notice that the red portion of the fraction is exactly the same as the rest of the fraction and so x may be substituted there and solved.

$$= x-1 \qquad = x($$

The reason why this expression is included in this concept is because it highlights one problem solving aspect of simplifying expressions that distinguishes PreCalculus from Algebra 1 and Algebra 2.

Review

Perform the indicated operation and simplify as much as possible.

1. $\frac{x^2+5x+4}{x^2+4x+3} \cdot \frac{5x^2+15x}{x+4}$
2. $\frac{x^2-4}{x^2+4x+4} \cdot \frac{7}{x-2}$

3. $\frac{4x^2-12x}{5x+10} \cdot \frac{x+2}{x} \div \frac{x-3}{1}$
4. $\frac{4x^3-4x}{x} \div \frac{2x-2}{x-4}$
5. $\frac{2x^3+8x}{x+1} \div \frac{x}{2x^2-2}$
6. $\frac{3x-1}{x^2+2x-15} - \frac{2}{x+5}$
7. $\frac{x^2-8x+7}{x^2-4x-21} \cdot \frac{x^2-9}{1-x^2}$
8. $\frac{2}{x+7} + \frac{1}{x-7}$
9. $\frac{6}{x-7} - \frac{6}{x+7}$
10. $\frac{3x+35}{x^2-25} + \frac{2}{x+5}$
11. $\frac{2x+20}{x^2-4x-12} + \frac{2}{x+2}$
12. $\frac{2}{x+6} - \frac{x-9}{x^2-3x-18}$
13. $-\frac{5x+30}{x^2+11x+30} + \frac{2}{x+5}$
14. $\frac{x+3}{x+2} + \frac{x^3+4x^2+5x+20}{2x^4+2x^2-40}$
15. $\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\dots}}}}}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.4.

3.4 Solving Rational Equations

Learning Objectives

Here you will extend your knowledge of linear and quadratic equations to rational equations in general. You will gain insight as to what extraneous solutions are and how to identify them.

The techniques for solving rational equations are extensions of techniques you already know. Recall that when there are fractions in an equation you can multiply through by the denominator to clear the fraction. The same technique helps turn rational expressions into polynomials that you already know how to solve. When you multiply by a constant there is no problem, but when you multiply through by a value that varies and could possibly be zero interesting things happen.

Since every equation is trivially true when both sides are multiplied by zero, how do you account for this when solving rational equations?

Finding Solutions to Rational Equations

The first step in solving rational equations is to transform the equation into a polynomial equation. This is accomplished by clearing the fraction which means multiplying the entire equation by the common denominator of all the rational expressions. Then you should solve using what you already know. The last thing to check once you have the solutions is that they do not make the denominators of any part of the equation equal to zero when substituted back into the original equation. If so, that solution is called **extraneous** and is a "fake" solution that was introduced when both sides of the equation were multiplied by a number that happened to be zero.

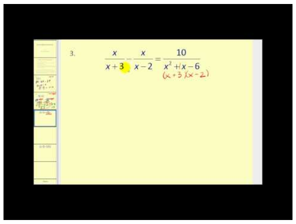
Take the following rational equation:

$x-\frac{5}{x+3}=12$

To find the solutions of the equation, first multiply all parts of the equation by $(x+3)$, the common denominator, and then simplify.

$$\begin{aligned}
x(x+3)-5&=12(x+3)\\
x^2+3x-5-12x-36&=0\\
x^2-9x-41&=0\\
x&=\frac{-(-9)\pm\sqrt{(-9)^2-4\cdot 1\cdot(-41)}}{2\cdot 1}\\
x&=\frac{9\pm 7\sqrt{5}}{2}
\end{aligned}$$

The only potential extraneous solution would have been -3 since that is the number that makes the denominator of the original equation zero. Therefore, both answers are possible.



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Examples

Example 1

Earlier, you were asked to account for the extra solutions introduced when both sides of an equation are multiplied by a variable. In order to deal with the possible extra solutions, you must check each solution to see if it makes the denominator of any fraction in the original equation zero. If it does, it is called an extraneous solution.

Example 2

Solve the following rational equation

$\frac{3x}{x+4} - \frac{1}{x+2} = -\frac{2}{x^2+6x+8}$

Multiply each part of the equation by the common denominator of $x^2+6x+8 = (x+2)(x+4)$.

$$\begin{aligned}
(x+2)(x+4)\left[\frac{3x}{x+4} - \frac{1}{x+2}\right] &= \left[\frac{-2}{(x+2)(x+4)}\right](x+2)(x+4) \\
3x(x+2) - (x+4) &= -2 \\
3x^2 + 6x - x - 2 &= 0 \\
3x^2 + 5x - 2 &= 0 \\
(3x-1)(x+2) &= 0 \\
x &= \frac{1}{3}, -2
\end{aligned}$$

Note that -2 is an extraneous solution. The only actual solution is $x = \frac{1}{3}$.

Example 3

Solve the following rational equation for y.

$x = 2 + \frac{1}{2+\frac{1}{y+1}}$

This question can be done multiple ways. You can use the clearing fractions technique twice.

$$\begin{aligned}
\left(2+\frac{1}{y+1}\right)x &= \left[2+\frac{1}{2+\frac{1}{y+1}}\right]\left(2+\frac{1}{y+1}\right)\\
2x+\frac{x}{y+1} &= 2\left(2+\frac{1}{y+1}\right)+1\\
2x+\frac{x}{y+1} &= 4+\frac{2}{y+1}+1\\
(y+1)\left[2x+\frac{x}{y+1}\right] &= \left[5+\frac{2}{y+1}\right](y+1)\\
2x(y+1)+x &= 5(y+1)+2\\
2xy+2x+x &= 5y+5+2
\end{aligned}$$

Now just get the y variable to one side of the equation and everything else to the other side.

$$\begin{aligned}
2xy-5y &= -3x+7\\
y(2x-5) &= -3x+7\\
y &= \frac{-3x+7}{2x-5}
\end{aligned}$$

Example 4

Solve the following rational equation.

$\frac{3x}{x-5}+4=x$

$\frac{3x}{x-5}+4=x$

$$\begin{aligned}
3x+4x-20 &= x^2-5x\\
0 &= x^2-12x+20\\
0 &= (x-2)(x-10)\\
x &= 2, 10
\end{aligned}$$

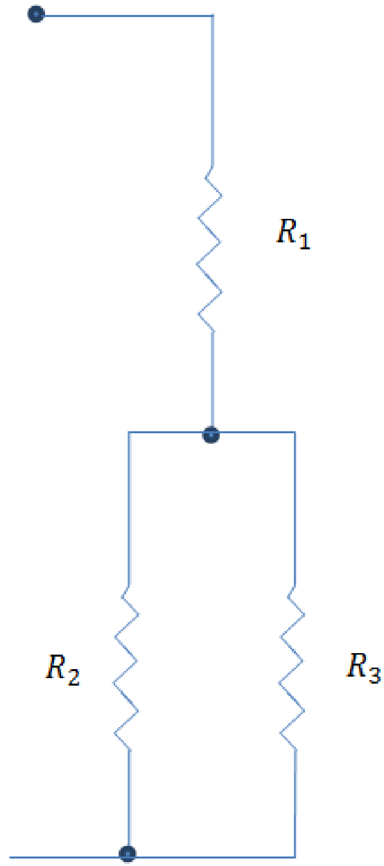
Neither solution is extraneous.

Example 5

In electrical circuits, resistance can be solved for using rational expressions. This is an electric circuit diagram with three resistors. The first resistor R_1 is run in series to the other two resistors R_2 and R_3 which are run in parallel. If the total resistance R is 100 ohms and R_1 and R_3 are each 22 ohms, what is the resistance of R_2?

The equation of value is:

$$R = R_1 + \frac{R_2R_3}{R_2+R_3}$$



$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$

$$\begin{aligned}
100 &= 22 + \frac{x \cdot 22}{x+22} \\
78(x+22) &= 22x \\
78x + 1716 &= 22x \\
56x &= -1716 \\
x &= -30.65
\end{aligned}$$

A follow up question would be to ask whether or not ohms can be negative which is beyond the scope of this text.

Review

Solve the following rational equations. Identify any extraneous solutions.

1. $\frac{2x-4}{x} = \frac{16}{x}$
2. $\frac{4}{x+1} - \frac{x}{x+1} = 2$
3. $\frac{5}{x+3} + \frac{2}{x-3} = 1$
4. $\frac{3}{x-4} - \frac{5}{x+4} = 6$
5. $\frac{x}{x+1} - \frac{6}{x+2} = 4$
6. $\frac{x}{x-4} - \frac{4}{x-4} = 8$
7. $\frac{4x}{x-2} + 3 = 1$
8. $\frac{-2x}{x+1} + 6 = -x$

9. $\frac{1}{x+2}+1=-2x$
10. $\frac{-6x-3}{x+1}-3=-4x$
11. $\frac{x+3}{x}-\frac{3}{x+3}=\frac{6}{x^2+3x}$
12. $\frac{x-4}{x}-\frac{2}{x-4}=\frac{8}{x^2-4x}$
13. $\frac{x+6}{x}-\frac{2}{x+6}=\frac{12}{x^2+6x}$
14. $\frac{x+5}{x}-\frac{3}{x+5}=\frac{15}{x^2+5x}$
15. Explain what it means for a solution to be extraneous.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.6.

3.5 Holes in Rational Functions

Learning Objectives

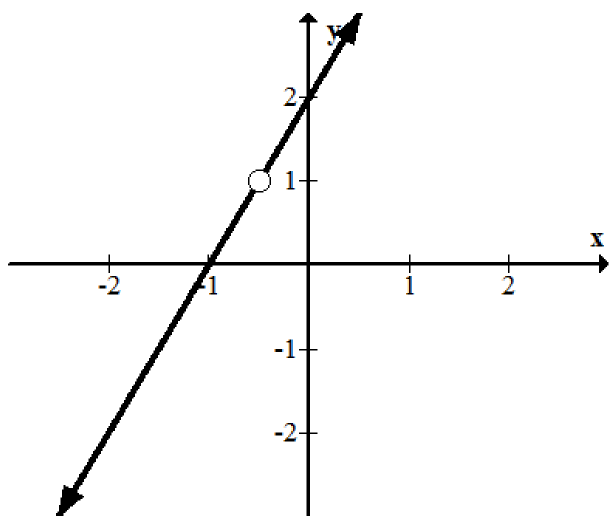
Here you will start factoring rational expressions that have holes known as removable discontinuities. In a function like $f(x)=\frac{(3x+1)(x-1)}{(x-1)}$, you should note that the factor $(x-1)$ clearly cancels leaving only $3x-1$. This appears to be a regular line. What happens to this line at $x=1$?

Holes and Rational Functions

A **hole** on a graph looks like a hollow circle. It represents the fact that the function approaches the point, but is not actually defined on that precise x value.

Take a look at the graph of the following equation:

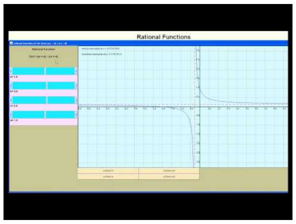
$f(x)=(2x+2)\cdot\frac{\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)}$



The reason why this function is not defined at $-\frac{1}{2}$ is because $-\frac{1}{2}$ is not in the domain of the function. As you can see, $f\left(-\frac{1}{2}\right)$ is undefined because it makes the denominator of the rational part of the function zero which makes the whole function undefined. Also notice that once the factors are canceled/removed then you are left with a normal function which in this case is $2x+2$. The hole in this situation is at $\left(-\frac{1}{2},1\right)$ because after removing the factors that cancel, $f\left(-\frac{1}{2}\right)=1$.

This is the essence of dealing with holes in rational functions. You should cancel what you can and graph the function like normal making sure to note what x values make the function undefined. Once the function is graphed without holes go back and insert the hollow circles indicating what x values are removed from the domain. This is why holes are called removable discontinuities.

Watch the first part of this video and focus on holes in rational equations.



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Examples

Example 1

Earlier, you were asked what happens to the equation $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$ at $x = 1$. Since this function that is not defined at $x = 1$ there is a removable discontinuity that is represented as a hollow circle on the graph. Otherwise the function behaves precisely as $3x+1$.

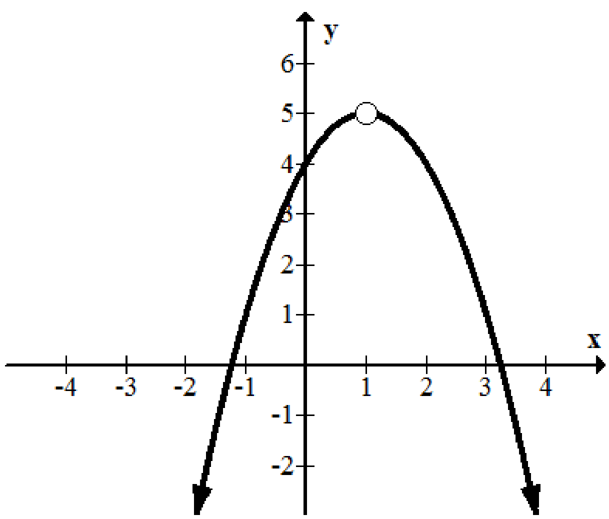
Example 2

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{-x^3+3x^2+2x-4}{x-1}$

This function requires some algebra to change it so that the removable factors become obvious. You should suspect that $(x-1)$ is a factor of the numerator and try polynomial or synthetic division to factor. When you do, the function becomes:

$f(x) = \frac{(-x^2+2x+4)(x-1)}{(x-1)}$



The removable discontinuity occurs at (1, 5).

Example 3

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{x^3-7x-6}$

This is probably one of the most challenging rational expressions with only holes that people ever try to graph by hand. There are multiple ways to start, but a good habit to get into is to factor everything you possibly can initially. The denominator seems less complicated with possible factors $(x \pm 1), (x \pm 2), (x \pm 3), (x \pm 6)$. Using polynomial division, you will find the denominator becomes:

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{(x+1)(x+2)(x-3)}$

The factors of the denominator are strong hints as to the factors of the numerator so use polynomial division and try each. When you fully factor the numerator you will have:

$f(x) = \frac{(x^3-6x^2+12x-9)(x+1)(x+2)(x-3)}{(x+1)(x+2)(x-3)}$

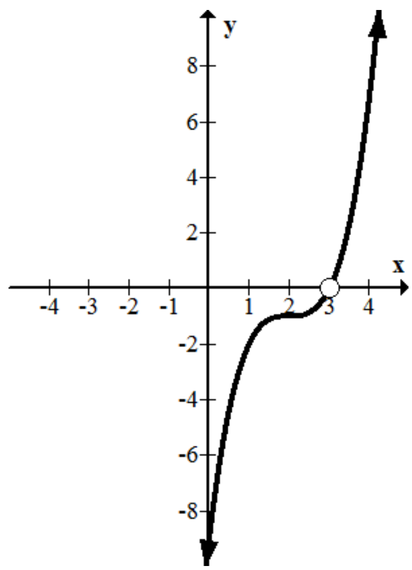
Note the factors that cancel $(x = -1, -2, 3)$ and then work with the cubic function that remains.

$f(x) = x^3 - 6x^2 + 12x - 9$

At this point it is probably reasonable to make a table and plot points to get a sense of where this cubic function lives. You also could notice that the coefficients are almost of the pattern 1 3 3 1 which is the binomial expansion. By separating the -9 into -8 -1 you can factor the first four terms.

$f(x) = x^3 - 6x^2 + 12x - 8 - 1 = (x-2)^3 - 1$

This is a cubic function that has been shifted right by two units and down one unit.



Note that there are two holes that do not fit in the graph window. When this happens you still need to note where they would appear given a properly sized window. To do this, substitute the invalid x values: $x = -1, -2, 3$ into the factored cubic that remained after canceling.

$f(x) = (x-2)^3 - 1$

Holes: (3, 0); (-1, -28); (-2, -65)

Example 4

Without graphing, identify the location of the holes of the following function.

$f(x) = \frac{x^3+4x^2+x-6}{x^2+5x+6}$

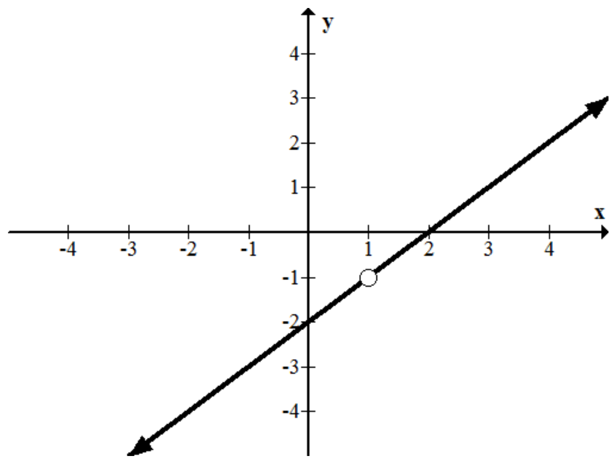
First factor everything. Then, identify the x values that make the denominator zero and use those values to find the exact location of the holes.

$f(x) = \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)}$

Holes: (-3, -4); (-2, -3)

Example 5

What is a possible equation for the following rational function?



The function seems to be a line with a removable discontinuity at (1, -1). The line is has slope 1 and y-intercept of -2 and so has the equation:

$f(x) = x - 2$

The removable discontinuity must not allow the x to be 1 which implies that it is of the form $\frac{x-1}{x-1}$. Therefore, the function is:

$f(x) = \frac{(x-2)(x-1)}{x-1}$

Review

1. How do you find the holes of a rational function?

2. What's the difference between a hole and a removable discontinuity?

3. If you see a hollow circle on a graph, what does that mean?

Without graphing, identify the location of the holes of the following functions.

4. $f(x) = \frac{x^2+3x-4}{x-1}$

5. $g(x) = \frac{x^2+8x+15}{x+3}$

6. $h(x) = \frac{x^3+6x^2+2x-8}{x^2+x-2}$

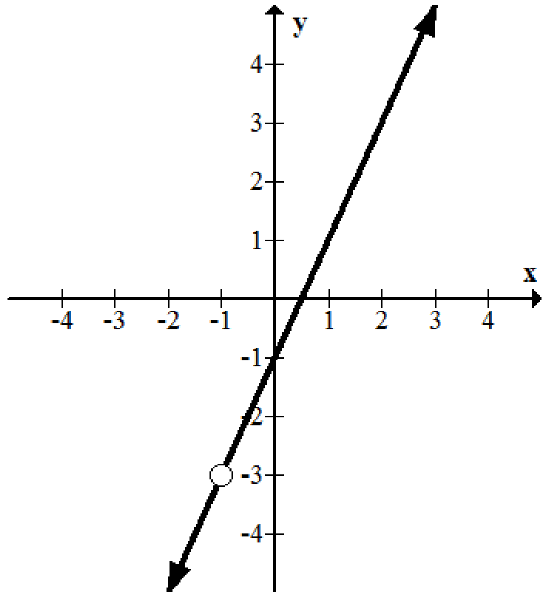
7. $k(x) = \frac{x^3+6x^2+2x-8}{x^2-3x+2}$

8. $j(x) = \frac{x^3+4x^2-17x-60}{x^2-9}$

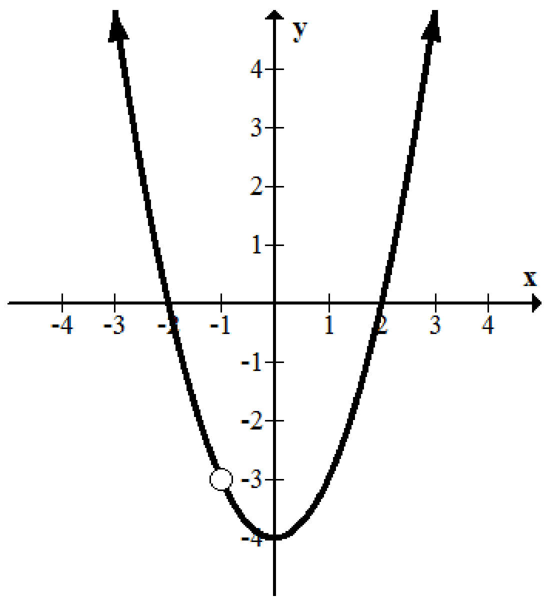
9. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-5x+4}$

10. $g(x) = \frac{x^3-4x^2-19x-14}{x^2-8x+7}$

11. What is a possible equation for the following rational function?



12. What is a possible equation for the following rational function?



Sketch the following rational functions.

13. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-x-12}$

14. $g(x) = \frac{x^3+4x^2-17x-60}{x^2+8x+15}$

15. $h(x) = \frac{x^3-4x^2-19x-14}{x^2-6x-7}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.7.

3.6 Zeroes of Rational Functions

Learning Objectives

Here you will revisit how to find zeroes of functions by focusing on rational expressions and what to do in special cases where the zeroes and holes seem to overlap.

The **zeroes** of a function are the collection of x values where the height of the function is zero. How do you find these values for a rational function and what happens if the zero turns out to be a hole?

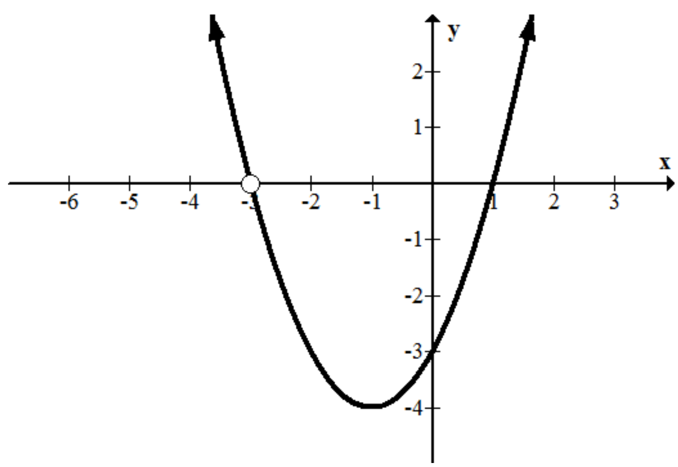
Finding Zeroes of Rational Functions

Zeroes are also known as x-intercepts, solutions or roots of functions. They are the x values where the height of the function is zero. For rational functions, you need to set the numerator of the function equal to zero and solve for the possible x values. If a hole occurs on the x value, then it is not considered a zero because the function is not truly defined at that point.

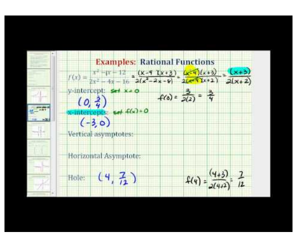
Take the following rational function:

$f(x) = \frac{(x-1)(x+3)(x+3)}{x+3}$

Notice how one of the $x+3$ factors seems to cancel and indicate a removable discontinuity. Even though there are two $x+3$ factors, the only zero occurs at $x = 1$ and the hole occurs at (-3, 0).



Watch the video below and focus on the portion of this video discussing holes and x-intercepts.



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Examples

Example 1

Earlier, you were asked how to find the zeroes of a rational function and what happens if the zero is a hole. To find the zeroes of a rational function, set the numerator equal to zero and solve for the x values. When a hole and a zero occur at the same point, the hole wins and there is no zero at that point.

Example 2

Create a function with zeroes at $x = 1, 2, 3$ and holes at $x = 0, 4$.

There are an infinite number of possible functions that fit this description because the function can be multiplied by any constant. One possible function could be:

$f(x) = \frac{(x-1)(x-2)(x-3)x(x-4)}{x(x-4)}$

Note that 0 and 4 are holes because they cancel out.

Example 3

Identify the zeroes, holes and y intercepts of the following rational function without graphing.

$f(x) = \frac{x(x-2)(x-1)(x+1)(x+1)(x+2)}{(x-1)(x+1)}$

The holes occur at $x = -1, 1$. To get the exact points, these values must be substituted into the function with the factors canceled.

$$f(x) = x(x-2)(x+1)(x+2)$$
$$f(-1) = 0, f(1) = -6$$

The holes are (-1, 0); (1, 6). The zeroes occur at $x = 0, 2, -2$. The zero that is supposed to occur at $x = -1$ has already been demonstrated to be a hole instead.

Example 4

Identify the y intercepts, holes, and zeroes of the following rational function.

$f(x) = \frac{6x^3-7x^2-x+2}{x-1}$

After noticing that a possible hole occurs at $x = 1$ and using polynomial long division on the numerator you should get:

$f(x) = (6x^2 - x - 2) \cdot \frac{x-1}{x-1}$

A hole occurs at $x = 1$ which turns out to be the point (1, 3) because $6 \cdot 1^2 - 1 - 2 = 3$.

The y-intercept always occurs where $x = 0$ which turns out to be the point (0, -2) because $f(0) = -2$.

To find the x-intercepts you need to factor the remaining part of the function:

$(2x+1)(3x-2)$

Thus the zeroes (x-intercepts) are $x = -\frac{1}{2}, \frac{2}{3}$.

Example 5

Identify the zeroes and holes of the following rational function.

$f(x)=\frac{2(x+1)(x+1)(x+1)}{2(x+1)}$

The hole occurs at $x=-1$ which turns out to be a double zero. The hole still wins so the point (-1, 0) is a hole. There are no zeroes. The constant 2 in front of the numerator and the denominator serves to illustrate the fact that constant scalars do not impact the x values of either the zeroes or holes of a function.

Review

Identify the intercepts and holes of each of the following rational functions.

1. $f(x)=\frac{x^3+x^2-10x+8}{x-2}$
2. $g(x)=\frac{6x^3-17x^2-5x+6}{x-3}$
3. $h(x)=\frac{(x+2)(1-x)}{x-1}$
4. $j(x)=\frac{(x-4)(x+2)(x+2)}{x+2}$
5. $k(x)=\frac{x(x-3)(x-4)(x+4)(x+4)(x+2)}{(x-3)(x+4)}$
6. $f(x)=\frac{x(x+1)(x+1)(x-1)}{(x-1)(x+1)}$
7. $g(x)=\frac{x^3-x^2-x+1}{x^2-1}$
8. $h(x)=\frac{4-x^2}{x-2}$
9. Create a function with holes at $x=3,5,9$ and zeroes at $x=1,2$.
10. Create a function with holes at $x=-1,4$ and zeroes at $x=1$.
11. Create a function with holes at $x=0,5$ and zeroes at $x=2,3$.
12. Create a function with holes at $x=-3,5$ and zeroes at $x=4$.
13. Create a function with holes at $x=-2,6$ and zeroes at $x=0,3$.
14. Create a function with holes at $x=1,5$ and zeroes at $x=0,6$.
15. Create a function with holes at $x=2,7$ and zeroes at $x=3$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.8.

3.7 Vertical Asymptotes

Learning Objectives

Here you will learn to recognize when vertical asymptotes occur and what makes them different from removable discontinuities.

The basic rational function $f(x) = \frac{1}{x}$ is a hyperbola with a vertical asymptote at $x = 0$. More complicated rational functions may have multiple vertical asymptotes. These asymptotes are very important characteristics of the function just like holes. Both holes and vertical asymptotes occur at x values that make the denominator of the function zero. A driving question is: what makes vertical asymptotes different from holes?

Finding Vertical Asymptotes

Vertical asymptotes occur when a factor of the denominator of a rational expression does not cancel with a factor from the numerator. When you have a factor that does not cancel, instead of making a hole at that x value, there exists a vertical asymptote. The vertical asymptote is represented by a dotted vertical line. Most calculators will not identify vertical asymptotes and some will incorrectly draw a steep line as part of a function where the asymptote actually exists.

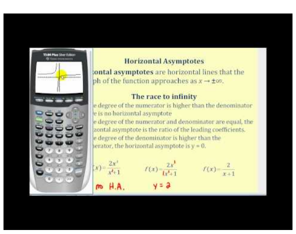
Your job is to be able to identify vertical asymptotes from a function and describe each asymptote using the equation of a vertical line.

Take the following rational function:

$f(x) = \frac{(2x-3)(x+1)(x-2)}{(x+2)(x+1)}$

To identify the holes and the equations of the vertical asymptotes, first decide what factors cancel out. The factor that cancels represents the removable discontinuity. There is a hole at (-1, 15). The vertical asymptote occurs at $x = -2$ because the factor $x+2$ does not cancel.

Watch the following video, focusing on the parts about vertical asymptotes.



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Examples

Example 1

Earlier, you were asked how asymptotes are different than holes. Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the domain of a rational function.

Example 2

Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 2 and 5
- Hole at (4, 2)

Each criteria helps build the function. The vertical asymptotes imply that the denominator has two factors that do not cancel with the numerator:

$\frac{1}{x\cdot(x-3)}$

The zeroes at 2 and 5 imply the numerator has two factors that do not cancel.

$\frac{(x-2)(x-5)}{x\cdot(x-3)}$

The hole at (4, 2) implies that there is a factor $x-4$ that cancels on the numerator and the denominator.

$\frac{(x-2)(x-5)(x-4)}{x\cdot(x-3)(x-4)}$

The tricky part is that the height of the function must be 2 after the $x-4$ factor has been canceled and the 4 is substituted in. Currently it is $-\frac{1}{2}$.

$\frac{(4-2)(4-5)}{4\cdot(4-3)}=-\frac{1}{2}$

In order to make the hole exist at a height of 2, you need to multiply the function by a scalar of -4.

$f(x)=\frac{-4(x-2)(x-5)(x-4)}{x\cdot(x-3)(x-4)}$

Example 3

Draw the vertical asymptotes for the following function.

$f(x)=\frac{1}{(x-4)(x-2)(x+3)}$

Note that you may not know the characteristics of what the function does inside these vertical lines. You will soon learn how to use sign tests as well as techniques you've already learned to fill in the four sections that this function is divided into.

Example 4

Identify the holes and equations of the vertical asymptotes of the following rational function.

$f(x)=\frac{3(x-1)(x+2)(x-3)(x+4)}{5\left(x+\frac{1}{2}\right)(2+x)(3-x)(x-8)}$

The vertical asymptotes occur at $x=-\frac{1}{2},x=8$. Holes occur when x is -2 and 3. To get the height of the holes at these points, remember to cancel what can be canceled and then substitute the values. A very common mistake is to forget to cancel $\frac{x-3}{3-x}=-1$.

$$g(x)=\frac{-3(x-1)(x+4)}{5\left(x+\frac{1}{2}\right)(x-8)}$$
$$g(-2)=\frac{6}{25}$$
$$g(3)=\frac{12}{25}$$

The holes are at $\left(-2,\frac{6}{25}\right),\left(3,\frac{12}{25}\right)$.

Example 5

Identify the domain of the following function and then identify the holes and vertical asymptotes.

$f(x) = \frac{(3x-4)(1-x)(x^2+4)}{(3x-2)(x-1)}$

The domain of the function written in interval notation is: $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$. Note that the domain is all real numbers except for where the denominator is zero.

There are two discontinuities: one is a hole and one is a vertical asymptote. The hole occurs at (1, 5). The vertical asymptote occurs at $x = \frac{2}{3}$.

Notice that holes are identified as points while vertical asymptotes are identified as lines of the form $x = a$ where a is some constant.

Review

1. Write a function that fits the following criteria:

- Vertical asymptotes at 1 and 4
- Zeroes at 3 and 5
- Hole at (6, 3)

2. Write a function that fits the following criteria:

- Vertical asymptotes at -2 and 2
- Zeroes at 1 and 5
- Hole at (3, -4)

3. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 1 and 2
- Hole at (8, 21)

4. Write a function that fits the following criteria:

- Vertical asymptotes at 2 and 6
- Zero at 5
- Hole at (4, 1)

5. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at (5, 10)

Give the equations of the vertical asymptotes for the following functions.

6. $f(x) = \frac{(2-x)}{(x-2)(x-4)}$

7. $g(x) = \frac{-x}{(x+1)(x-3)}$

8. $h(x) = 6 - \frac{x+2}{(x+1)(x-5)}$

9. $j(x) = \frac{10}{x-3} - \frac{x}{(x+2)(x-3)}$

10. $k(x) = 2 - \frac{(4-x)}{(x+3)(x-4)}$

Identify the holes and equations of the vertical asymptotes of the following rational functions.

11. $f(x) = \frac{3(x-1)(x+1)(x-4)(x+4)}{4(x+4)(2+x)(4-x)(x+1)}$

12. $g(x) = \frac{x(x-3)(x-8)(x-3)(x+4)}{7(x+1)(1+x)(3-x)(x-8)}$

State the domain of the following rational functions.

13. $h(x) = \frac{x(x+1)(x-3)(x+4)}{x(3-x)(x-1)}$

14. $j(x) = \frac{x^2+3x-4}{x^2-6x-16}$

15. $k(x) = \frac{2x-10}{x^3+4x^2+3x}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.9.

3.8 Horizontal Asymptotes

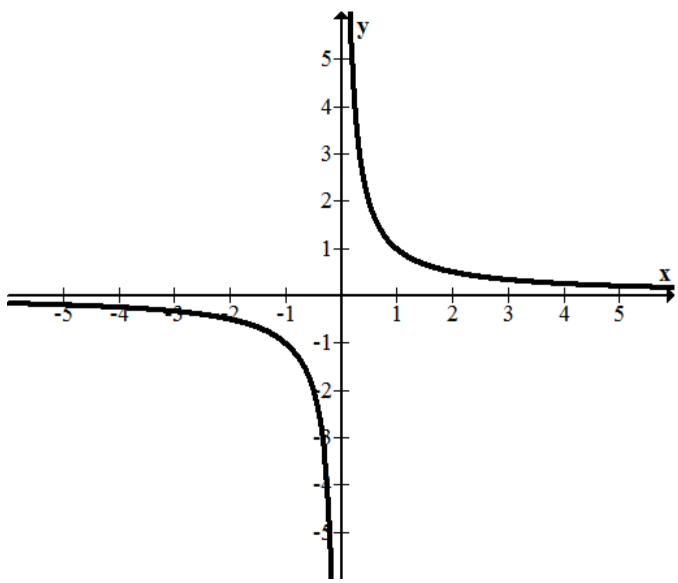
Learning Objectives

Here you will learn to identify when horizontal asymptotes exist, specify their height and write their equation.

Vertical asymptotes describe the behavior of a function as the values of x approach a specific number. **Horizontal asymptotes** describe the behavior of a function as the values of x become infinitely large and infinitely small. Since functions cannot touch vertical asymptotes, are they not allowed to touch horizontal asymptotes either?

Finding Horizontal Asymptotes

Horizontal asymptotes are a means of describing end behavior of a function. End behavior essentially is a description of what happens on either side of the graph as the function continues to the right and left infinitely. When you are determining the horizontal asymptotes, it is important to consider both the right and the left hand sides, because the horizontal asymptotes will not necessarily be the same in both places. Consider the reciprocal function and note how as x goes to the right and left it flattens to the line $y = 0$.



Sometimes functions flatten out and other times functions increase or decrease without bound. There are basically three cases.

Case 1: Degree of Numerator is Less than Degree of Denominator

The first case is the function flattens out to 0 as x gets infinitely large or infinitely small. This happens when the degree of the numerator is less than the degree of the denominator. The degree is determined by the greatest exponent of x.

$$f(x) = \frac{2x^8 + 3x^2 + 100}{x^9 - 12}$$

One way to reason through why this makes sense is because when x is a ridiculously large number then most parts of the function hardly make any impact. The 100 for example is nothing in comparison and neither is the $3x^2$. The two important terms to compare are x^8 and x^9. The 2 isn't even important now because if x is even just a million than the x^9 will be a million times bigger than the x^8 and the 2 hardly matters again. Essentially, when x gets big enough, this function acts like $\frac{1}{x}$ which has a horizontal asymptote of 0.

Case 2: Degree of Numerator is Equal to the Degree of Denominator

If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is equal to the ratio of the leading coefficients.

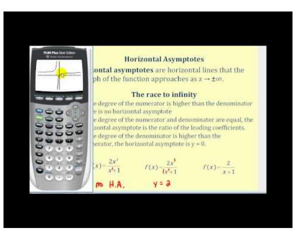
$$f(x) = \frac{6x^4 - 3x^3 + 12x^2 - 9}{3x^4 + 144x - 0.001}$$

Notice how the degree of both the numerator and the denominator is 4. This means that the horizontal asymptote is $y = \frac{6}{3} = 2$. One way to reason through why this makes sense is because when x gets to be a very large number all the smaller powers will not really make much of an impact. The biggest contributors are only the biggest powers. Then the value of the numerator will be about twice that of the denominator. As x gets even bigger, then the function will get even closer to 2.

Case 3: Degree of Numerator is Greater than the Degree of Denominator

If the degree of the numerator is greater than the degree of the denominator, there does not exist a horizontal asymptote. You must determine if the function increases or decreases without bound in both the left and right directions.

Watch the following video, focusing on the parts about horizontal asymptotes.



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Examples

Example 1

Earlier, you were asked if functions are allowed to touch their horizontal asymptotes. Functions may touch and pass through horizontal asymptotes without limit. This is a difference between vertical and horizontal asymptotes. In calculus, there are rigorous proofs to show that functions like the one in Example C do become arbitrarily close to the asymptote.

Example 2

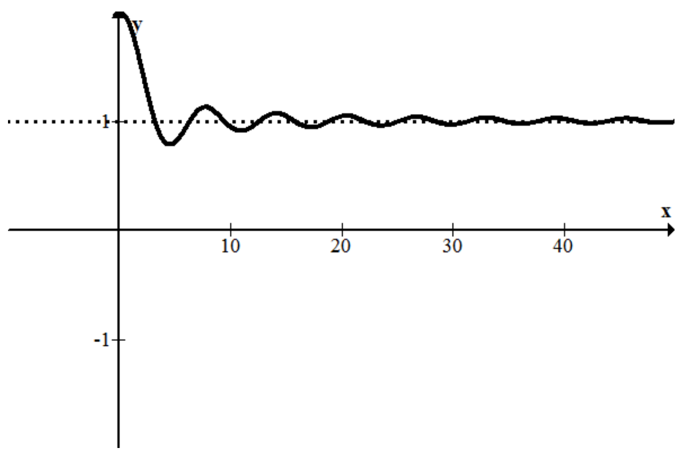
Identify the vertical and horizontal asymptotes of the following rational function.

$$f(x) = \frac{(x-2)(4x+3)(x-4)}{(x-1)(4x+3)(x-6)}$$

There is factor that cancels that is neither a horizontal or vertical asymptote. The vertical asymptotes occur at $x = 1$ and $x = 6$. To obtain the horizontal asymptote you could methodically multiply out each binomial, however since most of those terms do not matter, it is more efficient to first determine the relative powers of the numerator and the denominator. In this case they both happen to be 3. Next determine the coefficient of the cubic terms only. The numerator will have $4x^3$ and the denominator will have $4x^3$ and so the horizontal asymptote will occur at $y = \frac{4}{4} = 1$.

Example 3

Describe the right hand end behavior of the following function.



Notice how quickly this dampening wave function settles down. There seems to be an obvious horizontal axis on the right at $y = 1$

Example 4

Identify the horizontal asymptotes of the following function.

$$f(x) = \frac{(x-3)(x+2)}{|(x-5)| \cdot (x-1)}$$

First notice the absolute value surrounding one of the terms in the denominator. The degrees of both the numerator and the denominator will be 2 which means that the horizontal asymptote will occur at a number. As x gets infinitely large, the function is approximately:

$$f(x) = \frac{x^2}{x^2}$$

So the horizontal asymptote is $y = -1$ as x gets infinitely large.

On the other hand, as x gets infinitely small the function is approximately:

$$f(x) = \frac{x^2}{-x^2}$$

So the horizontal asymptote is $y = -1$ as x gets infinitely small.

In this case, you cannot blindly use the leading coefficient rule because the absolute value changes the sign.

Example 5

Identify the horizontal asymptotes if they exist for the following 3 functions.

1. $f(x) = \dfrac{3x^6 - 72x}{x^6 + 999}$

The degrees of the numerator and the denominatro are equal so the horizontal asymptote is$y = 3$.

2. $h(x) = \dfrac{ax^4 + bx^3 + cx^2 + dx + e}{fx^4 + gx^3 + hx^2}$

The degrees of the numerator and the denominator are equal again so the horizontal asymptote is $y = \frac{a}{f}$

3. $g(x) = \dfrac{f(x)}{h(x)}$

As x gets infinitely large,

$$g(x) = \frac{f(x)}{h(x)} = \frac{\frac{3x^6-72x}{x^6+999}}{\frac{ax^4+bx^3+cx^2+dx+e}{fx^4+gx^3+hx^2}} \approx \frac{3}{\frac{a}{f}} = \frac{3f}{a}$$

When you study calculus, you will learn the rigorous techniques that enable you to feel more confident about results like this.

Review

Identify the horizontal asymptotes, if they exist, for the following functions.

1. $f(x) = \dfrac{5x^4 - 2x}{x^4 + 32}$
2. $g(x) = \dfrac{3x^4 - 2x^6}{-x^4 + 2}$
3. $h(x) = \dfrac{3x^4 - 5x}{8x^3 + 3x^4}$
4. $j(x) = \dfrac{2x^3 - 15x}{-x^4 + 3}$
5. $k(x) = \dfrac{2x^5 - 3x}{5x^2 + 3x^4 + 2x - 7x^5}$
6. $f(x) = \dfrac{ax^{14} + bx^{23} + cx^{12} + dx + e}{fx^{24} + gx^{23} + hx^{21}}$

7. $g(x) = \frac{(x-1)(x+4)}{|(x-2)|\cdot(x-1)}$
8. Write a function that fits the following criteria:
    - Vertical asymptotes at $x = 1$ and $x = 4$
    - Zeroes at 3 and 5
    - Hole when $x = 6$
    - Horizontal asymptote at $y = \frac{2}{3}$
9. Write a function that fits the following criteria:
    - Vertical asymptotes at $x = -2$ and $x = 2$
    - Zeroes at 1 and 5
    - Hole when $x = 3$
    - Horizontal asymptote at $y = 1$
10. Write a function that fits the following criteria:
    - Vertical asymptotes at $x = 0$ and $x = 3$
    - Zeroes at 1 and 2
    - Hole when $x = 8$
    - Horizontal asymptote at $y = 2$
11. Write a function that fits the following criteria:
    - Vertical asymptotes at 2 and 6
    - Zero at 5
    - Hole when $x = 4$
    - Horizontal asymptote at $y = 0$
12. Write a function that fits the following criteria:
    - Vertical asymptote at 4
    - Zeroes at 0 and 3
    - Hole at when $x = 5$
    - No horizontal asymptotes

Identify the vertical and horizontal asymptotes of the following rational functions.

13. $f(x) = \frac{(x-5)(2x+1)(x-3)}{(x-3)(4x+5)(x-6)}$
14. $g(x) = \frac{x(x-1)(x+3)(x-5)}{3x(x-1)(4x+3)}$
15. $h(x) = \frac{(x-2)(x+3)(x-6)}{(x-4)(x+3)^2(x+2)}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

3.9 Oblique Asymptotes

Learning Objectives

Here you will extend your knowledge of horizontal and vertical asymptotes and learn to identify oblique (slanted) asymptotes. You will also be able to apply your knowledge of polynomial long division.

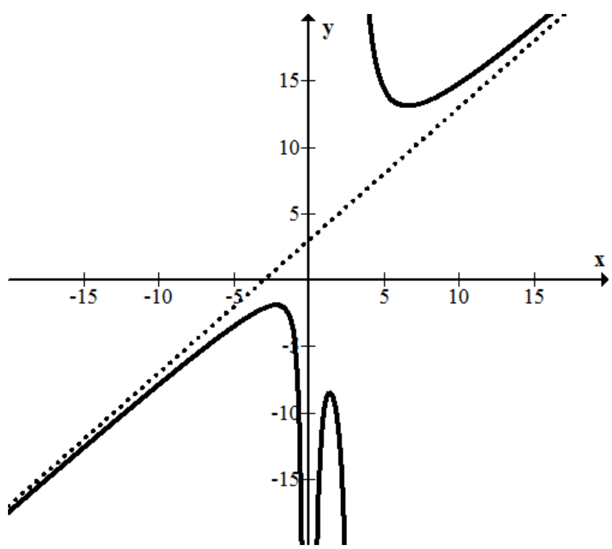
When the degree of the numerator of a rational function exceeds the degree of the denominator by one then the function has **oblique asymptotes**. In order to find these asymptotes, you need to use polynomial long division and the non-remainder portion of the function becomes the oblique asymptote. A natural question to ask is: what happens when the degree of the numerator exceeds that of the denominator by more than one?

Oblique Asymptotes

The following function is shown before and after polynomial long division is performed.

$f(x) = \frac{x^4+3x^2+2x+14}{x^3-3x^2} = x+3+\frac{12x^2+2x+14}{x^3-3x^2}$

Notice that the remainder portion will go to zero when x gets extremely large or extremely small because the power of the numerator is smaller than the power of the denominator. This means that while this function might go haywire with small absolute values of x, large absolute values of x are extremely close to the line $y = x+3$.



Oblique asymptotes are these slanted asymptotes that show exactly how a function increases or decreases without bound. Oblique asymptotes are also called slant asymptotes.

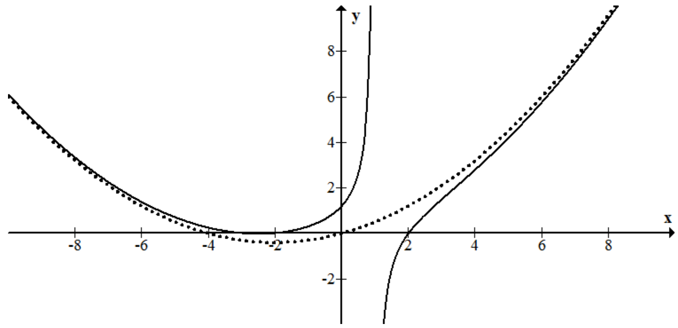
Sometimes a function will have an asymptote that does not look like a line. Take a look at the following function:

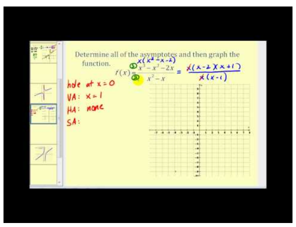
$f(x) = \frac{(x^2-4)(x+3)}{10(x-1)}$

The degree of the numerator is 3 while the degree of the denominator is 1 so the slant asymptote will not be a line. However when the graph is observed, there is still a clear pattern as to how this function increases without bound as x approaches very large and very small numbers.

$f(x) = \frac{1}{10}(x^2+4x) - \frac{12}{10(x-1)}$

As you can see, this looks like a parabola with a remainder. This rational function has a parabola backbone. A backbone is a function that a graph tends towards. This is not technically an oblique asymptote because it is not a line.





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Examples

Example 1

Earlier, you were asked what happens when the power of the numerator exceeds the power of the denominator by more than one. As seen above, when the power of numerator exceeds the power of the denominator by more than one, the function develops a backbone that be shaped like any polynomial. Oblique asymptotes are always lines.

Example 2

Find the asymptotes and intercepts of the function:

$f(x) = \frac{x^3}{x^2-4}$

The function has vertical asymptotes at $x = \pm 2$.

After long division, the function becomes:

$f(x) = x + \frac{4}{x^2-4}$

This makes the oblique asymptote at $y = x$

Example 3

Identify the vertical and oblique asymptotes of the following rational function.

$f(x) = \frac{x^3-x^2-x-1}{(x-3)(x+4)}$

After using polynomial long division and rewriting the function with a remainder and non-remainder portion it looks like this:

$f(x) = x-2+\frac{13x-25}{x^2+x-12} = x-2+\frac{13x-25}{(x-3)(x+4)}$

The oblique asymptote is $y = x-2$. The vertical asymptotes are at $x = 3$ and $x = -4$ which are easier to observe in last form of the function because they clearly don't cancel to become holes.

Example 4

Create a function with an oblique asymptote at $y = 3x-1$, vertical asymptotes at $x = 2, -4$ and includes a hole where x is 7.

While there are an infinite number of functions that match these criteria, one example is:

$f(x) = 3x-1+\frac{(x-7)}{(x-2)(x+4)(x-7)}$

Example 5

Identify the backbone of the following function and explain why the function does not have an oblique asymptote.

$f(x) = \frac{5x^5+27}{x^3}$

While polynomial long division is possible, it is also possible to just divide each term by x^3.

$f(x) = \frac{5x^5+27}{x^3} = \frac{5x^5}{x^3}+\frac{27}{x^3} = 5x^2+\frac{27}{x^3}$

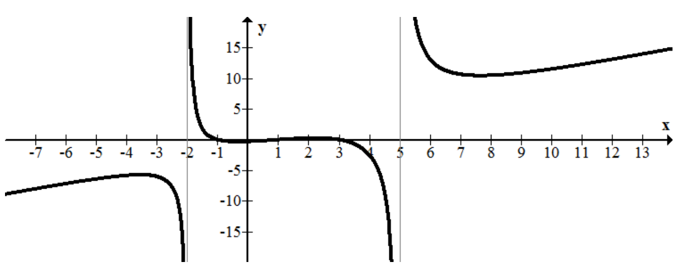
The backbone of this function is the parabola $y = 5x^2$. This is not an oblique asymptote because it is not a line.

Review

1. What is an oblique asymptote?
2. How can you tell by looking at the equation of a function if it will have an oblique asymptote or not?
3. Can a function have both an oblique asymptote and a horizontal asymptote? Explain.

For each of the following graphs, sketch the graph and then sketch in the oblique asymptote if it exists. If it doesn't exist, explain why not.

4.



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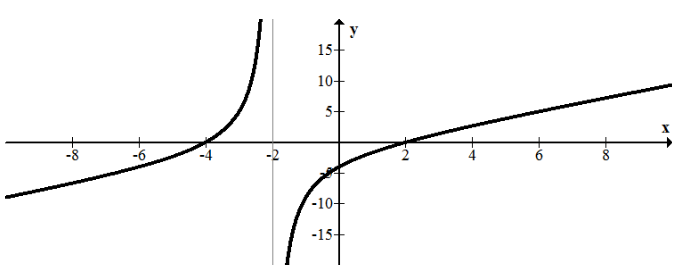

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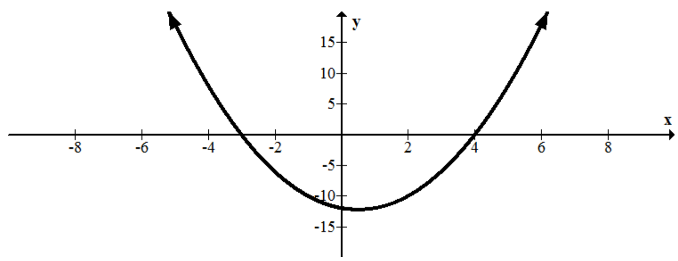

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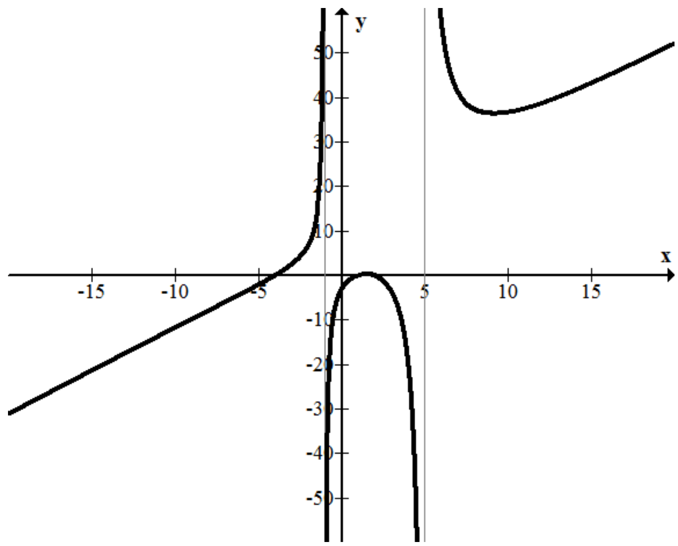

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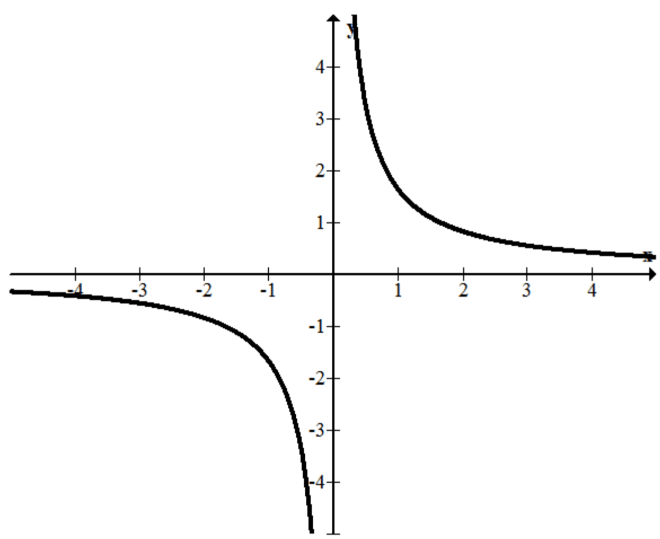


Find the equation of the oblique asymptote for each of the following rational functions. If there is not an oblique asymptote, explain why not and give an equation of the backbone of the function if one exists.

9. $f(x) = \frac{x^3-7x-6}{x^2-2x-15}$

10. $g(x) = \frac{x^3-7x-6}{x^4-3x^2-10}$

11. $h(x) = \frac{x^2+5x+6}{x^2+2x+1}$

12. $k(x) = \frac{x^4+9x^3+21x^2-x-30}{x^2+2x+1}$

13. Create a function with an oblique asymptotes at $y = 2x - 1$, a vertical asymptote at $x = 3$ and a hole where x is 7.

14. Create a function with an oblique asymptote at $y = x$, vertical asymptotes at $x = 1, -3$ and no holes.

15. Does a parabola have an oblique asymptote? What about a cubic function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.11.

3.10 Sign Test for Rational Function Graphs

Learning Objectives

Here you will learn to predict the nature of a rational function near the asymptotes.

The asymptotes of a rational function provide a very rigid structure in which the function must live. Once the asymptotes are known you must use the sign testing procedure to see if the function becomes increasingly positive or increasingly negative near the asymptotes. A driving question then becomes how close does near need to be in order for the sign test to work?

Sign Test for Rational Functions

Consider mentally substituting the number 2.99999 into the following rational expression.

$$f(x)=\frac{(x-1)(x+3)(x-5)(x+10)}{(x+2)(x-4)(x-3)}$$

Without doing any of the arithmetic, simply note the sign of each term:

$$f(x)=\frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(-)}$$

The only term where the value is close to zero is $(x-3)$ but careful subtraction still indicates a negative sign. The product of all of these signs is negative. This is strong evidence that this function approaches negative infinity as x approaches 3 from the left.

Next consider mentally substituting 3.00001 and going through the same process.

$$f(x)=\frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(+)}$$

The product of all of these signs is positive which means that from the right this function approaches positive infinity instead. This technique is called the sign test. The **sign test** is a procedure for determining only whether a function is above or below the axis at a particular value.

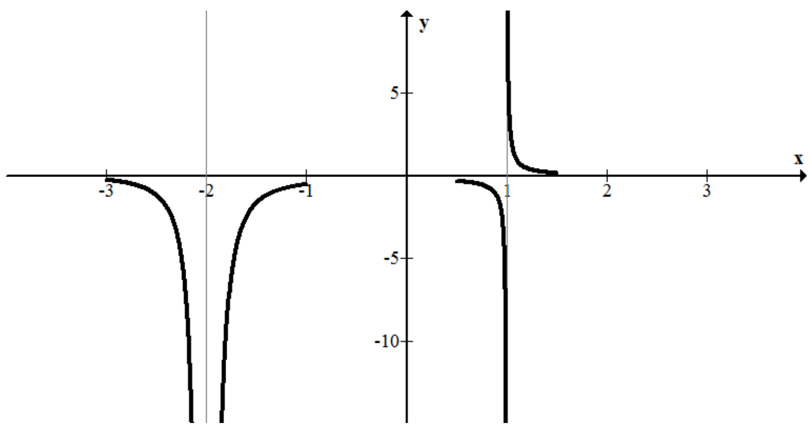
The sign test helps you sketch and graph a function. Look at the following function:

$$f(x)=\frac{1}{(x+2)^2\cdot(x-1)}$$

Your first step for sketching this is to identify the vertical asymptotes. The vertical asymptotes occur at $x=-2$ and $x=1$. Then, you use the asymptotes to perform a sign test. The points to use the sign testing procedure with are -2.001, -1.9999, 0.9999, 1.00001. The number of decimals does matter so long as the number is sufficiently close to the asymptote. Note that any real number squared is positive.

$$f(-2.001) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(0.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(1.0001) = \frac{(+)}{(+)\cdot(+)} = +$$

Later when you sketch everything you will use your knowledge of zeroes and intercepts. For now, focus on just the portions of the graph near the asymptotes. Note that the graph below is NOT complete.



Examples

Example 1

Earlier, you were asked how close the numbers need to be to perform the sign test. In order to truly answer the question about how close the numbers need to be, calculus should be used. For the purposes of PreCalculus, the testing number should be closer to the vertical asymptote than any other number in the problem. If the vertical asymptote occurs at 3 and 3.01 is in the problem elsewhere, do not choose 3.1 as a sign test number.

Example 2

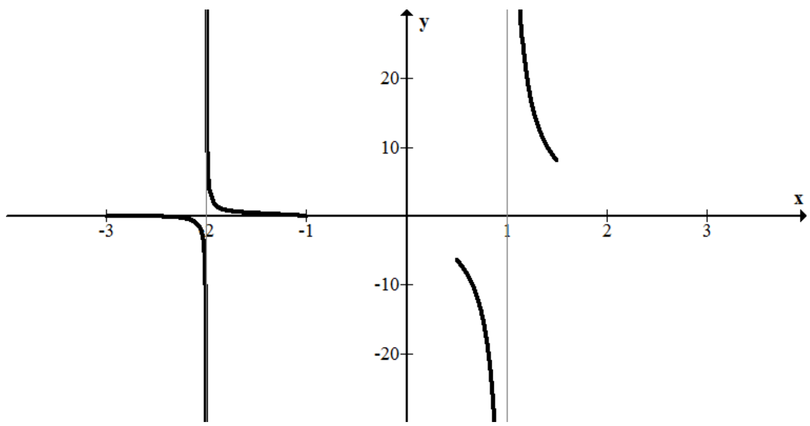
Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x+1)(x-4)^2(x-1)(x+3)^3}{100(x-1)^2(x+2)}$$

Note that $x = -2$ is clearly an asymptote. It may be initially unclear whether $x = 1$ is an asymptote or a hole. Just like holes have priority over zeroes, asymptotes have priority over holes. The four values to use the sign testing procedure are -2.001, -1.9999, 0.9999, 1.00001.

$$f(-2.001) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = +$$
$$f(0.9999) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = -$$
$$f(1.0001) = \frac{(+)\cdot(+)\cdot(+)\cdot(+)}{(+)\cdot(+)} = +$$

A sketch of the behavior of this function near the asymptotes is:

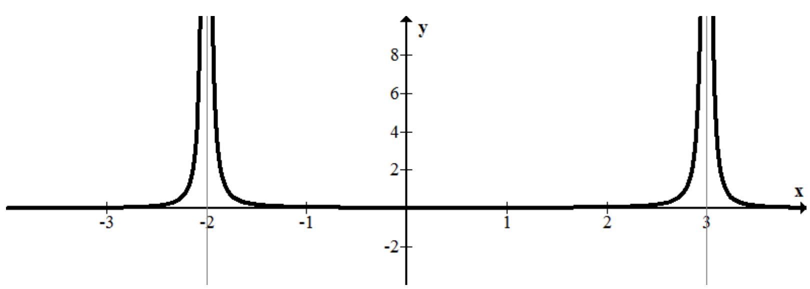


Example 3

Create a function with two vertical asymptotes at 3 and -2 such that the function approaches positive infinity from both directions at both vertical asymptotes.

Earlier, there was a function that approached negative infinity from both sides of the asymptote. This occurred because the term was squared in the denominator. An even powered term will always produce a positive term.

$$f(x) = \frac{1}{(x-3)^2(x+2)^2}$$



Example 4

Create a function with three vertical asymptotes such that the function approaches negative infinity for large and small values of x and has an oblique asymptote.

There are an infinite number of possible solutions. The key is to create a function that may work and then use the sign testing procedure to check. Here is one possibility.

$$f(x) = \frac{-x^7}{10(x-1)^2(x-2)^2(x-4)^2}$$

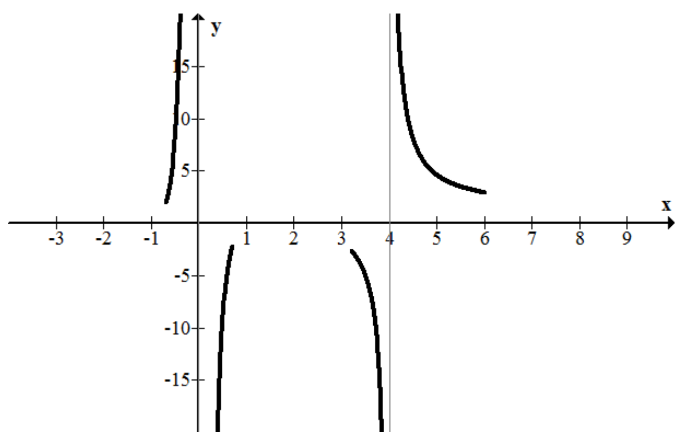
Example 5

Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x-2)^3(x-1)^2(x+1)(x+3)}{x^3\left(x+\frac{1}{2}\right)(x-1)(x-2)^2}$$

The vertical asymptotes occur at $x = 0, -\frac{1}{2}$. Therefore the x values to sign test are -.001, 0.001, 3.999, 4.0001.

$$\begin{aligned} f(-0.001) &= + \\ f(0.001) &= - \\ f(3.999) &= - \\ f(4.0001) &= + \end{aligned}$$



Review

Consider the function below for questions 1-4.

$$f(x) = \frac{(x-2)^4(x+1)(x+3)}{x^3(x+3)(x-4)}$$

1. Identify the vertical asymptotes.
2. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
3. What values will you need to use the sign test with in order to help you make a sketch of the graph?

4. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 5-8.

$$g(x) = \frac{3(x-2)^2(x-1)^2(x+1)(x+3)}{15x^2(x+5)(x+1)(x-3)^2}$$

5. Identify the vertical asymptotes.
6. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
7. What values will you need to use the sign test with in order to help you make a sketch of the graph?
8. Use the sign test and sketch the graph near the vertical asymptote(s).

Consider the function below for questions 9-12.

$$h(x) = \frac{9x^4 - 102x^3 + 349x^2 - 340x + 100}{x^3 - 9x^2 + 24x - 16}$$

9. Identify the vertical asymptotes.
10. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
11. What values will you need to use the sign test with in order to help you make a sketch of the graph?
12. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 13-16.

$$k(x) = \frac{2x^3 - 5x^2 - 11x - 4}{3x^3 + 11x^2 + 5x - 3}$$

13. Identify the vertical asymptotes.
14. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
15. What values will you need to use the sign test with in order to help you make a sketch of the graph?
16. Use the sign test and sketch the graph near the vertical asymptotes.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.12.

3.11 Graphs of Rational Functions by Hand

Learning Objectives

Here you will use your knowledge of zeroes, intercepts, holes and asymptotes to sketch rational functions by hand.

Sketching rational functions by hand is a mental workout because it combines so many different specific skills to produce a single coherent image. It will require you to closely examine the equation of the function in a variety of different ways in order to find clues as to the shape of the overall function. Since computers can graph these complicated functions much more accurately than people can, why is sketching by hand important?

Graphing Rational Functions

While there is no strict procedure for graphing rational functions by hand there is a flow of clues to look for in the function. In general, it will make sense to identify different pieces of information in this order and record them on a sketch.

Steps for Graphing Rational Functions By Hand

1. Examine the denominator of the rational function to determine the domain of the function. Distinguish between holes which are factors that can be canceled and vertical asymptotes that cannot. Plot the vertical asymptotes.
2. Identify the end behavior of the function by comparing the degrees of the numerator and denominator and determine if there exists a horizontal or oblique asymptote. Plot the horizontal or oblique asymptotes.
3. Identify the holes of the function and plot them.
4. Identify the zeroes and intercepts of the function and plot them.
5. Use the sign test to determine the behavior of the function near the vertical asymptotes.
6. Connect everything as best you can.

Now, apply those steps in graphing the following function:

$$f(x) = \frac{4x^3 - 2x^2 + 3x - 1}{8(x-1)^2(x+2)}$$

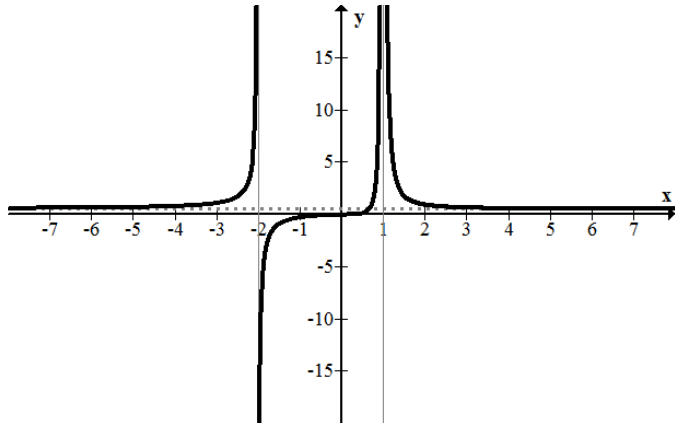
After attempting to factor the numerator you may realize that both $x = 1$ and $x = -2$ are vertical asymptotes rather than holes. The horizontal asymptote is $y = \frac{1}{2}$. There are no holes. The y-intercept is:

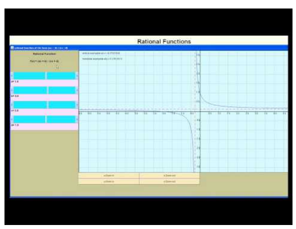
$f(0) = -\frac{1}{8 \cdot 2} = \frac{1}{16}$

The numerator is not factorable, but there is a zero between 0 and 1. You know this because there are no holes or asymptotes between 0 and 1 and the function switches from negative to positive in this region.

$$\begin{aligned}
4(0)^3 - 2(0)^2 + 3(0) - 1 &= -1 \\
4(1)^3 - 2(1)^2 + 3(1) - 1 &= 4
\end{aligned}$$

Putting all of this together in a sketch:





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Examples

Example 1

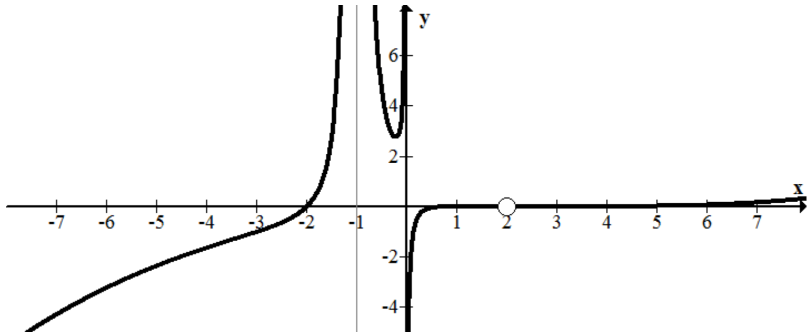
Earlier, you were asked why sketching graphs by hand is important. Computers can graph rational functions more accurately than people. However, computers may not be able to explain why a function behaves in certain ways. By being a detective and looking for clues in the equation of a function, you are applying high level analytical skills and powers of deduction. These analytical skills are vastly more important and transferable than the specific techniques involved with rational functions.

Example 2

Completely plot the following rational function.

$$f(x) = \frac{(x-3)^2(x-2)^3(x-1)(x+2)}{300(x+1)^2(x-2)x}$$

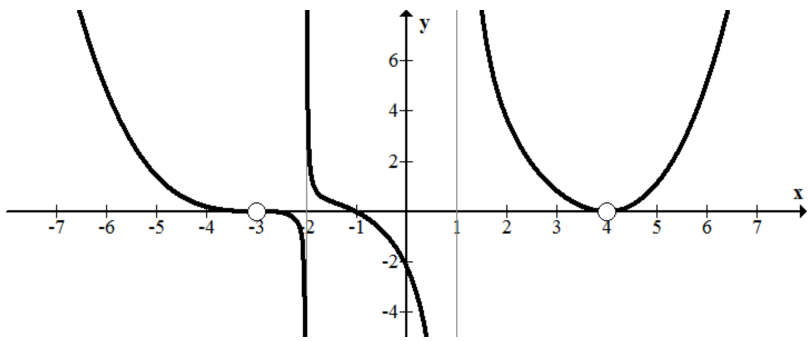
Since this function is already factored, much of the work is already done. There is a hole at (2, 0). There are two vertical asymptotes at $x = -1, 0$. There are no horizontal or oblique asymptotes because the degree of the numerator is much bigger than the degree of the denominator. As x gets large this function grows without bound. As x get very small, this function decreases without bound. The function has no y intercept because that is where a vertical asymptote is. Besides the hole at (2, 0), there are zeroes at (-2, 0), (1, 0) and (3, 0). This is what the graph ends up looking like.



Notice on the right portion of the graph the curve seems to stay on the x-axis. In fact it does go slightly above and below x-axis, crossing through it at (1, 0), (2, 0) and (3, 0) before starting to increase.

Example 3

Estimate a function that would have the following graphical characteristics:



First think about the vertical asymptotes and how they affect the equation of the function. Then consider zeroes and holes, and the way the graph looks at these places. Finally, use the y-intercept to refine your equation.

- The function has two vertical asymptotes at $x = -2, 1$ so the denominator must have the factors $(x+2)(x-1)$.
- There is one zero at $x = -1$, so the numerator must have a factor of $(x+1)$.
- There are two holes that appear to override zeroes which means the numerator and denominator must have the factors $(x+3)$ and $(x-4)$.
- Because the graph goes from above the x-axis to below the x-axis at $x = -3$, the degree of the exponent of the $(x+3)$ factor must be ultimately odd.
- Because the graph stays above the x-axis before and after $x = 4$, the degree of the $(x-4)$ factor must be ultimately even.

A good estimate for the function is:

$$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{(x+2)(x-1)(x+3)(x-4)}$$

This function has all the basic characteristics, however it isn't scaled properly. When $x = 0$ this function has a y-intercept of -216 when it should be about -2. Thus you must divide by 108 so that the y intercept matches. Here is a better estimate for the function:

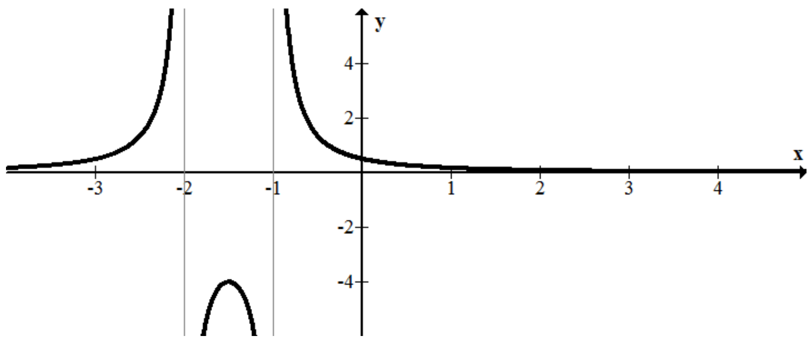
$$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{108(x+2)(x-1)(x+3)(x-4)}$$

Example 4

Graph the following rational function:

$$f(x) = \frac{1}{x^2 + 3x + 2}$$

Here is the graph:

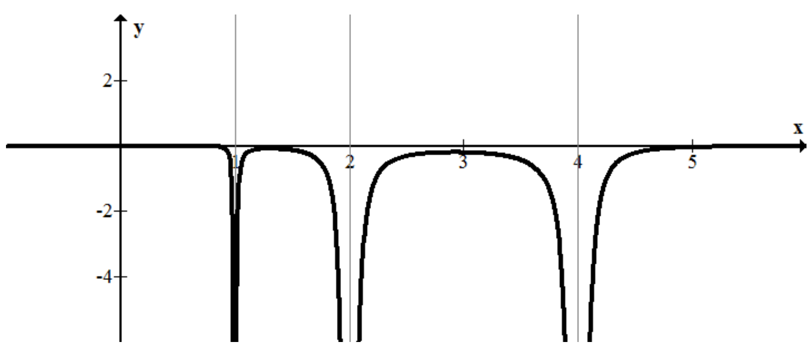


Example 5

Graph the following rational function:

$$f(x) = \frac{-x^4}{10(x-1)^2(x-2)^2(x-4)^2}$$

Here is the graph:



Review

Use the function below for 1-7.

$$f(x) = \frac{2(x+4)(x-3)(x+1)}{8(x-1)^2(x+2)}$$

1. Identify the vertical asymptotes and holes for the function.

2. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.
3. Identify any horizontal or oblique asymptotes for the function.
4. Describe the end behavior of the function.
5. Find the zeroes of the function.
6. Find the y-intercept of the function.
7. Use the information from 1-6 to sketch the function.

Use the function below for 8-14.

$$g(x) = \frac{(x^2-9)(x^2-4)}{5(x-2)^2(x+1)^2}$$

8. Identify the vertical asymptotes and holes for the function.
9. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.
10. Identify any horizontal or oblique asymptotes for the function.
11. Describe the end behavior of the function.
12. Find the zeroes of the function.
13. Find the y-intercept of the function.
14. Use the information from 8-13 to sketch the function.
15. Graph the function below by hand.

$$h(x) = \frac{x^3+5x^2+2x-8}{x^2-3x-10}$$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.13.

3.12 Special Right Triangles

Learning Objectives

Here you will review properties of 30-60-90 and 45-45-90 right triangles.

The Pythagorean Theorem is great for finding the third side of a right triangle when you already know two other sides. There are some triangles like 30-60-90 and 45-45-90 triangles that are so common that it is useful to know the side ratios without doing the Pythagorean Theorem each time. Using these patterns also allows you to totally solve for the missing sides of these special triangles when you only know one side length.

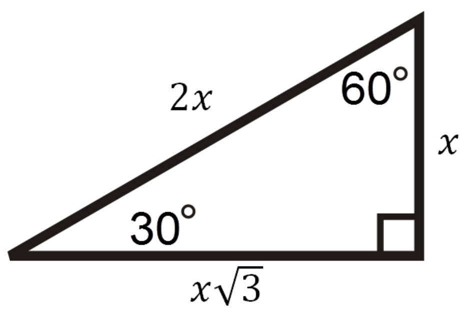
Given a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches, what is the value of x?

Special Right Triangles

There are three types of special right triangles, 30-60-90 triangles, 45-45-90 triangles, and Pythagorean triple triangles.

30-60-90 Triangles

A 30-60-90 right triangle has side ratios $x, x\sqrt{3}, 2x$.

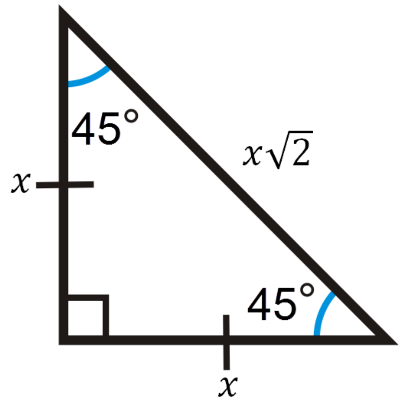


Confirm with Pythagorean Theorem:

$$\begin{aligned}x^2 + \left(x\sqrt{3}\right)^2 &= (2x)^2\\ x^2 + 3x^2 &= 4x^2\\ 4x^2 &= 4x^2\end{aligned}$$

45-45-90 Triangles

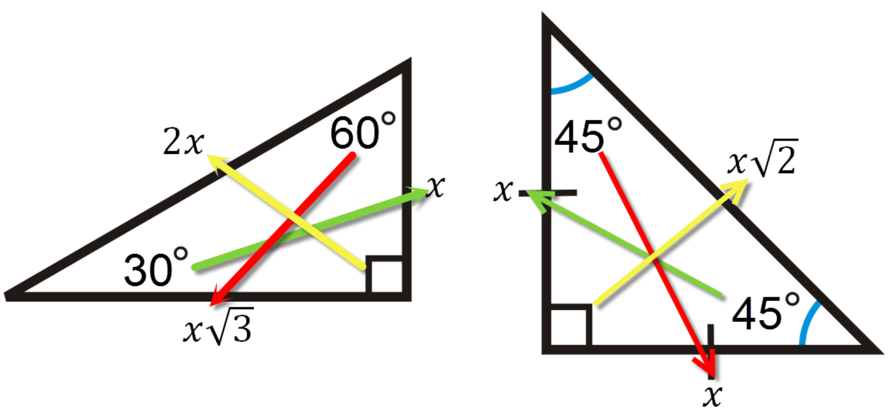
A 45-45-90 right triangle has side ratios $x, x, x\sqrt{2}$.



Confirm with Pythagorean Theorem:

$$x^2+x^2=\left(x\sqrt{2}\right)^2$$
$$2x^2=2x^2$$

Note that the order of the side ratios $x, x\sqrt{3}, 2x$ and $x, x, x\sqrt{2}$ is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to smallest angles and largest sides always correspond to the largest angles.



Pythagorean Triple Triangles

Pythagorean number triples are special right triangles with integer sides. While the angles are not integers, the side ratios are very useful to know because they show up everywhere. Knowing these number triples also saves a lot of time from doing the Pythagorean Theorem repeatedly. Here are some examples of Pythagorean number triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

More Pythagorean number triples can be found by scaling any other Pythagorean number triple. For example:

$3,4,5 \rightarrow 6,8,10$ (scaled by a factor of 2)

Even more Pythagorean number triples can be found by taking any odd integer like 11, squaring it to get 121, halving the result to get 60.5. The original number 11 and the two numbers that are 0.5 above and below (60 and 61) will always be a Pythagorean number triple.

$11^2 + 60^2 = 61^2$

Examples

Example 1

Earlier you were asked about a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches. If you can recognize the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches. $x = 6\sqrt{2}$.

Example 2

A 30-60-90 right triangle has hypotenuse of length 10. What are the lengths of the other two sides?

The hypotenuse is the side opposite 90. Sometimes it is helpful to draw a picture or make a table.

TABLE 3.1:

30	60	90
x	$x\sqrt{3}$	$2x$
		10

From the table you can write very small subsequent equations to solve for the missing sides.

$$10 = 2x$$
$$x = 5$$
$$x\sqrt{3} = 5\sqrt{3}$$

Example 3

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths.

TABLE 3.2:

30	60	90
x	$x\sqrt{3}$	$2x$

TABLE 3.2: (continued)

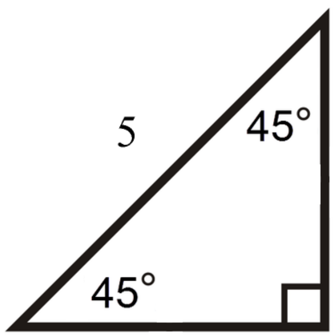
	18	

$$18 = x\sqrt{3}$$
$$\frac{18}{\sqrt{3}} = x$$
$$x = \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$

Note that you need to rationalize denominators.

Example 4

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.



The other sides are each $\frac{5\sqrt{2}}{2}$.

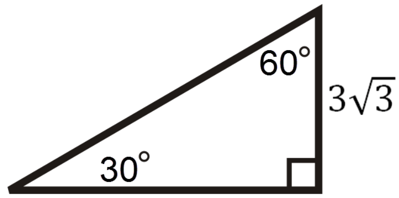
TABLE 3.3:

45	45	90
x	x	$x\sqrt{2}$
		5

$$x\sqrt{2} = 5$$
$$x = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

Example 5

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.



The other sides are 9 and $6\sqrt{3}$.

TABLE 3.4:

30	60	90
x	$x\sqrt{3}$	$2x$
$3\sqrt{3}$		

$$x = 3\sqrt{3}$$
$$2x = 6\sqrt{3}$$
$$x\sqrt{3} = 3\sqrt{3}\cdot\sqrt{3} = 9$$

Review

For 1-4, find the missing sides of the 45-45-90 triangle based on the information given in each row.

TABLE 3.5:

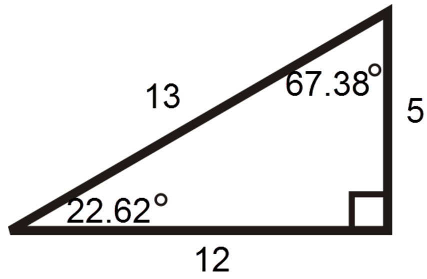
Problem Number	Side Opposite 45°	Side Opposite 45°	Side Opposite 90°
1.	3		
2.		7.2	
3.			16
4.	$5\sqrt{2}$		

For 5-8, find the missing sides of the 30-60-90 triangle based on the information given in each row.

TABLE 3.6:

Problem Number	Side Opposite 30°	Side Opposite 60°	Side Opposite 90°
5.	$3\sqrt{2}$		
6.		4	
7.			15
8.			$12\sqrt{3}$

Use the picture below for 9-11.



9. Which angle corresponds to the side that is 12 units?

10. Which side corresponds to the right angle?

11. Which angle corresponds to the side that is 5 units?

12. A right triangle has an angle of $\frac{\pi}{6}$ radians and a hypotenuse of 20 inches. What are the lengths of the other two sides of the triangle?

13. A triangle has two angles that measure $\frac{\pi}{4}$ radians. The longest side is 3 inches long. What are the lengths of the other two sides?

For 14-19, verify the Pythagorean Number Triple using the Pythagorean Theorem.

14. 3, 4, 5

15. 5, 12, 13

16. 7, 24, 25

17. 8, 15, 17

18. 9, 40, 41

19. 6, 8, 10

20. Find another Pythagorean Number Triple using the method explained for finding "11, 60, 61".

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

3.13 Right Triangle Trigonometry

Learning Objectives

Here you will learn the six right triangle ratios and how to use them to completely solve for the missing sides and angles of any right triangle.

Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

Imagine trying to measure the height of a flag pole. It would be very difficult to measure vertically because it could be several stories tall. Instead walk 10 feet away and notice that the flag pole makes a 65 degree angle with your feet. Using this information, what is the height of the flag pole?

Trigonometric Functions

The six trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. **Opp** stands for the side opposite of the angle θ, **hyp** stands for hypotenuse and **adj** stands for side adjacent to the angle θ.

$\sin\theta = \frac{opp}{hyp}$

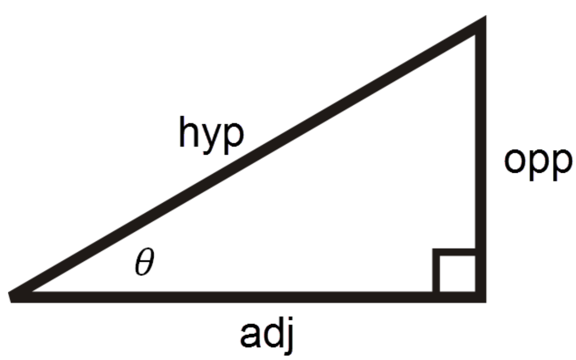
$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

$\cot\theta = \frac{adj}{opp}$

$\sec\theta = \frac{hyp}{adj}$

$\csc\theta = \frac{hyp}{opp}$



The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. Trigonometric functions are used by identifying two known pieces of information on a triangle and one unknown, setting up and solving for the unknown. Calculators are important because the operations of sin, cos and tan are already programmed in. The other three (cot, sec and csc) are not usually in calculators because there is a **reciprocal relationship** between them and tan, cos and sec.

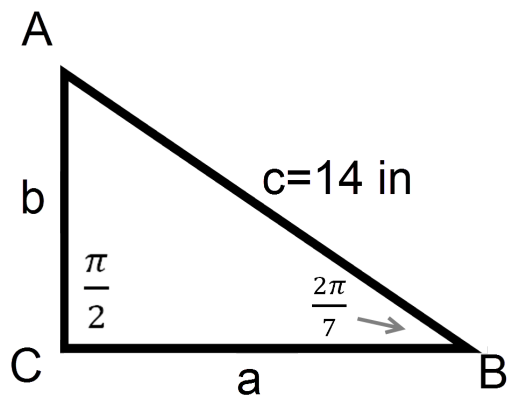
$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$

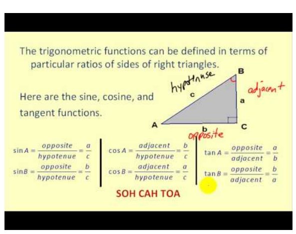
$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$

Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

Note that the images throughout this concept are not drawn to scale. If you were given the following triangle and asked to solve for side b, you would use sine to find b.



$$\sin\left(\frac{2\pi}{7}\right) = \frac{b}{14}$$
$$b = 14 \cdot \sin\left(\frac{2\pi}{7}\right) \approx 10.9\ in$$



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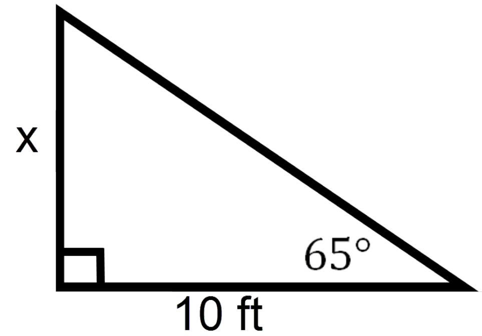
Examples

Example 1

Earlier, you were asked about the height of a flagpole that you are 10 feet away from. You notice that the flag pole makes a 65° angle with your feet.

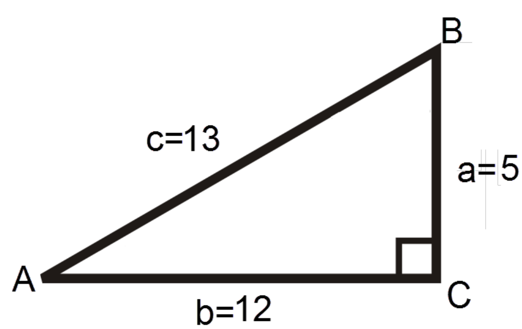
If you are 10 feet from the base of a flagpole and assume that the flagpole makes a 90° angle with the ground, you can use the following triangle to model the situation.



$$\tan 65^\circ = \frac{x}{10}$$
$$x = 10 \tan 65^\circ \approx 30.8\ ft$$

Example 2

Solve for angle A.



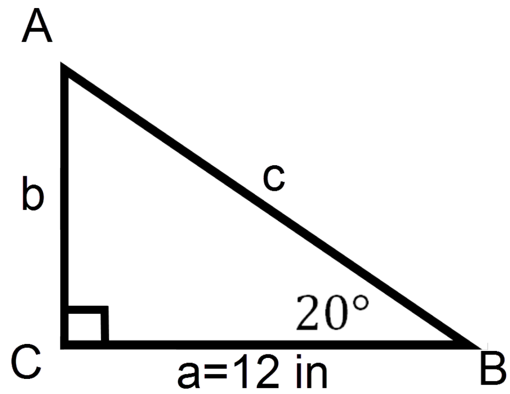
This problem can be solved using sin, cos or tan because the opposite, adjacent and hypotenuse lengths are all given.

The argument of a sin function is always an angle. The arcsin or $\sin^{-1}\theta$ function on the calculator on the other hand has an argument that is a side ratio. It is useful for finding angles that have that side ratio.

$$\sin A = \frac{5}{13}$$
$$A = \sin^{-1}\left(\frac{5}{13}\right) \approx 0.39\ radian \approx 22.6^\circ$$

Example 3

Given a right triangle with $a = 12\ in$, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.

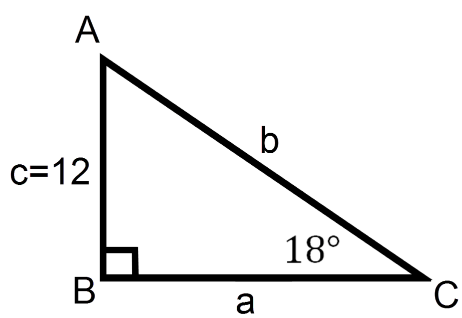


$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77\ in$$

Example 4

Given $\triangle ABC$ where B is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is a?

Drawing out this triangle, it looks like:

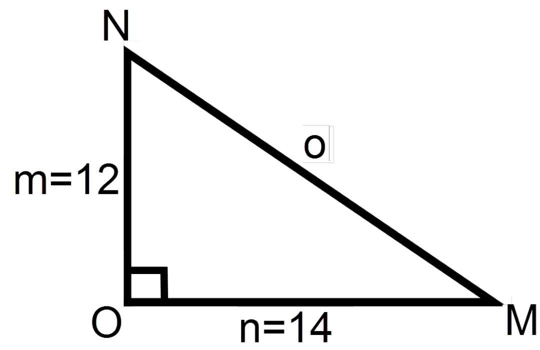


$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

Example 5

Given $\triangle MNO$ where O is a right angle, $m = 12$, and $n = 14$. What is the measure of angle M?

Drawing out the triangle, it looks like:



$$\tan M = \frac{12}{14}$$
$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 0.7\ radian \approx 40.6^\circ$$

Review

For 1-15, information about the sides and/or angles of right triangle *ABC* is given. Completely solve the triangle (find all missing sides and angles) to 1 decimal place.

TABLE 3.7:

Problem Number	A	B	C	a	b	c
1.	90°				4	7
2.	90°		37°	18		
3.		90°	15°		32	
4.			90°	6		11
5.	90°	12°		19		
6.		90°			17	10
7.	90°	10°			2	
8.	4°	90°		0.3		
9.	$\frac{\pi}{2}$ radian		1 radian			15
10.		$\frac{\pi}{2}$ radian		12	15	
11.			$\frac{\pi}{2}$ radian		9	14
12.	$\frac{\pi}{4}$ radian	$\frac{\pi}{4}$ radian			5	
13.	$\frac{\pi}{2}$ radian			26	13	
14.		$\frac{\pi}{2}$ radian			19	16
15.			$\frac{\pi}{2}$ radian	10		$10\sqrt{2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.4.

3.14 Polynomial Long Division and Synthetic Division

Learning Objectives

Here you will learn how to perform long division with polynomials. You will see how synthetic division abbreviates this process. In addition to mastering this procedure, you will see the how the remainder root theorem and the rational root theorem operate. While you may be experienced in factoring, there will always be polynomials that do not readily factor using basic or advanced techniques. How can you identify the roots of these polynomials?

Rational Roots and Dividing Polynomials

There are numerous theorems that point out relationships between polynomials and their factors. For example there is a theorem that a polynomial of degree n must have exactly n solutions/factors that may or may not be real numbers. The Rational Root Theorem and the Remainder Theorem are two theorems that are particularly useful starting places when manipulating polynomials.

The Rational Root Theorem

The **Rational Root Theorem** states that in a polynomial, every rational solution can be written as a reduced fraction $\left(x=\frac{p}{q}\right)$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient.

Let's identify all the possible rational solutions of the following polynomial using the Rational Root Theorem.

$12x^{18}-91x^{17}+x^{16}+\cdots+2x^2-14x+5=0$

The integer factors of 5 are 1, 5. The integer factors of 12 are 1, 2, 3, 4, 6 and 12. Since pairs of factors could both be negative, remember to include $\pm$.

$\pm\frac{p}{q}=\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6},\frac{1}{12},\frac{5}{1},\frac{5}{2},\frac{5}{3},\frac{5}{4},\frac{5}{6},\frac{5}{12}$

While narrowing the possible solutions down to 24 possible rational answers may not seem like a big improvement, it surely is. This is especially true considering there are only a handful of integer solutions. If this question required you to find a solution, then the Rational Root Theorem would give you a great starting place. Once you have one root, you can use either polynomial long division or synthetic to divide the factor out and to keep reducing the expression. We will use the Rational Root Theorem in Example 3.

Polynomial Long Division and the Remainder Theorem

Polynomial long division is identical to regular long division except the dividend and divisor are both polynomials instead of numbers.

The **Remainder Theorem** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division.

Let's put this knowledge to use and use polynomial long division to divide:

$\frac{x^3+2x^2-5x+7}{x-3}$

First note that it is clear that 3 is not a root of the polynomial because of the Rational Root Theorem and so there will definitely be a remainder. Start a polynomial long division question by writing the problem like a long division problem with regular numbers:

$x-3\overline{)x^3+2x^2-5x+7}$

Just like with regular numbers ask yourself "how many times does x go into x^3?" which in this case is x^2.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \end{array}$$

Now multiply the x^2 by $x-3$ and copy below. Remember to subtract the entire quantity.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)} \end{array}$$

Combine the rows, bring down the next number and repeat.

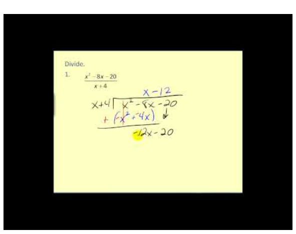
$$\begin{array}{r} x^2+5x+10 \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)} \\ 5x^2-5x \\ \underline{-(5x^2-15x)} \\ 10x+7 \\ \underline{-(10x-30)} \\ 37 \end{array}$$

The number 37 is the remainder. There are two things to think about at this point. First, interpret in an equation:

$\frac{x^3+2x^2-5x+7}{x-3} = (x^2+5x+10) + \frac{37}{x-3}$

Second, check your result with the Remainder Theorem which states that the original function evaluated at 3 must be 37. Notice the notation indicating to substitute 3 in for x.

$(x^3+2x^2-5x+7)|_{x=3} = 3^3+2\cdot 3^2-5\cdot 3+7 = 27+18-15+7 = 37$



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Synthetic Division

Synthetic division is an abbreviated version of polynomial long division where only the coefficients are used. Synthetic division is mostly used when the leading coefficients of the numerator and denominator are equal to 1 and the divisor is a first degree binomial. Let's use synthetic division to divide the same expression that we divided above with polynomial long division:

$\frac{x^3+2x^2-5x+7}{x-3}$

Instead of continually writing and rewriting the x symbols, synthetic division relies on an ordered spacing.

$\underline{+3|}$ 1 2 -5 7

Notice how only the coefficients for the denominator are used and the divisor includes a positive three rather than a negative three. The first coefficient is brought down and then multiplied by the three to produce the value which goes beneath the 2.

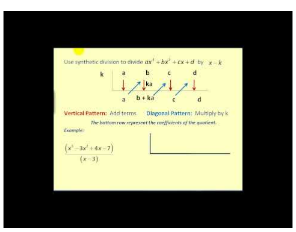
$$\begin{array}{r|rrrr} \underline{+3} & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & & \\ \hline & 1 & & & \end{array}$$

Next the new column is added. $2+3=5$, which goes beneath the 2^{nd} column. Now, multiply $5 \cdot +3 = 15$, which goes underneath the -5 in the 3^{rd} column. And the process repeats...

$$\begin{array}{r|rrrr} \underline{+3} & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & 15 & 30 \\ \hline & 1 & 5 & 10 & 37 \end{array}$$

The last number, 37, is the remainder. When writing out the resulting expression, you will put this remainder over the divisor. The three other numbers represent the quadratic that is identical to the solution to the result from dividing the expression using polynomial long division. Note that when writing out the expression, you decrease the exponent of the leading coefficient of the original by 1.

$(1x^2+5x+10)+\frac{37}{x-3}$



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Examples

Example 1

Earlier, you were asked about identifying roots of polynomials that do not readily factor using the techniques you have learned so far. Identifying roots of polynomials by hand can be tricky business. The best way to identify roots is to use the rational root theorem to quickly identify likely candidates for solutions and then use synthetic or polynomial long division to quickly and effectively test them to see if their remainders are truly zero.

Example 2

Divide the following polynomials.

$\frac{x^3+2x^2-4x+8}{x-2}$

Since the leading coefficients of the numerator and denominator are both 1 and the denominator is a binomial, synthetic division is a good method to use here.

$\frac{x^3+2x^2-4x+8}{x-2} = x^2+4x+4+\frac{16}{x-2}$

Example 3

Completely factor the following polynomial.

$x^4+6x^3+3x^2-26x-24$

Notice that possible roots are $\pm 1, 2, 3, 4, 6, 8, 24$. Of these 14 possibilities, four will yield a remainder of zero. When you find one, use long division or synthetic division to factor out the root that you found. Then find another zero and repeat the process.

$$\begin{aligned}
&x^4+6x^3+3x^2-26x-24\\
&=(x+1)(x^3+5x^2-2x-4)\\
&=(x+1)(x-2)(x^2+7x+12)\\
&=(x+1)(x-2)(x+3)(x+4)
\end{aligned}$$

The first zero found was -1. It was divided out of the original expression to find the remaining non-factored portion of the expression. The second zero found was 2 from the remaining piece and was divided out. Once you get down to a quadratic expression, you can use the other factoring techniques you know to factor the rest of the expression.

Example 4

Divide the following polynomials.

$\frac{3x^5-2x^2+10x-5}{x-1}$

Since the first coefficient of the numerator is not 1, polynomial long division is a good method to use here.

$\frac{3x^5-2x^2+10x-5}{x-1} = 3x^4+3x^3+3x^2+x+11+\frac{6}{x-1}$

Review

Identify all possible rational solutions of the following polynomials using the Rational Root Theorem.

1. $15x^{14}-12x^{13}+x^{12}+\cdots+2x^2-5x+5=0$
2. $18x^{11}+42x^{10}+x^9+\cdots+x^2-3x+7=0$
3. $12x^{16}+11x^{15}+3x^{14}+\cdots+6x^2-2x+11=0$
4. $14x^7-7x^6+x^5+\cdots+x^2+6x+3=0$
5. $9x^9-10x^8+3x^7+\cdots+4x^2-2x+2=0$

Completely factor the following polynomials.

6. $2x^4 - x^3 - 21x^2 - 26x - 8$

7. $x^4 + 7x^3 + 5x^2 - 31x - 30$

8. $x^4 + 3x^3 - 8x^2 - 12x + 16$

9. $4x^4 + 19x^3 - 48x^2 - 117x - 54$

10. $2x^4 + 17x^3 - 8x^2 - 173x + 210$

Divide the following polynomials.

11. $\frac{x^4+7x^3+5x^2-31x-30}{x+4}$

12. $\frac{x^4+7x^3+5x^2-31x-30}{x+2}$

13. $\frac{x^4+3x^3-8x^2-12x+16}{x+3}$

14. $\frac{2x^4-x^3-21x^2-26x-8}{x^3-x^2-10x-8}$

15. $\frac{x^4+8x^3+3x^2-32x-28}{x^3+10x^2+23x+14}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.5.

3.15 Properties of Exponents

Objective

Using the properties of exponents to simplify numeric and algebraic expressions.

Review Queue

Simplify the following expressions.

1. 5^2
2. 3^3
3. $2^3 \cdot 2^2$

For questions 4-6, $x = -3, y = 2$, and $z = -4$. Evaluate the following.

4. xy^2
5. $(xy)^2$
6. $\frac{x^2z}{y^2}$

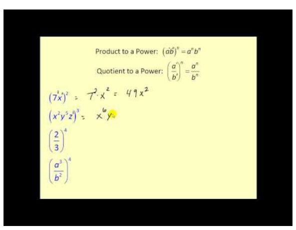
Product and Quotient Properties

Objective

To use and understand the multiplication and quotient properties of exponents.

Watch This

Watch the first part of this video, until about 3:30.



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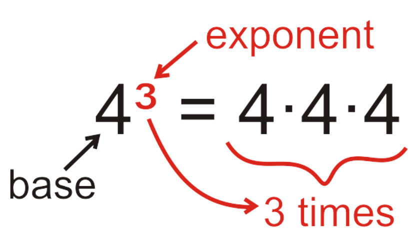
James Sousa: Properties of Exponents

Guidance

To review, the power (or exponent) of a number is the little number in the superscript. The number that is being raised to the power is called the **base**. The **exponent** indicates how many times the base is multiplied by itself.



There are several properties of exponents. We will investigate two in this concept.

Example A

Expand and solve 5^6.

Solution: 5^6 means 5 times itself six times.

$$5^6 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$$

Investigation: Product Property

1. Expand $3^4 \cdot 3^5$.

$\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{3^4} \cdot \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{3^5}$

2. Rewrite this expansion as one power of three.

3^9

3. What is the sum of the exponents?

$4 + 5 = 9$

4. Fill in the blank: $a^m \cdot a^n = a^{\_ + \_}$

$a^m \cdot a^n = a^{m+n}$

Rather than expand the exponents every time or find the powers separately, we can use this property to simplify the product of two exponents with the same base.

Example B

Simplify:

(a) $x^3 \cdot x^8$

(b) $xy^2x^2y^9$

Solution: Use the Product Property above.

(a) $x^3 \cdot x^8 = x^{3+8} = x^{11}$

(b) If a number does not have an exponent, you may assume the exponent is 1. Reorganize this expression so the x's are together and y's are together.

$$xy^2x^2y^9 = x^1 \cdot x^2 \cdot y^2 \cdot y^9 = x^{1+2} \cdot y^{2+9} = x^3y^{11}$$

Investigation: Quotient Property

1. Expand $2^8 \div 2^3$. Also, rewrite this as a fraction.

$\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}$

2. Cancel out the common factors and write the answer one power of 2.

$\frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = 2^5$

3. What is the difference of the exponents?

$8 - 3 = 5$

4. Fill in the blank: $\frac{a^m}{a^n} = a^{\_ - \_}$

$\frac{a^m}{a^n} = a^{m-n}$

Example C

Simplify:

(a) $\frac{5^9}{5^7}$

(b) $\frac{x^4}{x^2}$

(c) $\frac{xy^5}{x^6y^2}$

Solution: Use the Quotient Property from above.

(a) $\frac{5^9}{5^7} = 5^{9-7} = 5^2 = 25$

(b) $\frac{x^4}{x^2} = x^{4-2} = x^2$

(c) $\frac{x^{10}y^5}{x^6y^2} = x^{10-6}y^{5-2} = x^4y^3$

Guided Practice

Simplify the following expressions. Evaluate any numerical answers.

1. $7 \cdot 7^2$

2. $\frac{3^7}{3^3}$

3. $\frac{16x^4y^5}{4x^2y^2}$

Answers

1. $7 \cdot 7^2 = 7^{1+2} = 7^3 = 343$

2. $\frac{3^7}{3^3} = 3^{7-3} = 3^4 = 81$

3. $\frac{16x^4y^5}{4x^2y^2} = 4x^{4-2}y^{5-3} = 4x^2y^2$

Vocabulary

Product of Powers Property
$a^m \cdot a^n = a^{m+n}$

Quotients of Powers Property
$\frac{a^m}{a^n} = a^{m-n}; a \neq 0$

Problem Set

Expand the following numbers and evaluate.

1. 2^6
2. 10^3
3. $(-3)^5$
4. $(0.25)^4$

Simplify the following expressions. Evaluate any numerical answers.

5. $4^2 \cdot 4^7$
6. $6 \cdot 6^3 \cdot 6^2$
7. $\frac{8^3}{8}$
8. $\frac{2^4 \cdot 3^5}{2 \cdot 3^2}$

9. $b^6 \cdot b^3$
10. $5^2 x^4 \cdot x^9$
11. $\frac{y^{12}}{y^5}$
12. $\frac{a^8 \cdot b^6}{b \cdot a^4}$
13. $\frac{3^7 x^6}{3^3 x^3}$
14. $d^5 f^3 d^9 f^7$
15. $\frac{2^8 m^{18} n^{14}}{2^5 m^{11} n^4}$
16. $\frac{9^4 p^5 q^8}{9^2 p q^2}$

Investigation Evaluate the powers of negative numbers.

17. Find:
    a. $(-2)^1$
    b. $(-2)^2$
    c. $(-2)^3$
    d. $(-2)^4$
    e. $(-2)^5$
    f. $(-2)^6$
18. Make a conjecture about even vs. odd powers with negative numbers.
19. Is $(-2)^4$ different from -2^4? Why or why not?

Negative and Zero Exponent Properties

Objective

To evaluate and use negative and zero exponents.

Guidance

In this concept, we will introduce negative and zero exponents. First, let's address a zero in the exponent through an investigation.

Investigation: Zero Exponents

1. Evaluate $\frac{5^6}{5^6}$ by using the Quotient of Powers property.

$\frac{5^6}{5^6} = 5^{6-6} = 5^0$

2. What is a number divided by itself? Apply this to #1.

$\frac{5^6}{5^6} = 1$

3. Fill in the blanks. $\frac{a^m}{a^m} = a^{m-m} = a^{-} = \_$

$a^0 = 1$

Investigation: Negative Exponents

1. Expand $\frac{3^2}{3^7}$ and cancel out the common 3's and write your answer with positive exponents.

$\frac{3^2}{3^7} = \frac{\cancel{3}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{3}\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3} = \frac{1}{3^5}$

2. Evaluate $\frac{3^2}{3^7}$ by using the Quotient of Powers property.

$\frac{3^2}{3^7} = 3^{2-7} = 3^{-5}$

3. Are the answers from #1 and #2 equal? Write them as a single statement.

$\frac{1}{3^5} = 3^{-5}$

4. Fill in the blanks. $\frac{1}{a^m} = a-$ and $\frac{1}{a^{-m}} = a-$

$\frac{1}{a^m} = a^{-m}$ and $\frac{1}{a^{-m}} = a^m$

From the two investigations above, we have learned two very important properties of exponents. First, anything to the zero power is one. Second, negative exponents indicate placement. If an exponent is negative, it needs to be moved from where it is to the numerator or denominator. We will investigate this property further in the Problem Set.

Example A

Simplify the following expressions. Your answer should only have positive exponents.

(a) $\frac{5^2}{5^5}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7}$

(c) $\frac{a^4b^0}{a^8b}$

Solution: Use the two properties from above. An easy way to think about where the "leftover" exponents should go, is to look at the fraction and determine which exponent is greater. For example, in b, there are more x's in the denominator, so the leftover should go there.

(a) $\frac{5^2}{5^5} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7} = \frac{y^{1-1}z^{12-7}}{x^{12-7}} = \frac{y^0z^5}{x^5} = \frac{z^5}{x^5}$

(c) $\frac{a^4b^0}{a^8b} = a^{4-8}b^{0-1} = a^{-4}b^{-1} = \frac{1}{a^4b}$

Alternate Method: Part c

$\frac{a^4b^0}{a^8b} = \frac{1}{a^{8-4}b} = \frac{1}{a^4b}$

Example B

Simplify the expressions. Your answer should only have positive exponents.

(a) $\frac{xy^5}{8y^{-3}}$

(b) $\frac{27g^{-7}h^0}{18g}$

Solution: In these expressions, you will need to move the negative exponent to the numerator or denominator and then change it to a positive exponent to evaluate. Also, simplify any numerical fractions.

(a) $\frac{xy^5}{8y^{-3}} = \frac{xy^5y^3}{8} = \frac{xy^{5+3}}{8} = \frac{xy^8}{8}$

(b) $\frac{27g^{-7}h^0}{18g} = \frac{3}{2g^1g^7} = \frac{3}{2g^{1+7}} = \frac{3}{2g^8}$

Example C

Multiply the two fractions together and simplify. Your answer should only have positive exponents.

$$\frac{4x^{-2}y^5}{20x^8} \cdot \frac{-5x^6y}{15y^{-9}}$$

Solution: The easiest way to approach this problem is to multiply the two fractions together first and then simplify.

$$\frac{4x^{-2}y^5}{20x^8}\cdot\frac{-5x^6y}{15y^{-9}}=-\frac{20x^{-2+6}y^{5+1}}{300x^8y^{-9}}=-\frac{x^{-2+6-8}y^{5+1+9}}{15}=-\frac{x^{-4}y^{15}}{15}=-\frac{y^{15}}{15x^4}$$

Guided Practice

Simplify the expressions.

1. $\frac{8^6}{8^9}$
2. $\frac{3x^{10}y^2}{21x^7y^{-4}}$
3. $\frac{2a^8b^{-4}}{16a^{-5}}\cdot\frac{4^3a^{-3}b^0}{a^4b^7}$

Answers

1. $\frac{8^6}{8^9}=8^{6-9}=\frac{1}{8^3}=\frac{1}{512}$
2. $\frac{3x^{10}y^2}{21x^7y^{-4}}=\frac{x^{10-7}y^{2-(-4)}}{7}=\frac{x^3y^6}{7}$
3. $\frac{2a^8b^{-4}}{16a^{-5}}\cdot\frac{4^3a^{-3}b^0}{a^4b^7}=\frac{128a^{8-3}b^{-4}}{16a^{-5+4}b^7}=\frac{8a^{5+1}}{b^{7+4}}=\frac{8a^6}{b^{11}}$

Vocabulary

Zero Exponent Property
$a^0=1, a\neq 0$

Negative Exponent Property
$\frac{1}{a^m}=a^{-m}$ and $\frac{1}{a^{-m}}=a^m, a\neq 0$

Problem Set

Simplify the following expressions. Answers cannot have negative exponents.

1. $\frac{8^2}{8^4}$
2. $\frac{x^6}{x^{15}}$
3. $\frac{7^{-3}}{7^{-2}}$
4. $\frac{y^{-9}}{y^{10}}$
5. $\frac{x^0y^5}{xy^7}$
6. $\frac{a^{-1}b^8}{a^5b^7}$
7. $\frac{14c^{10}d^{-4}}{21c^6d^{-3}}$
8. $\frac{8g^0h}{30g^{-9}h^2}$
9. $\frac{5x^4}{10y^{-2}}\cdot\frac{y^7x}{x^{-1}y}$
10. $\frac{g^9h^5}{6gh^{12}}\cdot\frac{18h^3}{g^8}$
11. $\frac{4a^{10}b^7}{12a^{-6}}\cdot\frac{9a^{-5}b^4}{20a^{11}b^{-8}}$
12. $\frac{-g^8h}{6g^{-8}}\cdot\frac{9g^{15}h^9}{-h^{11}}$
13. Rewrite the following exponential pattern with positive exponents: $5^{-4},5^{-3},5^{-2},5^{-1},5^0,5^1,5^2,5^3,5^4$.
14. Evaluate each term in the pattern from #13.
15. Fill in the blanks.

As the numbers increase, you _______________ the previous term by 5.

As the numbers decrease, you ______________ the previous term by 5.

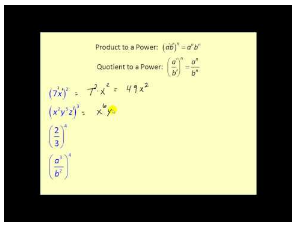
Power Properties

Objective

To discover and use the power properties of exponents.

Watch This

Watch the second part of this video, starting around 3:30.



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James Sousa: Properties of Exponents

Guidance

The last set of properties to explore are the power properties. Let's investigate what happens when a power is raised to another power.

Investigation: Power of a Power Property

1. Rewrite $(2^3)^5$ as 2^3 five times.

$(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$

2. Expand each 2^3. How many 2's are there?

$(2^3)^5 = \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} = 2^{15}$

3. What is the *product* of the powers?

$3 \cdot 5 = 15$

4. Fill in the blank. $(a^m)^n = a^{\_ \cdot \_}$

$(a^m)^n = a^{mn}$

The other two exponent properties are a form of the distributive property.

Power of a Product Property: $(ab)^m = a^m b^m$

Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example A

Simplify the following.

(a) $(3^4)^2$

(b) $(x^2 y)^5$

Solution: Use the new properties from above.

(a) $(3^4)^2 = 3^{4 \cdot 2} = 3^8 = 6561$

(b) $(x^2 y)^5 = x^{2 \cdot 5} y^5 = x^{10} y^5$

Example B

Simplify $\left(\frac{3a^{-6}}{2^2a^2}\right)^4$ without negative exponents.

Solution: This example uses the Negative Exponent Property from the previous concept. Distribute the 4^{th} power first and then move the negative power of a from the numerator to the denominator.

$$\left(\frac{3a^{-6}}{2^2a^2}\right)^4=\frac{3^4a^{-6\cdot 4}}{2^{2\cdot 4}a^{2\cdot 4}}=\frac{81a^{-24}}{2^8a^8}=\frac{81}{256a^{8+24}}=\frac{81}{256a^{32}}$$

Example C

Simplify $\frac{4x^{-3}y^4z^6}{12x^2y}\div\left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2$ without negative exponents.

Solution: This example is definitely as complicated as these types of problems get. Here, all the properties of exponents will be used. Remember that dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned}\frac{4x^{-3}y^4z^6}{12x^2y}\div\left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2&=\frac{4x^{-3}y^4z^6}{12x^2y}\cdot\frac{225x^6z^{-4}}{25x^2y^{-2}}\\&=\frac{y^3z^6}{3x^5}\cdot\frac{9x^4y^2}{z^4}\\&=\frac{3x^4y^5z^6}{x^5z^4}\\&=\frac{3y^5z^2}{x}\end{aligned}$$

Guided Practice

Simplify the following expressions without negative exponents.

1. $\left(\frac{5a^3}{b^4}\right)^7$
2. $(2x^5)^{-3}(3x^9)^2$
3. $\frac{(5x^2y^{-1})^3}{10y^6}\cdot\left(\frac{16x^8y^5}{4x^7}\right)^{-1}$

Answers

1. Distribute the 7 to every power within the parenthesis.

$$\left(\frac{5a^3}{b^4}\right)^7=\frac{5^7a^{21}}{b^{28}}=\frac{78,125a^{21}}{b^{28}}$$

2. Distribute the -3 and 2 to their respective parenthesis and then use the properties of negative exponents, quotient and product properties to simplify.

$$(2x^5)^{-3}(3x^9)^2=2^{-3}x^{-15}3^2x^{18}=\frac{9x^3}{8}$$

3. Distribute the exponents that are outside the parenthesis and use the other properties of exponents to simplify. Anytime a fraction is raised to the -1 power, it is equal to the reciprocal of that fraction to the first power.

$$\begin{aligned}\frac{\left(5x^2y^{-1}\right)^3}{10y^6}\cdot\left(\frac{16x^8y^5}{4x^7}\right)^{-1} &= \frac{5^3x^{-6}y^{-3}}{10y^6}\cdot\frac{4x^7}{16x^8y^5}\\ &= \frac{500xy^{-3}}{160x^8y^{11}}\\ &= \frac{25}{8x^7y^{14}}\end{aligned}$$

Vocabulary

Power of Power Property
: $(a^m)^n = a^{mn}$

Power of a Product Property
: $(ab)^m = a^m b^m$

Power of a Quotient Property
: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Problem Set

Simplify the following expressions without negative exponents.

1. $(2^5)^3$
2. $(3x)^4$
3. $\left(\frac{4}{5}\right)^2$
4. $(6x^3)^3$
5. $\left(\frac{2a^3}{b^5}\right)^7$
6. $(4x^8)^{-2}$
7. $\left(\frac{1}{7^2h^9}\right)^{-1}$
8. $\left(\frac{2x^4y^2}{5x^{-3}y^5}\right)^3$
9. $\left(\frac{9m^5n^{-7}}{27m^6n^5}\right)^{-4}$
10. $\frac{(4x)^2(5y)^{-3}}{(2x^3y^5)^2}$
11. $(5r^6)^4\left(\frac{1}{3}r^{-2}\right)^5$
12. $(4t^{-1}s)^3(2^{-1}ts^{-2})^{-3}$
13. $\frac{6a^2b^4}{18a^{-3}b^4}\cdot\left(\frac{8b^{12}}{40a^{-8}b^5}\right)^2$
14. $\frac{2(x^4y^4)^0}{2^4x^3y^5z}\div\frac{8z^{10}}{32x^{-2}y^5}$
15. $\frac{5g^6}{15g^0h^{-1}}\cdot\left(\frac{h}{9g^{15}j^7}\right)^{-3}$
16. **Challenge** $\frac{a^7b^{10}}{4a^{-5}b^{-2}}\cdot\left[\frac{(6ab^{12})^2}{12a^9b^{-3}}\right]^2\div(3a^5b^{-4})^3$
17. Rewrite 4^3 as a power of 2.
18. Rewrite 9^2 as a power of 3.
19. Solve the equation for x. $3^2\cdot 3^x = 3^8$
20. Solve the equation for x. $(2^x)^4 = 4^8$

3.16 Adding, Subtracting and Multiplying Polynomials

Objective

To add, subtract, and multiply polynomials.

Review Queue

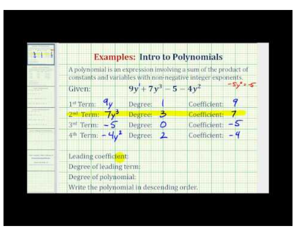
1. Multiply $(3x-1)(x+4)$.
2. Factor x^2-6x+9.
3. Multiply $(2x+5)(2x-11)$.
4. Combine like terms: $3x+15+2x-8-x$

Adding and Subtracting Polynomials

Objective

Adding and subtracting polynomials, as well as learning about the different parts of a polynomial.

Watch This



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James Sousa: Ex: Intro to Polynomials in One Variable

Guidance

A **polynomial** is an expression with multiple variable terms, such that the exponents are greater than or equal to zero. All quadratic and linear equations are polynomials. Equations with negative exponents, square roots, or variables in the denominator are not polynomials.

Polynomials	Not Polynomials
$2x^2+6x-9$	$10x^{-1}+6x^2$
$-x^3+9$	$\sqrt{x}-2$
$4x^4+5x^3-8x^2+12x+24$	$\frac{3}{x}+5$

Now that we have established what a polynomial is, there are a few important parts. Just like with a quadratic, a polynomial can have a **constant**, which is a number without a variable. The **degree** of a polynomial is the largest exponent. For example, all quadratic equations have a degree of 2. Lastly, the **leading coefficient** is the coefficient in front of the variable with the degree. In the polynomial $4x^4+5x^3-8x^2+12x+24$ above, the degree is 4 and the leading coefficient is also 4. Make sure that when finding the degree and leading coefficient you have the polynomial in standard form. **Standard form** lists all the variables in order, from greatest to least.

Example A

Rewrite $x^3-5x^2+12x^4+15-8x$ in standard form and find the degree and leading coefficient.

Solution: To rewrite in standard form, put each term in order, from greatest to least, according to the exponent. Always write the constant last.

$$x^3-5x^2+12x^4+15-8x \to 12x^4+x^3-5x^2-8x+15$$

Now, it is easy to see the leading coefficient, 12, and the degree, 4.

Example B

Simplify $(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)$

Solution: To add or subtract two polynomials, combine like terms. **Like terms** are any terms where the exponents of the variable are the same. We will regroup the polynomial to show the like terms.

$$\begin{aligned}&(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)\\&x^4+(4x^3-8x^3)-2x^2+(4x-9x)+(15-6)\\&x^4-4x^3-2x^2-5x+9\end{aligned}$$

Example C

Simplify $(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)$

Solution: When subtracting, distribute the negative sign to every term in the second polynomial, then combine like terms.

$$\begin{aligned}&(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)\\&2x^3+x^2-6x-7-5x^3+3x^2-10x+12\\&(2x^3-5x^3)+(x^2+3x^2)+(-6x-10x)+(-7+12)\\&-3x^3+4x^2-16x+5\end{aligned}$$

Guided Practice

1. Is $\sqrt{2x^3-5x}+6$ a polynomial? Why or why not?

2. Find the leading coefficient and degree of $6x^2-3x^5+16x^4+10x-24$.

Add or subtract.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)$

Answers

1. No, this is not a polynomial because x is under a square root in the equation.

2. In standard form, this polynomial is $-3x^5+16x^4+6x^2+10x-24$. Therefore, the degree is 5 and the leading coefficient is -3.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)=10x^3+5x^2-7x+8$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)=6x^3+2x^2+6x+15$

Vocabulary

Polynomial
An expression with multiple variable terms, such that the exponents are greater than or equal to zero.

Constant
A number without a variable in a mathematical expression.

Degree(of a polynomial)
The largest exponent in a polynomial.

Leading coefficient
The coefficient in front of the variable with the degree.

Standard form
Lists all the variables in order, from greatest to least.

Like terms
Any terms where the exponents of the variable are the same.

Problem Set

Determine if the following expressions are polynomials. If not, state why. If so, write in standard form and find the degree and leading coefficient.

1. $\frac{1}{x^2}+x+5$
2. $x^3+8x^4-15x+14x^2-20$
3. x^3+8
4. $5x^{-2}+9x^{-1}+16$
5. $x^2\sqrt{2}-x\sqrt{6}+10$
6. $\frac{x^4+8x^2+12}{3}$
7. $\frac{x^2-4}{x}$
8. $-6x^3+7x^5-10x^6+19x^2-3x+41$

Add or subtract the following polynomials.

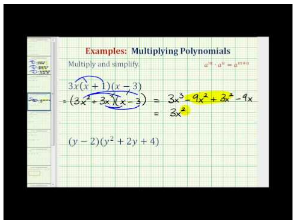
9. $(x^3+8x^2-15x+11)+(3x^3-5x^2-4x+9)$
10. $(-2x^4+x^3+12x^2+6x-18)-(4x^4-7x^3+14x^2+18x-25)$
11. $(10x^3-x^2+6x+3)+(x^4-3x^3+8x^2-9x+16)$
12. $(7x^3-2x^2+4x-5)-(6x^4+10x^3+x^2+4x-1)$
13. $(15x^2+x-27)+(3x^3-12x+16)$
14. $(2x^5-3x^4+21x^2+11x-32)-(x^4-3x^3-9x^2+14x-15)$
15. $(8x^3-13x^2+24)-(x^3+4x^2-2x+17)+(5x^2+18x-19)$

Multiplying Polynomials

Objective

To multiply together several different types of polynomials.

Watch This



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James Sousa: Ex: Polynomial Multiplication Involving Binomials and Trinomials

Guidance

Multiplying together polynomials is very similar to multiplying together factors. You can FOIL or we will also present an alternative method. When multiplying together polynomials, you will need to use the properties of exponents, primarily the Product Property $(a^m \cdot a^n = a^{m+n})$ and combine like terms.

Example A

Find the product of $(x^2 - 5)(x^3 + 2x - 9)$.

Solution: Using the FOIL method, you need be careful. First, take the x^2 in the first polynomial and multiply it by every term in the second polynomial.

$$(x^2 - 5)(x^3 + 2x - 9) = x^5 + 2x^3 - 9x^2$$

Now, multiply the -5 and multiply it by every term in the second polynomial.

$$(x^2 - 5)(x^3 + 2x - 9) = x^5 + 2x^3 - 9x^2 - 5x^3 - 10x + 45$$

Lastly, combine any like terms. In this example, only the x^3 terms can be combined.

$$\begin{aligned}(x^2 - 5)(x^3 + 2x - 9) &= x^5 \boxed{+ 2x^3} - 9x^2 \boxed{- 5x^3} - 10x + 45 \\ &= x^5 - 3x^3 - 9x^2 - 10x + 45\end{aligned}$$

Example B

Multiply $(x^2 + 4x - 7)(x^3 - 8x^2 + 6x - 11)$.

Solution: In this example, we will use the "box" method. Align the two polynomials along the top and left side of a rectangle and make a row or column for each term. Write the polynomial with more terms along the top of the rectangle.

	x^3	$-8x^2$	$+6x$	-11
x^2				
$4x$				
-7				

Multiply each term together and fill in the corresponding spot.

	x^3	$-8x^2$	$+6x$	-11
x^2	x^5	$-8x^4$	$6x^3$	$-11x^2$
$4x$	$4x^4$	$-32x^3$	$24x^2$	$-44x$
-7	$-7x^3$	$56x^2$	$-42x$	77

Finally, combine like terms. The final answer is $x^5 - 4x^4 - 33x^3 + 69x^2 - 86x + 77$. This method presents an alternative way to organize the terms. Use whichever method you are more comfortable with. Keep in mind, no matter which method you use, you will multiply every term in the first polynomial by every term in the second.

Example C

Find the product of $(x-5)(2x+3)(x^2+4)$.

Solution: In this example we have three binomials. When multiplying three polynomials, start by multiplying the first two binomials together.

$$\begin{aligned}(x-5)(2x+3) &= 2x^2+3x-10x-15\\ &= 2x^2-7x-15\end{aligned}$$

Now, multiply the answer by the last binomial.

$$\begin{aligned}(2x^2-7x-15)(x^2+4) &= 2x^4+8x^2-7x^3-28x-15x^2-60\\ &= 2x^4-7x^3-7x^2-28x-60\end{aligned}$$

Guided Practice

Find the product of the polynomials.

1. $-2x^2(3x^3-4x^2+12x-9)$
2. $(4x^2-6x+11)(-3x^3+x^2+8x-10)$
3. $(x^2-1)(3x-4)(3x+4)$

4. $(2x-7)^2$

Answers

1. Use the distributive property to multiply $-2x^2$ by the polynomial.

$$-2x^2(3x^3-4x^2+12x-9) = -6x^5+8x^4-24x^3+18x^2$$

2. Multiply each term in the first polynomial by each one in the second polynomial.

$$\begin{aligned}(4x^2-6x+11)(-3x^3+x^2+8x-10) &= -12x^5+4x^4+32x^3-40x^2\\ &\quad +18x^4-6x^3-48x^2+60x\\ &\quad -33x^3+11x^2+88x-110\\ &= -12x^5+22x^4-7x^3-77x^2+148x-110\end{aligned}$$

3. Multiply the first two binomials together.

$$(x^2-1)(3x-4) = 3x^3-4x^2-3x+4$$

Multiply this product by the last binomial.

$$\begin{aligned}(3x^3-4x^2-3x+4)(3x+4) &= 9x^4+12x^3-12x^3-16x^2-9x^2-12x+12x-16\\ &= 9x^4-25x^2-16\end{aligned}$$

4. The square indicates that there are two binomials. Expand this and multiply.

$$\begin{aligned}(2x-7)^2 &= (2x-7)(2x-7)\\ &= 4x^2-14x-14x+49\\ &= 4x^2-28x+49\end{aligned}$$

Problem Set

Find the product.

1. $5x(x^2-6x+8)$
2. $-x^2(8x^3-11x+20)$
3. $7x^3(3x^3-x^2+16x+10)$
4. $(x^2+4)(x-5)$
5. $(3x^2-4)(2x-7)$
6. $(9-x^2)(x+2)$
7. $(x^2+1)(x^2-2x-1)$
8. $(5x-1)(x^3+8x-12)$
9. $(x^2-6x-7)(3x^2-7x+15)$
10. $(x-1)(2x-5)(x+8)$

11. $(2x^2+5)(x^2-2)(x+4)$
12. $(5x-12)^2$
13. $-x^4(2x+11)(3x^2-1)$
14. $(4x+9)^2$
15. $(4x^3-x^2-3)(2x^2-x+6)$
16. $(2x^3-6x^2+x+7)(5x^2+2x-4)$
17. $(x^3+x^2-4x+15)(x^2-5x-6)$

3.17 Factoring and Solving Polynomial Equations

Objective

To solve and factor polynomials using several different methods.

Review Queue

Factor the following quadratics.

1. $x^2-9x-22$

2. $4x^2-25$

3. $6x^2+7x-5$

4. Set #3 equal to zero and solve.

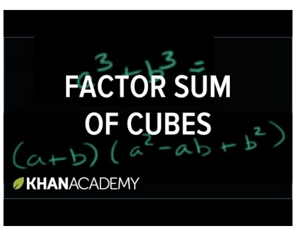
Sum and Difference of Cubes

Objective

To learn the sum and difference of cubes formulas for factoring certain types of polynomials.

Watch This

First watch this video.



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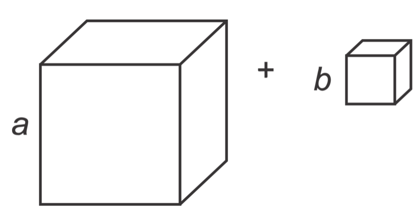
Khan Academy: Factoring Sum of Cubes

Guidance

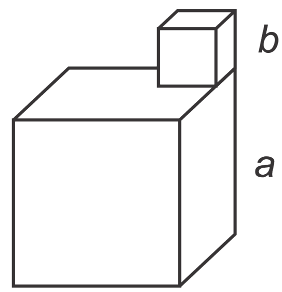
In the previous chapter, you learned how to factor several different types of quadratic equations. Here, we will expand this knowledge to certain types of polynomials. The first is the sum of cubes. The sum of cubes is what it sounds like, the sum of two cube numbers or a^3+b^3. We will use an investigation involving volume to find the factorization of this polynomial.

Investigation: Sum of Cubes Formula

1. Pictorially, the sum of cubes looks like this:



Or, we can put one on top of the other.

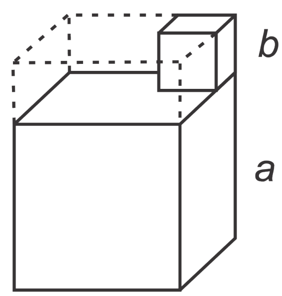


2. Recall that the formula for volume is $length \times width \times depth$. Find the volume of the sum of these two cubes.

$V = a^3 + b^3$

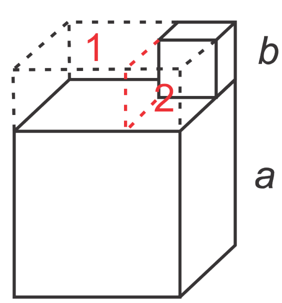
3. Now, we will find the volume in a different way. Using the second picture above, will add in imaginary lines so that these two cubes look like one large prism. Find the volume of this prism.

$$\begin{aligned} V &= a \times a \times (a+b) \\ &= a^2(a+b) \end{aligned}$$



4. Subtract the imaginary portion on top. In the picture, they are prism 1 and prism 2.

$$V = a^2(a+b) - \left[\underbrace{ab(a-b)}_{Prism\ 1} + \underbrace{b^2(a-b)}_{Prism\ 2}\right]$$



5. Pull out any common factors within the brackets.

$V = a^2(a+b) - b(a-b)[a+b]$

6. Notice that both terms have a common factor of $(a+b)$. Pull this out, put it in front, and get rid of the brackets.

$V = (a+b)(a^2 - b(a-b))$

7. Simplify what is inside the second set of parenthesis.

$V = (a+b)(a^2 - ab + b^2)$

In the last step, we found that $a^3 + b^3$ factors to $(a+b)(a^2 - ab + b^2)$. This is the **Sum of Cubes Formula**.

Example A

Factor $8x^3 + 27$.

Solution: First, determine if these are "cube" numbers. A cube number has a cube root. For example, the cube root of 8 is 2 because $2^3 = 8$. $3^3 = 27, 4^3 = 64, 5^3 = 125$, and so on.

$$\begin{aligned} a^3 &= 8x^3 = (2x)^3 & b^3 &= 27 = 3^3 \\ a &= 2x & b &= 3 \end{aligned}$$

In the formula, we have:

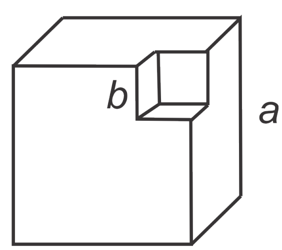
$$\begin{aligned} (a+b)(a^2 - ab + b^2) &= (2x+3)((2x)^2 - (2x)(3) + 3^2) \\ &= (2x+3)(4x^2 - 6x + 9) \end{aligned}$$

Therefore, $8x^3 + 27 = (2x+3)(4x^2 - 6x + 9)$. The second factored polynomial does not factor any further.

Investigation: Difference of Cubes

1. Pictorially, the difference of cubes looks like this:

Imagine the smaller cube is taken out of the larger cube.

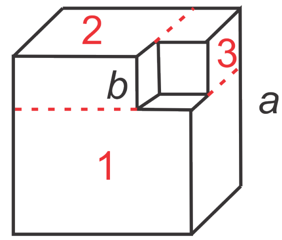


2. Recall that the formula for volume is $length \times width \times depth$. Find the volume of the difference of these two cubes.

$V = a^3 - b^3$

3. Now, we will find the volume in a different way. Using the picture here, will add in imaginary lines so that the shape is split into three prisms. Find the volume of prism 1, prism 2, and prism 3.

$$\begin{aligned} &\text{Prism 1}: a \cdot a \cdot (a-b) \\ &\text{Prism 2}: a \cdot b \cdot (a-b) \\ &\text{Prism 3}: b \cdot b \cdot (a-b) \end{aligned}$$



4. Add the volumes together to get the volume of the entire shape.

$V = a^2(a-b)+ab(a-b)+b^2(a-b)$

5. Pull out any common factors and simplify.

$V = (a-b)(a^2+ab+b^2)$

In the last step, we found that a^3-b^3 factors to $(a-b)(a^2+ab+b^2)$. This is the **Difference of Cubes Formula**.

Example B

Factor x^5-125x^2.

Solution: First, take out any common factors.

$$x^5-125x^2 = x^2(x^3-125)$$

What is inside the parenthesis is a difference of cubes. Use the formula.

$$\begin{aligned} x^5-125x^2 &= x^2(x^3-125) \\ &= x^2(x^3-5^3) \\ &= x^2(x-5)(x^2+5x+25) \end{aligned}$$

Example C

Find the real-number solutions of $x^3-8=0$.

Solution: Factor using the difference of cubes.

$$\begin{aligned} x^3-8 &= 0 \\ (x-2)(x^2+2x+4) &= 0 \\ x &= 2 \end{aligned}$$

In the last step, we set the first factor equal to zero. The second factor, x^2+2x+4, will give imaginary solutions. For both the sum and difference of cubes, this will always happen.

Guided Practice

Factor using the sum or difference of cubes.

1. x^3-1

2. $3x^3+192$

3. $125-216x^3$

4. Find the real-number solution to $27x^3+8=0$.

Answers

1. Factor using the difference of cubes.

$$\begin{aligned} x^3-1 &= x^3-1^3 \\ &= (x-1)(x^2+x+1) \end{aligned}$$

2. Pull out the 3, then factor using the sum of cubes.

$$\begin{aligned} 3x^3+192 &= 3(x^3+64) \\ &= 3(x^3+4^3) \\ &= 3(x+4)(x^2-4x+16) \end{aligned}$$

3. Factor using the difference of cubes.

$$\begin{aligned} 125-216x^3 &= 5^3-(6x)^3 \\ &= (5-6x)(5^2+(5)(6x)+(6x)^2) \\ &= (5-6x)(25+30x+36x^2) \end{aligned}$$

4. Factor using the sum of cubes and then solve.

$$\begin{aligned} 27x^3+8 &= 0 \\ (3x)^3+2^3 &= 0 \\ (3x+2)(9x^2-6x+4) &= 0 \\ x &= -\frac{2}{3} \end{aligned}$$

Vocabulary

Sum of Cubes Formula
$a^3+b^3=(a+b)(a^2-ab+b^2)$

Difference of Cubes Formula
$a^3-b^3=(a-b)(a^2+ab+b^2)$

Problem Set

Factor each polynomial by using the sum or difference of cubes.

1. x^3-27
2. $64+x^3$
3. $32x^3-4$
4. $64x^3+343$
5. $512-729x^3$
6. $125x^4+8x$

7. $648x^3 + 81$
8. $5x^6 - 135x^3$
9. $686x^7 - 1024x^4$

Find the real-number solutions for each equation.

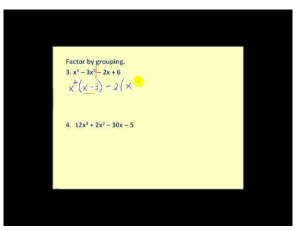
10. $125x^3 + 1 = 0$
11. $64 - 729x^3 = 0$
12. $8x^4 - 343x = 0$
13. **Challenge** Find ALL solutions (real and imaginary) for $5x^5 + 625x^2 = 0$.
14. **Challenge** Find ALL solutions (real and imaginary) for $686x^3 + 2000 = 0$.
15. **Real Life Application** You have a piece of cardboard that you would like to fold up and make an open (no top) box out of. The dimensions of the cardboard are $36'' \times 42''$. Write a factored equation for the volume of this box. Find the volume of the box when $x = 1, 3,$ and 5.

Factoring by Grouping

Objective

To factor and solve certain polynomials by grouping.

Watch This



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James Sousa: Factor By Grouping

Guidance

In the *Factoring when* $a \neq 1$ concept (in the previous chapter), we introduced factoring by grouping. We will expand this idea to other polynomials here.

Example A

Factor $x^4 + 7x^3 - 8x - 56$ by grouping.

Solution: First, group the first two and last two terms together. Pull out any common factors.

$$\underbrace{x^4 + 7x^3}_{x^3(x+7)}\underbrace{-8x - 56}_{-8(x+7)}$$

Notice what is inside the parenthesis is *the same*. This should always happen when factoring by grouping. Pull out this common factor.

$$x^3(x+7) - 8(x+7)$$
$$(x+7)(x^3 - 8)$$

Look at the factors. Can they be factored any further? Yes. The second factor is a difference of cubes. Use the formula.

$$\begin{aligned}&(x+7)(x^3-8)\\&(x+7)(x-2)(x^2+2x+4)\end{aligned}$$

Example B

Factor x^3+5x^2-x-5 by grouping.

Solution: Follow the steps from above.

$$\begin{aligned}&x^3+5x^2-x-5\\&x^2(x+5)-1(x+5)\\&(x+5)(x^2-1)\end{aligned}$$

Look to see if we can factor either factor further. Yes, the second factor is a difference of squares.

$$\begin{aligned}&(x+5)(x^2-1)\\&(x+5)(x-1)(x+1)\end{aligned}$$

Example C

Find all real-number solutions of $2x^3-3x^2+8x-12=0$.

Solution: Follow the steps from Example A.

$$\begin{aligned}2x^3-3x^2+8x-12&=0\\x^2(2x-3)+4(2x-3)&=0\\(2x-3)(x^2+4)&=0\end{aligned}$$

Now, determine if you can factor further. No, x^2+4 is a sum of squares and not factorable. Setting the first factor equal to zero, we get $x=\frac{3}{2}$.

Guided Practice

Factor the following polynomials by grouping.

1. $x^3+7x^2-2x-14$

2. $2x^4-5x^3+2x-5$

3. Find all the real-number solutions of $4x^3-8x^2-x+2=0$.

Answers

Each of these problems is done in the same way: Group the first two and last two terms together, pull out any common factors, what is inside the parenthesis is the same, factor it out, then determine if either factor can be factored further.

1.

$$\begin{aligned}&x^3+7x^2-2x-14\\&x^2(x+7)-2(x+7)\\&(x+7)(x^2-2)\end{aligned}$$

x^2-2 is not a difference of squares because 2 is not a square number. Therefore, this cannot be factored further.

2.

$$\begin{aligned}&2x^4-5x^3+2x-5\\&x^3(2x-5)+1(2x-5)\\&(2x-5)(x^3+1)\qquad \text{Sum of cubes, factor further.}\\&(2x-5)(x+1)(x^2+x+1)\end{aligned}$$

3. Factor by grouping.

$$\begin{aligned}4x^3-8x^2-x+2&=0\\4x^2(x-2)-1(x-2)&=0\\(x-2)(4x^2-1)&=0\\(x-2)(2x-1)(2x+1)&=0\\x&=2,\frac{1}{2},-\frac{1}{2}\end{aligned}$$

Problem Set

Factor the following polynomials using factoring by grouping. Factor each polynomial completely.

1. $x^3-4x^2+3x-12$
2. $x^3+6x^2-9x-54$
3. $3x^3-4x^2+15x-20$
4. $2x^4-3x^3-16x+24$
5. $4x^3+4x^2-25x-25$
6. $4x^3+18x^2-10x-45$
7. $24x^4-40x^3+81x-135$
8. $15x^3+6x^2-10x-4$
9. $4x^3+5x^2-100x-125$

Find all the real-number solutions of the polynomials below.

10. $9x^3-54x^2-4x+24=0$
11. $x^4+3x^3-27x-81=0$
12. $x^3-2x^2-4x+8=0$
13. **Challenge** Find ALL the solutions of $x^6-9x^4-x^2+9=0$.
14. **Challenge** Find ALL the solutions of $x^3+3x^2+16x+48=0$.

Factoring Polynomials in Quadratic Form

Objective

To factor and solve polynomials that are in "quadratic form."

Guidance

The last type of factorable polynomial are those that are in quadratic form. **Quadratic form** is when a polynomial looks like a trinomial or binomial and can be factored like a quadratic. One example is when a polynomial is in the form ax^4+bx^2+c. Another possibility is something similar to the difference of squares, a^4-b^4. This can be factored to $(a^2-b^2)(a^2+b^2)$ or $(a-b)(a+b)(a^2+b^2)$. Always keep in mind that the greatest common factors should be factored out first.

Example A

Factor $2x^4-x^2-15$.

Solution: This particular polynomial is factorable. Let's use the method we learned in the *Factoring when* $a \neq 1$ concept. First, $ac=-30$. The factors of -30 that add up to -1 are -6 and 5. Expand the middle term and then use factoring by grouping.

$$\begin{aligned}
&2x^4-x^2-15\\
&2x^4-6x^2+5x^2-15\\
&2x^2(x^2-3)+5(x^2-3)\\
&(x^2-3)(2x^2+5)
\end{aligned}$$

Both of the factors are not factorable, so we are done.

Example B

Factor $81x^4-16$.

Solution: Treat this polynomial equation like a difference of squares.

$$\begin{aligned}
&81x^4-16\\
&(9x^2-4)(9x^2+4)
\end{aligned}$$

Now, we can factor $9x^2-4$ using the difference of squares a second time.

$$(3x-2)(3x+2)(9x^2+4)$$

$9x^2+4$ cannot be factored because it is a sum of squares. This will have imaginary solutions.

Example C

Find all the real-number solutions of $6x^5-51x^3-27x=0$.

Solution: First, pull out the GCF among the three terms.

$$\begin{aligned}
6x^5-51x^3-27x&=0\\
3x(2x^4-17x^2-9)&=0
\end{aligned}$$

Factor what is inside the parenthesis like a quadratic equation. $ac=-18$ and the factors of -18 that add up to -17 are -18 and 1. Expand the middle term and then use factoring by grouping.

$$\begin{aligned}6x^5-51x^3-27x&=0\\3x(2x^4-17x^2-9)&=0\\3x(2x^4-18x^2+x^2-9)&=0\\3x[2x^2(x^2-9)+1(x^2-9)]&=0\\3x(x^2-9)(2x^2+1)&=0\end{aligned}$$

Factor x^2-9 further and solve for x where possible. $2x^2+1$ is not factorable.

$$\begin{aligned}3x(x^2-9)(2x^2+1)&=0\\3x(x-3)(x+3)(2x^2+1)&=0\\x&=-3,0,3\end{aligned}$$

Guided Practice

Factor the following polynomials.

1. $3x^4+14x^2+8$

2. $36x^4-25$

3. Find all the real-number solutions of $8x^5+26x^3-24x=0$.

Answers

1. $ac=24$ and the factors of 24 that add up to 14 are 12 and 2.

$$\begin{aligned}&3x^4+14x^2+8\\&3x^4+12x^2+2x^2+8\\&3x^2(x^2+4)+2(x^4+4)\\&(x^2+4)(3x^2+2)\end{aligned}$$

2. Factor this polynomial like a difference of squares.

$$\begin{aligned}&36x^4-25\\&(6x^2-5)(6x^2+5)\end{aligned}$$

6 and 5 are not square numbers, so this cannot be factored further.

3. Pull out a $2x$ from each term.

$$\begin{aligned}8x^5+26x^3-24x&=0\\2x(4x^4+13x-12)&=0\\2x(4x^4+16x^2-3x^2-12)&=0\\2x[4x^2(x^2+4)-3(x^2+4)]&=0\\2x(x^2+4)(4x^2-3)&=0\end{aligned}$$

Set each factor equal to zero.

$$\begin{aligned} 2x &= 0 & x^2+4 &= 0 & & & 4x^2-3 &= 0 \\ & & & & \text{and} & & x^2 &= \frac{3}{4} \\ x &= 0 & x^2 &= -4 & & & x &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Notice the second factor will give imaginary solutions.

Vocabulary

Quadratic form
When a polynomial looks a trinomial or binomial and can be factored like a quadratic equation.

Problem Set

Factor the following quadratics completely.

1. x^4-6x^2+8
2. x^4-4x^2-45
3. $4x^4-11x^2-3$
4. $6x^4+19x^2+8$
5. x^4-81
6. $16x^4-1$
7. $6x^5+26x^3-20x$
8. $4x^6-36x^2$
9. $625-81x^4$

Find all the real-number solutions to the polynomials below.

10. $2x^4-5x^2-12=0$
11. $16x^4-49=0$
12. $12x^6+69x^4+45x^2=0$

3.18 Dividing Polynomials

Objective

To divide one polynomial by another using long or synthetic division.

Review Queue

Divide the following numbers by hand.

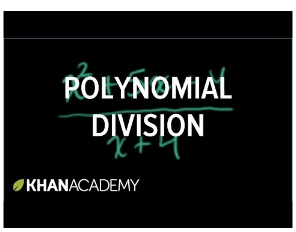
1. $60 \div 4$
2. $\sqrt[18]{1512}$
3. $825 \div 5$
4. $\sqrt[7.6]{3214.8}$

Long Division

Objective

To use long division to divide polynomials.

Watch This



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Khan Academy: Polynomial Division

Guidance

Even though it does not seem like it, factoring is a form of division. Each factor goes into the larger polynomial evenly, without a remainder. For example, take the polynomial $2x^3-3x^2-8x+12$. If we use factoring by grouping, we find that the factors are $(2x-3)(x-2)(x+2)$. If we multiply these three factors together, we will get the original polynomial. So, if we divide by $2x-3$, we should get x^2-4.

$$2x-3\overline{\smash{)}2x^3-3x^2-8x+12}$$

How many times does $2x$ go into $2x^3$? It goes in x^2 times.

$$\begin{array}{r|l} & x^2 \\ \hline 2x-3 & 2x^3-3x^2-8x+12 \\ & \underline{2x^3-3x^2} \\ & \quad 0 \end{array}$$

Place x^2 above the x^2 term in the polynomial.

Multiply x^2 by both terms in the **divisor** ($2x$ and -3) and place them until their like terms. *Subtract* from the **dividend** $(2x^3-3x^2-8x+12)$. Pull down the next two terms and repeat.

$$\begin{array}{r|l} & \quad x^2+0x-4 \\ \hline 2x-3 & 2x^3-3x^2-8x+12 \\ & \underline{2x^3-3x^2} \\ & \quad\ 0 \quad -8x+12 \\ & \qquad\quad \underline{-8x+12} \\ & \qquad\qquad\quad 0 \end{array}$$

$-8x$ divided by $2x = -4$

After multiplying both terms in the divisor by -4, place that under the terms you brought down. When subtracting we notice that everything cancels out. Therefore, just like we thought, x^2-4 is a factor.

When dividing polynomials, not every divisor will go in evenly to the dividend. If there is a remainder, write it as a fraction over the divisor.

Example A

$(2x^3-6x^2+5x-20) \div (x^2-5)$

Solution: Set up the problem using a long division bar.

$$x^2-5\ \overline{)\ 2x^3-6x^2+5x-20}$$

How many times does x^2 go into $2x^3$? $2x$ times.

$$\begin{array}{r|l} & 2x \\ \hline x^2-5 & 2x^3 \ -6x^2+5x-20 \\ & \underline{2x^3-10x^2} \\ & \qquad\ 4x^2+5x-20 \end{array}$$

Multiply $2x$ by the divisor. *Subtract* that from the dividend.

Repeat the previous steps. Now, how many times does x^2 go into $4x^2$? 4 times.

$$\begin{array}{r|l} & 2x \quad +4 \\ \hline x^2-5 & 2x^3 \ -6x^2+5x-20 \\ & 2x^3-10x^2 \\ \hline & \qquad 4x^2+5x-20 \\ & \qquad 4x^2 \qquad -20 \\ \hline & \qquad\qquad 5x \end{array}$$

At this point, we are done. x^2 cannot go into $5x$ because it has a higher degree. Therefore, $5x$ is a remainder. The complete answer would be $2x+4+\frac{5x}{x^2-5}$.

Example B

$(3x^4+x^3-17x^2+19x-6) \div (x^2-2x+1)$. Determine if x^2-2x+1 goes evenly into $3x^4+x^3-17x^2+19x-6$. If so, try to factor the divisor and quotient further.

Solution: First, do the long division. If x^2-2x+1 goes in evenly, then the remainder will be zero.

$$\begin{array}{r|l} & 3x^2 \ +7x \quad -6 \\ \hline x^2-2x+1 & 3x^4 \ +x^3-17x^2+19x-6 \\ & 3x^4-6x^3 \quad 3x^2 \\ \hline & \quad 7x^3-20x^2+19x \\ & \quad 7x^3-14x^2 \ +7x \\ \hline & \qquad -6x^2+12x-6 \\ & \qquad -6x^2+12x-6 \\ \hline & \qquad\qquad 0 \end{array}$$

This means that x^2-2x+1 and $3x^2+7x-6$ both go evenly into $3x^4+x^3-17x^2+19x-6$. Let's see if we can factor either x^2-2x+1 or $3x^2+7x-6$ further.

$x^2-2x+1=(x-1)(x-1)$ and $3x^2+7x-6=(3x-2)(x+3)$.

Therefore, $3x^4+x^3-17x^2+19x-6=(x-1)(x-1)(x+3)(3x-2)$. You can multiply these to check the work. A binomial with a degree of one is a **factor** of a larger polynomial, $f(x)$, if it goes evenly into it. In this example, $(x-1)(x-1)(x+3)$ and $(3x-2)$ are all factors of $3x^4+x^3-17x^2+19x-6$. We can also say that 1, 1, -3, and $\frac{2}{3}$ are all solutions of $3x^4+x^3-17x^2+19x-6$.

Factor Theorem: A polynomial, $f(x)$, has a factor, $(x-k)$, if and only if $f(k)=0$.

In other words, if k is a **solution** or a **zero**, then the factor, $(x-k)$ divides evenly into $f(x)$.

Example C

Determine if 5 is a solution of $x^3+6x^2-8x+15$.

Solution: To see if 5 is a solution, we need to divide the factor into $x^3+6x^2-8x+15$. The factor that corresponds with 5 is $(x-5)$.

$$
\begin{array}{r|l}
 & x^2 + 11x \quad +5 \\ \hline
x-5 & x^3 + 6x^2 - 50x + 15 \\
 & \underline{x^3 - 5x^2} \\
 & \quad 11x^2 - 50x \\
 & \quad \underline{11x^2 - 55x} \\
 & \qquad\qquad 5x + 15 \\
 & \qquad\qquad \underline{5x - 25} \\
 & \qquad\qquad\quad 40
\end{array}
$$

Since there is a remainder, 5 is not a solution.

Guided Practice

1. $(5x^4 + 6x^3 - 12x^2 - 3) \div (x^2 + 3)$

2. Is $(x+4)$ a factor of $x^3 - 2x^2 - 51x - 108$? If so, find any other factors.

3. What are the real-number solutions to #2?

4. Determine if 6 is a solution to $2x^3 - 9x^2 - 12x - 24$.

Answers

1. Make sure to put a placeholder in for the $x-$term.

$$
\begin{array}{r|l}
 & 5x^2 \ +6x \quad -27 \\ \hline
x^2+3 & 5x^4 + 6x^3 - 12x^2 \ {\color{red}+0x} \ -3 \\
 & \underline{5x^4 \qquad\quad +15x^2} \\
 & \qquad 6x^3 - 27x^2 \ {\color{red}+0x} \\
 & \qquad \underline{6x^3 \qquad\quad +18x} \\
 & \qquad\qquad -27x^2 - 18x \ -3 \\
 & \qquad\qquad \underline{-27x^2 \qquad\quad -81} \\
 & \qquad\qquad\qquad -18x + 78
\end{array}
$$

The final answer is $5x^2 + 6x - 27 - \frac{18x-78}{x^2+3}$.

2. Divide $(x+4)$ into $x^3 - 2x^2 - 51x - 108$ and if the remainder is zero, it is a factor.

$$
\begin{array}{r|l}
 & x^2 \ -6x \ -27 \\ \hline
x+4 & x^3 - 2x^2 - 51x - 108 \\
 & \underline{x^3 + 4x^2} \\
 & \ -6x^2 - 51x \\
 & \ \underline{-6x^2 - 24x} \\
 & \qquad -27x - 108 \\
 & \qquad \underline{-27x - 108} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$x+4$ is a factor. Let's see if $x^2 - 6x - 27$ factors further. Yes, the factors of -27 that add up to -6 are -9 and 3. Therefore, the factors of $x^3 - 2x^2 - 51x - 108$ are $(x+4), (x-9)$, and $(x+3)$.

3. The solutions would be -4, 9, and 3; the opposite sign of each factor.

4. To see if 6 is a solution, we need to divide $(x-6)$ into $2x^3-9x^2-12x-24$.

$$\begin{array}{r|l}
 & 2x^2 \quad +3x \quad +6 \\
\hline
x-6\ ) & 2x^3 \quad -9x^2 -12x -24 \\
 & 2x^3 -12x^2 \\
\hline
 & \qquad 3x^2 -12x \\
 & \qquad 3x^2 -18x \\
\hline
 & \qquad\qquad 6x -24 \\
 & \qquad\qquad 6x -36 \\
\hline
 & \qquad\qquad\quad 12
\end{array}$$

Because the remainder is not zero, 6 is not a solution.

Vocabulary

Long division (of polynomial)
The process of dividing polynomials where the divisor has two or more terms.

Divisor
The polynomial that divides into another polynomial.

Dividend
The polynomial that the divisor goes into. The polynomial under the division bar.

Quotient
The answer to a division problem.

Problem Set

Divide the following polynomials using long division.

1. $(2x^3+5x^2-7x-6)\div(x+1)$
2. $(x^4-10x^3+15x-30)\div(x-5)$
3. $(2x^4-8x^3+4x^2-11x-1)\div(x^2-1)$
4. $(3x^3-4x^2+5x-2)\div(3x+2)$
5. $(3x^4-5x^3-21x^2-30x+8)\div(x-4)$
6. $(2x^5-5x^3+6x^2-15x+20)\div(2x^2+3)$

Determine all the real-number solutions to the following polynomials, given one zero.

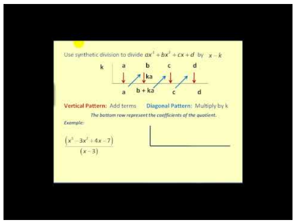
7. $x^3-9x^2+27x-15;-5$
8. $6x^3-37x^2+5x+6;6$
9. Find a polynomial with the zeros 4, -2, and $\frac{3}{2}$.
10. **Challenge** Find *two* polynomials with the zeros 8, 5, 1, and -1.

Synthetic Division

Objective

Use synthetic division as a short-cut and alternative to long division (in certain cases) and to find zeros.

Watch This



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James Sousa: Polynomial Division: Synthetic Division

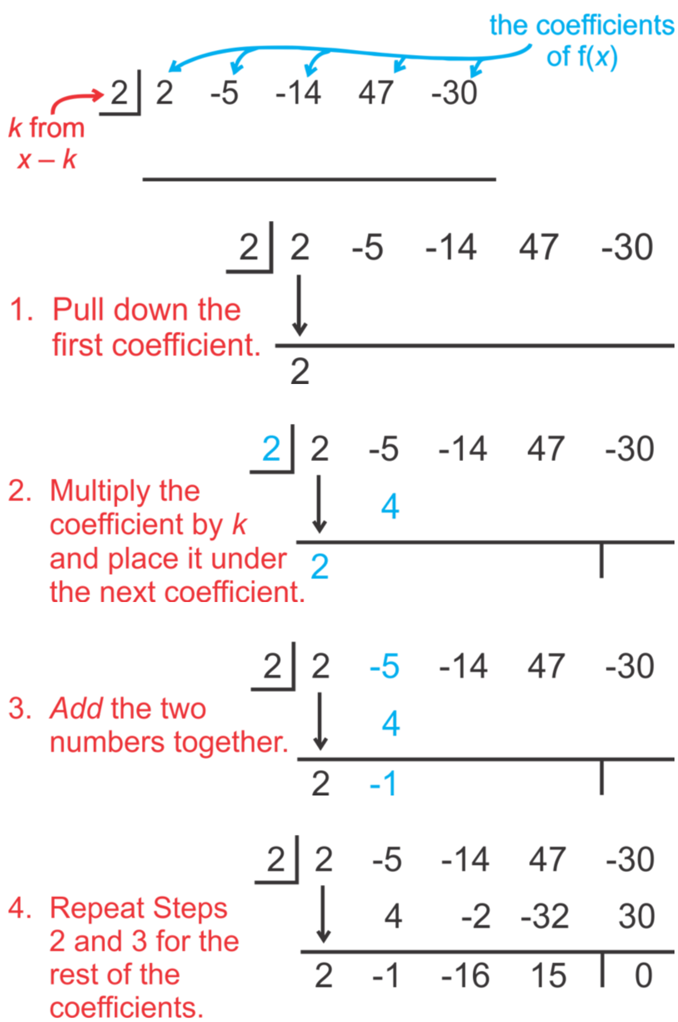
Guidance

Synthetic division is an alternative to long division from the previous concept. It can also be used to divide a polynomial by a possible factor, $x-k$. However, synthetic division cannot be used to divide larger polynomials, like quadratics, into another polynomial.

Example A

Divide $2x^4-5x^3-14x^2-37x-30$ by $x-2$.

Solution: Using synthetic division, the setup is as follows:



To "read" the answer, use the numbers as follows:

$$\begin{array}{c|ccccc} 2 & 2 & -5 & -14 & 47 & -30 \\ & \downarrow & 4 & -2 & -32 & 30 \\ \hline & 2 & -1 & -16 & 15 & \vert\; 0 \end{array}$$

coefficients of factored polynomial

last number is the remainder

Therefore, 2 is a solution, because the remainder is zero. The factored polynomial is $2x^3 - x^2 - 16x + 15$. Notice that when we synthetically divide by k, the "leftover" polynomial is one degree less than the original. We could also write $(x-2)(2x^3 - x^2 - 16x + 15) = 2x^4 - 5x^3 - 14x^2 + 47x - 30$.

Example B

Determine if 4 is a solution to $f(x) = 5x^3 + 6x^2 - 24x - 16$.

Using synthetic division, we have:

$$\begin{array}{c|cccc} 4 & 5 & 6 & -24 & -16 \\ & \downarrow & 20 & 104 & 320 \\ \hline & 5 & 26 & 80 & \vert\; 304 \end{array}$$

The remainder is 304, so 4 is not a solution. Notice if we substitute in $x = 4$, also written $f(4)$, we would have $f(4) = 5(4)^3 + 6(4)^2 - 24(4) - 16 = 304$. This leads us to the Remainder Theorem.

Remainder Theorem: If $f(k) = r$, then r is also the remainder when dividing by $(x-k)$.

This means that if you substitute in $x = k$ or divide by k, what comes out of $f(x)$ is the same. r is the remainder, but also is the corresponding $y-$value. Therefore, the point (k, r) would be on the graph of $f(x)$.

Example C

Determine if $(2x-5)$ is a factor of $4x^4 - 9x^2 - 100$.

Solution: If you use synthetic division, the factor is not in the form $(x-k)$. We need to solve the possible factor for zero to see what the possible solution would be. Therefore, we need to put $\frac{5}{2}$ up in the left-hand corner box. Also, not every term is represented in this polynomial. When this happens, you must put in zero placeholders. In this example, we need zeros for the x^3-term and the $x-$term.

$$\begin{array}{c|ccccc} 5/2 & 4 & 0 & -9 & 0 & -100 \\ & \downarrow & 10 & 25 & 40 & 100 \\ \hline & 4 & 10 & 16 & 40 & \vert\; 0 \end{array}$$

This means that $\frac{5}{2}$ is a zero and its corresponding binomial, $(2x-5)$, is a factor.

Guided Practice

1. Divide $x^3 + 9x^2 + 12x - 27$ by $(x+3)$. Write the resulting polynomial with the remainder (if there is one).

2. Divide $2x^4 - 11x^3 + 12x^2 + 9x - 2$ by $(2x+1)$. Write the resulting polynomial with the remainder (if there is one).

3. Is 6 a solution for $f(x) = x^3 - 8x^2 + 72$? If so, find the real-number zeros (solutions) of the resulting polynomial.

Answers

1. Using synthetic division, divide by -3.

$$\begin{array}{r|rrrr} -3 & 1 & 9 & 12 & -27 \\ & \downarrow & -3 & -18 & 18 \\ \hline & 1 & 6 & -6 & |\,-9 \end{array}$$

The answer is $x^2+6x-6-\frac{9}{x+3}$.

2. Using synthetic division, divide by $-\frac{1}{2}$.

$$\begin{array}{r|rrrrr} -1/2 & 2 & -11 & 12 & 9 & -2 \\ & \downarrow & -1 & 6 & -9 & 0 \\ \hline & 2 & -12 & 18 & 0 & |\,-2 \end{array}$$

The answer is $2x^3-12x^2+18x-\frac{2}{2x+1}$.

3. Put a zero placeholder for the $x-$term. Divide by 6.

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 0 & 72 \\ & \downarrow & 6 & -12 & -72 \\ \hline & 1 & -2 & -12 & |\,0 \end{array}$$

The resulting polynomial is $x^2-2x-12$. While this quadratic does not factor, we can use the Quadratic Formula to find the other roots.

$$x=\frac{2\pm\sqrt{2^2-4(1)(-12)}}{2}=\frac{2\pm\sqrt{4+48}}{2}=\frac{2\pm 2\sqrt{13}}{2}=1\pm\sqrt{13}$$

The solutions to this polynomial are 6, $1+\sqrt{13}\approx 4.61$ and $1-\sqrt{13}\approx -2.61$.

Vocabulary

Synthetic Division

An alternative to long division for dividing $f(x)$ by k where only the coefficients of $f(x)$ are used.

Remainder Theorem

If $f(k)=r$, then r is also the remainder when dividing by $(x-k)$.

Problem Set

Use synthetic division to divide the following polynomials. Write out the remaining polynomial.

1. $(x^3+6x^2+7x+10)\div(x+2)$

2. $(4x^3 - 15x^2 - 120x - 128) \div (x - 8)$
3. $(4x^2 - 5) \div (2x + 1)$
4. $(2x^4 - 15x^3 - 30x^2 - 20x + 42) \div (x + 9)$
5. $(x^3 - 3x^2 - 11x + 5) \div (x - 5)$
6. $(3x^5 + 4x^3 - x - 2) \div (x - 1)$
7. Which of the division problems above generate no remainder? What does that mean?
8. What is the difference between a zero and a factor?
9. Find $f(-2)$ if $f(x) = 2x^4 - 5x^3 - 10x^2 + 21x - 4$.
10. Now, divide $2x^4 - 5x^3 - 10x^2 + 21x - 4$ by $(x + 2)$ synthetically. What do you notice?

Find all real zeros of the following polynomials, given one zero.

11. $12x^3 + 76x^2 + 107x - 20; -4$
12. $x^3 - 5x^2 - 2x + 10$

3.19 Finding all Solutions of Polynomial Functions

Objective

To find all the solutions to higher-degree polynomials using synthetic division, factoring, and the Quadratic Formula.

Review Queue

Factor the following polynomials completely.

1. $8x^3 - 27$
2. $3x^2 - 20x - 7$
3. $6x^4 + 17x^2 + 7$
4. $x^3 - 9x^2 - 4x + 36$
5. $x^4 + 2x^3 + x + 2$
6. $162x^5 - 512x$

Finding Rational and Real Zeros

Objective

To find all the rational and real-number zeros of a higher-degree polynomial.

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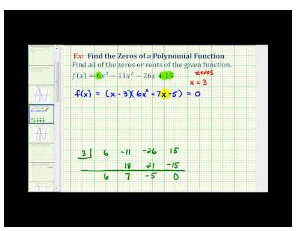

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James Sousa: Ex 2: Find the Zeros of a Polynomial Function - Real Rational Zeros

Guidance

Recall from the Quadratic Functions chapter, that every quadratic equation has two solutions. The degree of a quadratic equation is 2, thus leading us towards the notion that it has 2 solutions. *The degree will always tell us the maximum number of solutions a polynomial has.* Quadratic equations also have a few different possibilities for solutions; two real-number solutions (parabola passes through the $x-$axis twice), one real-number solution (where the solution is the vertex, called a repeated root), or two imaginary solutions (where the graph does not touch the $x-$axis at all).

When it comes to solutions for polynomials, all these options are possibilities. There can be rational, irrational and imaginary solutions. *Irrational and imaginary solutions will always come in pairs.* This is due to the fact that to find these types of solutions, you must use the Quadratic Formula and the $\pm$ sign will give two solutions. In this concept we will only address real-number solutions.

Now, you might be wondering, how do we find all these solutions? One way is to use the Rational Root Theorem.

Rational Root Theorem: For a polynomial, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \cdots a_0$ are integers, the rational roots can be determined from the factors of a_n and a_0. More specifically, if p is a factor of a_0 and q is a factor of a_n, then all the rational factors will have the form $\pm \frac{p}{q}$.

In other words, the factors of the constant divided by the factors of the leading coefficient will yield all the possible rational solutions to $f(x)$.

Example A

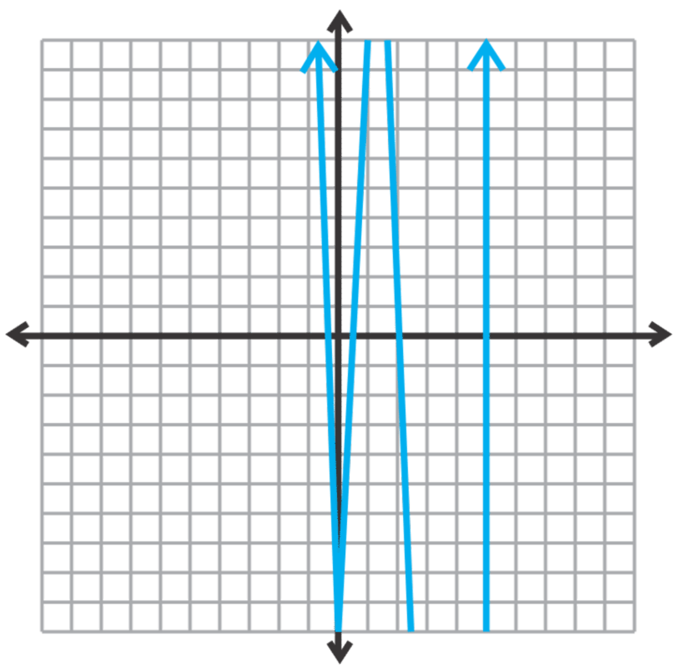
Find all the possible rational solutions to $f(x) = 6x^4 - 43x^3 + 66x^2 - 3x - 10$.

Solution: All the possible factors of 10 are 1, 2, 5, and 10. All the possible factors of 6 are 1, 2, 3, and 6. The possible combinations are $\frac{\pm 1.\pm 2,\pm 5,\pm 10}{\pm 1,\pm 2,\pm 3,\pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 2, \pm \frac{2}{3}, \pm 5, \pm \frac{5}{2}, \pm \frac{5}{3}, \pm \frac{5}{6}, \pm 10, \pm \frac{10}{3}$. Therefore, there are 24 possibilities.

Example B

Find the rational solutions to $f(x) = 6x^4 - 43x^3 + 66x^2 - 3x - 10$.

Solution: Before the days of graphing calculators, you would have to test all 24 possible solutions to find the correct solutions. Now, we can graph the function and eliminate any possibilities that seem unreasonable. Because the degree of the function is 4, there will be 4 solutions. Here is the graph:



Looking back at Example A, the reasonable solutions appear to be: $5, 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}$, or $\pm \frac{5}{6}$. By just looking at the graph, the solutions between -1 and 1 are difficult to see. This is why we have listed all the solutions between -1 and 1 to test. Let's test 5 and 2 using synthetic division.

$$\begin{array}{r|rrrrr} 5 & 6 & -43 & 66 & -3 & -10 \\ & \downarrow & 30 & -65 & 5 & 10 \\ \hline & 6 & -13 & 1 & 2 & \vert\ 0 \end{array}$$

The remainder is zero, like we thought.

Now, rather than starting over with the division by 2, *continue* with the leftover polynomial.

5	6	-43	66	-3	-10	
	↓	30	-65	5	10	
2	6	-13	1	2	0	
	↓	12	-2	-2		
	6	-1	-1	0		

Again, the remainder is zero. Both 5 and 2 are zeros.

To find the last two zeros, we can test all the fractions above using synthetic division. OR, we can factor this leftover polynomial. Because we started with a polynomial of degree 4, this leftover polynomial is a quadratic. It is $6x^2 - x - 1$. $ac = -6$ and the factors of -1 that add up to -6 are -3 and 2. Expand the $x-$term and factor by grouping.

$$\underbrace{6x^2 - 3x} + \underbrace{2x - 1}$$
$$3x(2x-1) + 1(2x-1)$$
$$(2x-1)(3x+1)$$

Setting these two factors equal to zero, we have $x = \frac{1}{2}$ and $-\frac{1}{3}$. Therefore, the solutions to this polynomial are 5, 2, $\frac{1}{2}$ and $-\frac{1}{3}$.

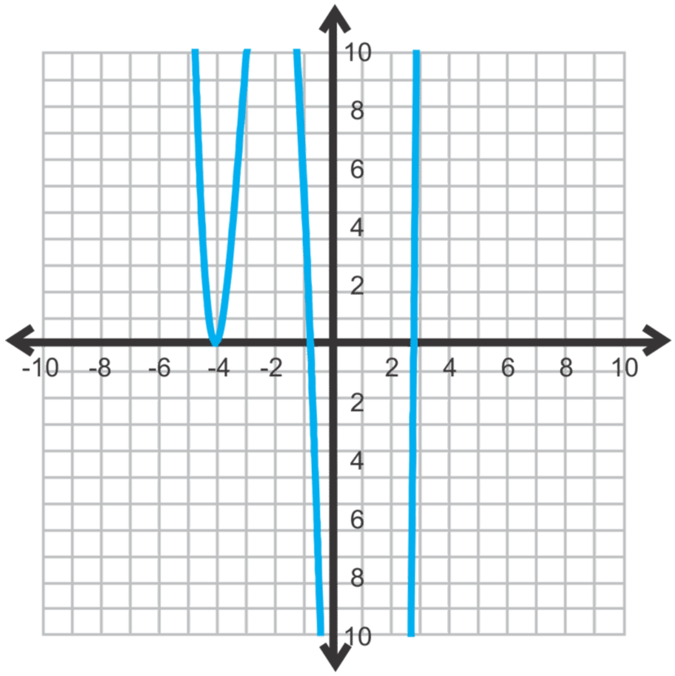
Check your answer: To check your work, you can multiply the factors together to see if you get the original polynomial.

$$\underbrace{(2x-1)(3x+1)}\underbrace{(x-5)(x-2)}$$
$$(6x^2 - x - 1)(x^2 - 7x + 10)$$
$$6x^4 - 43x^3 + 66x^2 - 3x - 10$$

Example C

Find all the real solutions to $f(x) = x^4 + 6x^3 - 2x^2 - 48x - 32$.

Solution: First, sketch a graph.



Now, use the Rational Root Theorem to determine all possible rational roots.

$$\frac{\text{factors of -32}}{\text{factors of 1}} = \frac{\pm 32, \pm 16, \pm 8, \pm 4, \pm 2, \pm 1}{\pm 1}$$

Using the graph, it looks like -4 is the only possible rational solution. Also, notice that the graph touches at -4 and *does not pass through the x−axis.* That means that this solution is a repeated root. Let's do synthetic division.

$$\begin{array}{r|rrrr|r}
-4 & 1 & 6 & -2 & -48 & -32 \\
 & \downarrow & -4 & -8 & 40 & -32 \\
\hline
-4 & 1 & 2 & -10 & -8 & 0 \\
 & \downarrow & -4 & 8 & 8 & \\
\hline
 & 1 & -2 & -2 & 0 &
\end{array}$$

Because the root is repeated, we did synthetic division twice. At the end of the synthetic division, the leftover polynomial is $x^2 - 2x - 2$ which is not factorable. Therefore, to find the last two real solutions, we must do the Quadratic Formula.

$$\begin{aligned}
x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \approx 2.73, -0.73
\end{aligned}$$

The roots, or zeros, of $f(x) = x^4 + 6x^3 - 2x^2 - 48x - 32$ are -4 (twice), 2.73, and -0.73. Looking back at the graph, we see that this is where the function crosses the $x-$axis. The graph is always a good way to double-check your work.

Guided Practice

Find all the real solutions of the following functions.

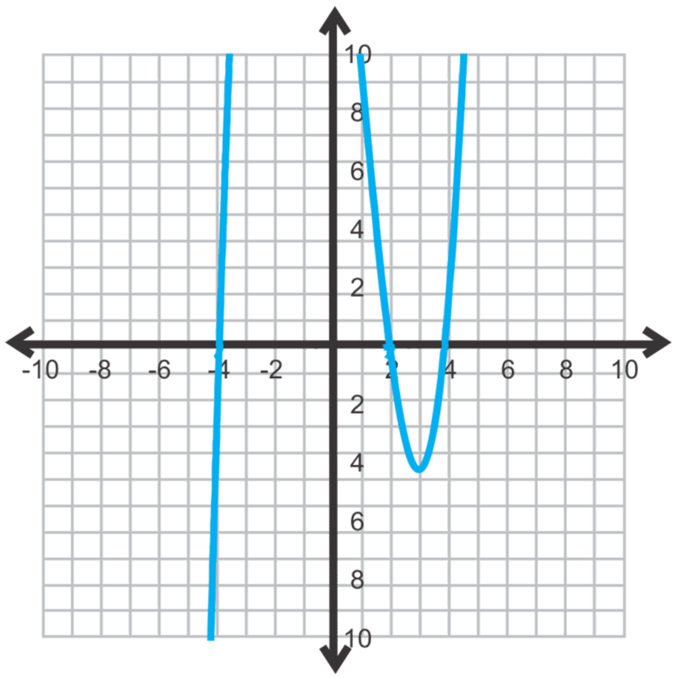
1. $f(x) = x^3 - 2x^2 - 15x + 30$

2. $f(x) = 6x^3 + 19x^2 + 11x - 6$

3. $f(x) = x^5 - 4x^4 - 18x^3 + 38x^2 - 11x - 6$

Answers

1. Using the Rational Root Theorem, the possible rational roots are: $\pm 30, \pm 15, \pm 10, \pm 6, \pm 5, \pm 3, \pm 2, \pm 1$. Now, graph the function.



By looking at the graph, the only reasonable rational root is 2. We can rule out 4 and -4 because they are not included in the list of rational roots. Therefore, these two roots will be irrational. Do the synthetic division for 2.

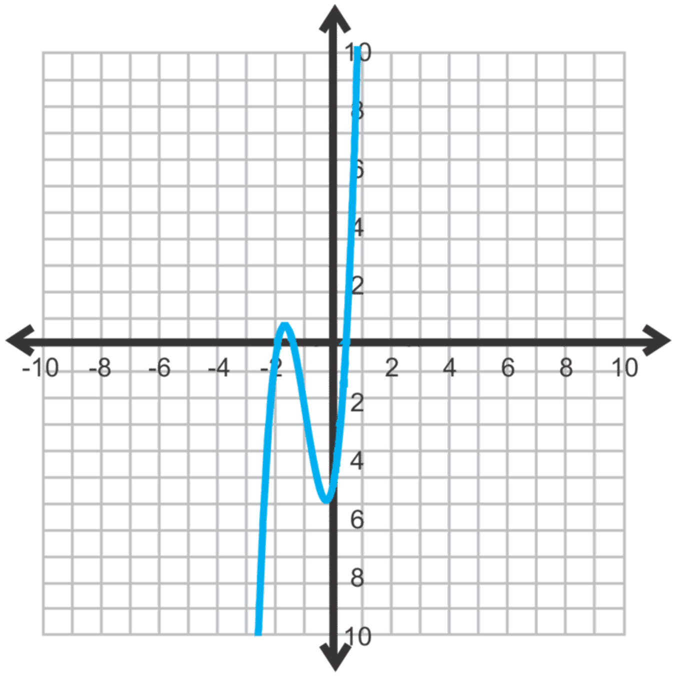
$$\begin{array}{r|rrrr} 2 & 1 & -2 & -15 & 30 \\ & \downarrow & 2 & 0 & -30 \\ \hline & 1 & 0 & -15 & 0 \end{array}$$

The leftover polynomial is $x^2 - 15 = 0$. This polynomial can be solved by using square roots.

$$\begin{aligned} x^2 - 15 &= 0 \\ x^2 &= 15 \\ x &= \pm\sqrt{15} \approx \pm 3.87 \end{aligned}$$

*Instead of using the Rational Root Theorem and synthetic division, this problem could have also been solved using factoring by grouping.

2. Using the Rational Root Theorem, the possible rational roots are: $\pm 6, \pm 3, \pm 2, \pm \frac{3}{2}, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}$.



By looking at the graph, the reasonable rational roots are $-2, -\frac{3}{2}, \frac{1}{3}$ or $\frac{1}{6}$. The rational answers are difficult to see because they do not cross exactly the $x-$axis on an integer. Therefore, we will do the synthetic division for -2 first.

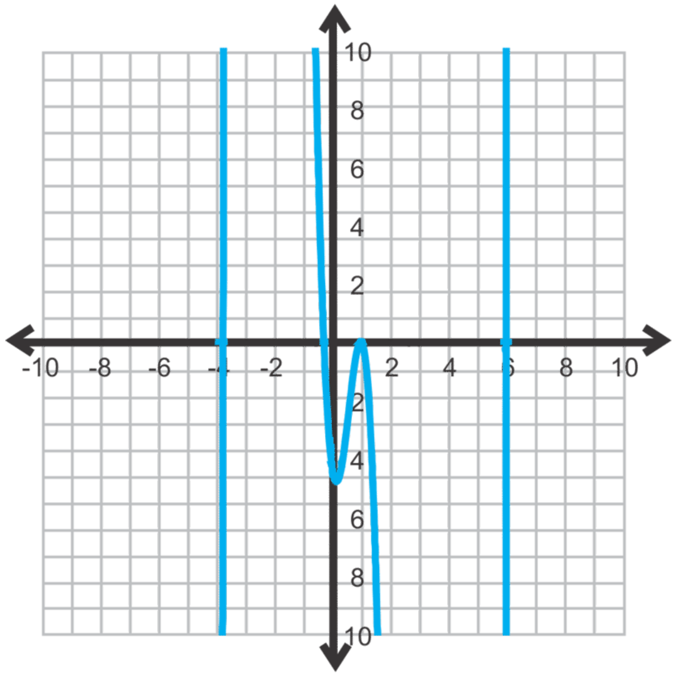
$$\begin{array}{r|rrrr}
-2 & 6 & 19 & 11 & -6 \\
 & \downarrow & -12 & -14 & 6 \\
\hline
 & 6 & 7 & -3 & \vert\ 0
\end{array}$$

The leftover polynomial is $6x^2 + 7x - 3$, which is factorable. You can decide if you would like to factor this polynomial, use the Quadratic Formula, or test the rational possibilities from above. Let's factor.

$$\begin{aligned}
&6x^2 + 7x - 3 \\
&6x^2 + 9x - 2x - 3 \\
&3x(2x+3) - 1(2x+3) \\
&(3x-1)(2x+3)
\end{aligned}$$

From these factors, the rational solutions are $\frac{1}{3}$ and $-\frac{3}{2}$.

3. Using the Rational Root Theorem, the possible rational roots are: $\pm 6, \pm 3, \pm 2, \pm 1$.



From the graph, the possible roots are 6 and 1. It looks like 1 is a double root because the function reaches the $x-$axis at 1, but does not pass through it. Do synthetic division with 6, 1, and 1 again.

6	1	-4	-18	38	-11	-6
	↓	6	12	-36	12	6
1	1	2	-6	2	1	0
	↓	1	3	-3	1	
1	1	3	-3	-1	0	
	↓	1	4	1		
	1	4	1	0		

The leftover polynomial is x^2+4x+1. This is not a factorable polynomial, so use the Quadratic Formula to find the last two roots.

$$x=\frac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}=\frac{-4\pm\sqrt{12}}{2}=\frac{-4\pm 2\sqrt{3}}{2}=-2\pm\sqrt{3}\approx -0.27,-3.73$$

Vocabulary

Rational Root Theorem: For a polynomial, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $a_n,a_{n-1},\cdots a_0$ are integers, the rational roots can be determined from the factors of a_n and a_0. More specifically, if p is a factor of a_0 and q is a factor of a_n, then all the rational factors will have the form $\pm\frac{p}{q}$.

Problem Set

Find all the possible rational solutions for the following polynomials. Use the Rational Root Theorem.

1. $f(x)=x^3+6x^2-18x+20$
2. $f(x)=4x^4+x^2-15$

3. $f(x) = -2x^3 + 7x^2 - x + 8$
4. $f(x) = x^4 - 3x^3 - 4x^2 + 15x + 9$

Find all the real-number solutions for each function below. Use any method you like.

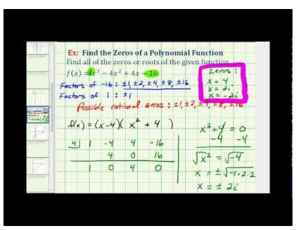
5. $f(x) = 6x^3 - 17x^2 + 11x - 2$
6. $f(x) = x^4 + 7x^3 + 6x^2 - 32x - 32$
7. $f(x) = 16x^3 + 40x^2 - 25x - 3$
8. $f(x) = 2x^3 - 9x^2 + 21x - 18$
9. $f(x) = 4x^3 - 16x^2 + 39x - 295$
10. $f(x) = 18x^4 + 3x^3 - 17x^2 + 17x - 55$
11. $f(x) = x^5 + 7x^4 - 3x^3 - 65x^2 - 8x - 156$
12. Solve $f(x) = 3x^4 - x^2 - 14$ by factoring. How many real solutions does this function have? What type of solution(s) could the others be?

Finding Imaginary Solutions

Objective

To find all the solutions to any polynomial, including imaginary solutions.

Watch This



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/60119

James Sousa: Ex 4: Find the Zeros of a Polynomial Function with Imaginary Zeros

Guidance

In #12 from the previous problem set, there are two imaginary solutions. *Imaginary solutions always come in pairs.* To find the imaginary solutions to a function, use the Quadratic Formula. If you need a little review on imaginary numbers and how to solve a quadratic equation with complex solutions see the *Quadratic Equations* chapter.

Example A

Solve $f(x) = 3x^4 - x^2 - 14$. (#12 from the previous problem set.)

Solution: First, this quartic function can be factored just like a quadratic equation. See the *Factoring Polynomials in Quadratic Form* concept from this chapter for review.

$$\begin{aligned}
f(x) &= 3x^4 - x^2 - 14 \\
0 &= 3x^4 - 7x^2 + 6x^2 - 14 \\
0 &= x^2(3x^2 - 7) + 2(3x^2 - 7) \\
0 &= (x^2 + 2)(3x^2 - 7)
\end{aligned}$$

Now, because neither factor can be factored further and there is no $x-$term, we can set each equal to zero and solve.

$$\begin{aligned} x^2+2&=0 \\ x^2&=-2 \\ x&=\pm\sqrt{-2} \text{ or } \pm i\sqrt{2} \end{aligned} \qquad \textit{and} \qquad \begin{aligned} 3x^2-7&=0 \\ 3x^2&=7 \\ x^2&=\frac{7}{3} \\ x&=\pm\sqrt{\frac{7}{3}} \text{ or } \pm\frac{\sqrt{21}}{3} \end{aligned}$$

Including the imaginary solutions, there are four, which is what we would expect because the degree of this function is four.

Example B

Find all the solutions of the function $g(x) = x^4 + 21x^2 + 90$.

Solution: When graphed, this function does not touch the $x-$axis. Therefore, all the solutions are imaginary. To solve, this function can be factored like a quadratic equation. The factors of 90 that add up to 21 are 6 and 15.

$$\begin{aligned} g(x) &= x^4+21x^2+90 \\ 0 &= (x^2+6)(x^2+15) \end{aligned}$$

Now, set each factor equal to zero and solve.

$$\begin{aligned} x^2+6&=0 \\ x^2&=-6 \\ x&=\pm i\sqrt{6} \end{aligned} \qquad \textit{and} \qquad \begin{aligned} x^2+15&=0 \\ x^2&=-15 \\ x&=\pm i\sqrt{15} \end{aligned}$$

Example C

Find the function that has the solution 3, -2, and $4+i$.

Solution: Notice that one of the given solutions is imaginary. Imaginary solutions always come in pairs, so $4-i$ is also a factor, they are **complex conjugates.** Now, translate each solution into a factor and multiply them all together.

Remember!
$i^2 = -1$

$$(x-3)(x+2)(x-(4+i))(x-(4-i))$$
$$(x^2+2x-3x-6)(x^2-4x+\cancel{ix}-4x+16-\cancel{4i}-\cancel{ix}+\cancel{4i}-i^2)$$
$$(x^2-x-6)(x^2-8x+17)$$
$$(x^4-8x^3+17x^2-x^3+8x^2-17x-6x^2+48x-102)$$
$$(x^4-9x^3+19x^2+31x-102)$$

Any multiple of this function would also have these roots. For example, $2x^4-18x^3+38x^2+62x-204$ would have these roots as well.

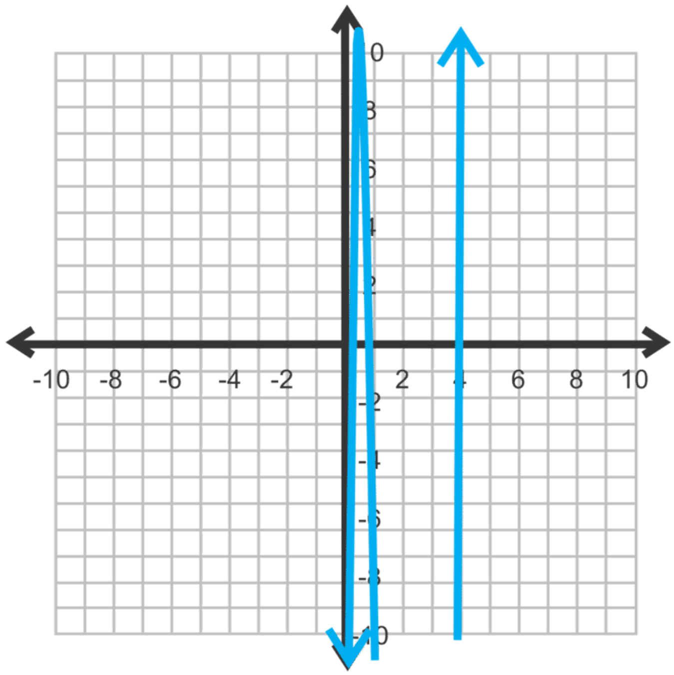
Guided Practice

Find all the solutions to the following functions.

1. $f(x) = 25x^3 - 120x^2 + 81x - 4$
2. $f(x) = 4x^4 + 35x^2 - 9$
3. Find the equation of a function with roots 4, $\sqrt{2}$ and $1 - i$.

Answers

1. First, graph the function.



Using the Rational Root Theorem, the possible realistic zeros could be $\frac{1}{25}$, 1 or 4. Let's try these three possibilities using synthetic division.

$$\begin{array}{c|cccc} \frac{1}{25} & 25 & -120 & 81 & -4 \\ & \downarrow & 1 & -118.96 & -1.5184 \\ \hline & 25 & -119 & -37.96 & -5.5184 \end{array} \qquad \begin{array}{c|cccc} 1 & 25 & -120 & 81 & -4 \\ & \downarrow & 25 & -95 & -14 \\ \hline & 25 & -95 & -14 & -18 \end{array}$$

$$\begin{array}{c|cccc} 4 & 25 & -120 & 81 & -4 \\ & \downarrow & 100 & -80 & 4 \\ \hline & 25 & -20 & 1 & 0 \end{array}$$

Of these three possibilities, only 4 is a zero. The leftover polynomial, $25x^2 - 20x + 1$ is not factorable, so we need to do the Quadratic Formula to find the last two zeros.

$$\begin{aligned} x &= \frac{20 \pm \sqrt{20^2 - 4(25)(1)}}{2(25)} \\ &= \frac{20 \pm \sqrt{400 - 100}}{50} \\ &= \frac{20 \pm 10\sqrt{3}}{50} \ \mathit{or} \ \frac{2 \pm \sqrt{3}}{5} \approx 0.746 \ \mathit{and} \ 0.054 \end{aligned}$$

*__Helpful Hint:__ Always find the decimal values of each zero to make sure they match up with the graph.

2. $f(x) = 4x^4 + 35x^2 - 9$ is factorable. $ac = -36$.

$$\begin{aligned}
&4x^4 + 35x^2 - 9\\
&4x^4 + 36x^2 - x^2 - 9\\
&4x^2(x^2+9) - 1(x^2+9)\\
&(x^2+9)(4x^2-1)
\end{aligned}$$

Setting each factor equal to zero, we have:

$$\begin{array}{lcl}
 & & 4x^2 - 1 = 0\\
x^2 + 9 = 0 & & 4x^2 = 1\\
x^2 = -9 & or & x^2 = \frac{1}{4}\\
x = \pm 3i & & x = \pm\frac{1}{2}
\end{array}$$

*This problem could have also been by using the same method from #1.

3. Recall that irrational and imaginary roots come in pairs. Therefore, all the roots are 4, $\sqrt{2}, -\sqrt{2}, 1+i, 1-i$. Multiply all 5 roots together.

$$\begin{aligned}
&(x-4)(x-\sqrt{2})(x+\sqrt{2})(x-(1+i))(x-(1-i))\\
&(x-4)(x^2-2)(x^2-2x+2)\\
&(x^3-4x^2-2x+8)(x^2-2x+2)\\
&x^5-6x^4+8x^3-4x^2-20x+16
\end{aligned}$$

Problem Set

Find all solutions to the following functions. Use any method.

1. $f(x) = x^4 + x^3 - 12x^2 - 10x + 20$
2. $f(x) = 4x^3 - 20x^2 - 3x + 15$
3. $f(x) = 2x^4 - 7x^2 - 30$
4. $f(x) = x^3 + 5x^2 + 12x + 18$
5. $f(x) = 4x^4 + 4x^3 - 22x^2 - 8x + 40$
6. $f(x) = 3x^4 + 4x^2 - 15$
7. $f(x) = 2x^3 - 6x^2 + 9x - 27$
8. $f(x) = 6x^4 - 7x^3 - 280x^2 - 419x + 280$
9. $f(x) = 9x^4 + 6x^3 - 28x^2 + 2x + 11$
10. $f(x) = 2x^5 - 19x^4 + 30x^3 + 97x^2 - 20x + 150$

Find a function with the following roots.

11. $4, i$
12. $-3, -2i$

13. $\sqrt{5}, -1+i$
14. $2, \frac{1}{3}, 4-\sqrt{2}$
15. **Writing** Write down the steps you use to find all the zeros of a polynomial function.
16. **Writing** Why do imaginary and irrational roots always come in pairs?
17. **Challenge** Find all the solutions to $f(x) = x^5 + x^3 + 8x^2 + 8$.

3.20 Using Rational Exponents and nth Roots

Objective

To introduce rational exponents and nth roots. Then, we will apply the properties of exponents to rational functions and nth roots.

Review Queue

Evaluate each expression.

1. $(5x)^2$
2. $\frac{4x^5y^7}{12xy^9}$
3. $\sqrt{81x^2y^5}$
4. $\sqrt{\frac{75}{96}}$

Defining nth Roots

Objective

To define and use n^{th} roots.

Guidance

So far, we have seen exponents with integers and the square root. In this concept, we will link roots and exponents. First, let's define additional roots. Just like the square and the square root are inverses of each other, the inverse of a cube is the cubed root. The inverse of the fourth power is the fourth root.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3, \ \sqrt[5]{32} = \sqrt[5]{2^5} = 2$$

The n^{th} **root** of a number, x^n, is x, $\sqrt[n]{x^n} = x$. And, just like simplifying square roots, we can simplify n^{th} roots.

Example A

Find $\sqrt[6]{729}$.

Solution: To simplify a number to the sixth root, there must be 6 of the same factor to pull out of the root.

$729 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$

Therefore, $\sqrt[6]{729} = \sqrt[6]{3^6} = 3$. The sixth root and the sixth power cancel each other out. We say that 3 is the sixth root of 729.

From this example, we can see that it does not matter where the exponent is placed, it will always cancel out with the root.

$$\sqrt[6]{3^6} = \sqrt[6]{3}^6 \ or \ \left(\sqrt[6]{3}\right)^6$$
$$\sqrt[6]{729} = (1.2009\ldots)^6$$
$$3 = 3$$

So, it does not matter if you evaluate the root first or the exponent.

The n^{th} Root Theorem: For any real number a, root n, and exponent m, the following is always true: $\sqrt[n]{a^m} = \sqrt[n]{a}^m = \left(\sqrt[n]{a}\right)^m$.

Example B

Evaluate without a calculator:

a) $\sqrt[5]{32^3}$

b) $\sqrt{16}^3$

Solution:

a) If you solve this problem as written, you would first find 32^3 and then apply the 5^{th} root.

$\sqrt[5]{32^3} = \sqrt[5]{38768} = 8$

However, this would be very difficult to do without a calculator. This is an example where it would be easier to apply the root and then the exponent. Let's rewrite the expression and solve.

$\sqrt[5]{32}^3 = 2^3 = 8$

b) This problem does not need to be rewritten. $\sqrt{16} = 4$ and then $4^3 = 64$.

Example C

Simplify:

a) $\sqrt[4]{64}$

b) $\sqrt[3]{\frac{54x^3}{125y^5}}$

Solution:

a) To simplify the fourth root of a number, there must be 4 of the same factor to pull it out of the root. Let's write the prime factorization of 64 and simplify.

$\sqrt[4]{64} = \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = 2\sqrt[4]{4}$

Notice that there are 6 2's in 64. We can pull out 4 of them and 2 2's are left under the radical.

b) Just like simplifying fractions with square roots, we can separate the numerator and denominator.

$$\sqrt[3]{\frac{54x^3}{125y^5}} = \frac{\sqrt[3]{54x^3}}{\sqrt[3]{125y^5}} = \frac{\sqrt[3]{2 \cdot 3 \cdot 3 \cdot 3 \cdot x^3}}{\sqrt[3]{5 \cdot 5 \cdot 5 \cdot y^3 \cdot y^2}} = \frac{3x\sqrt[3]{2}}{5y\sqrt[3]{y^2}}$$

Notice that because the x is cubed, the cube and cubed root cancel each other out. With the y-term, there were five, so three cancel out with the root, but two are still left under radical.

Guided Practice

Simplify each expression below, without a calculator.

1. $\sqrt[4]{625z^8}$
2. $\sqrt[7]{32x^5y}$
3. $\sqrt[5]{9216}$
4. $\sqrt[3]{\frac{40}{175}}$

Answers

1. First, you can separate this number into two different roots, $\sqrt[4]{625} \cdot \sqrt[4]{z^8}$. Now, simplify each root.

$\sqrt[4]{625} \cdot \sqrt[4]{z^8} = \sqrt[4]{5^4} \cdot \sqrt[4]{z^4 \cdot z^4} = 5z^2$

When looking at the z^8, think about how many z^4 you can even pull out of the fourth root. The answer is 2, or a z^2, outside of the radical.

2. $32 = 2^5$, which means there are not 7 2's that can be pulled out of the radical. Same with the x^5 and the y. Therefore, you cannot simplify the expression any further.

3. Write out 9216 in the prime factorization and place factors into groups of 5.

$$\begin{aligned}\sqrt[5]{9216} &= \sqrt[5]{\boxed{2\cdot 2\cdot 2\cdot 2\cdot 2}\cdot\boxed{2\cdot 2\cdot 2\cdot 2\cdot 2}\cdot 3\cdot 3} \\ &= \sqrt[5]{2^5\cdot 2^5\cdot 3^2} \\ &= 2\cdot 2\sqrt[5]{3^2} \\ &= 4\sqrt[5]{9}\end{aligned}$$

4. Reduce the fraction, separate the numerator and denominator and simplify.

$\sqrt[3]{\frac{40}{175}} = \sqrt[3]{\frac{8}{35}} = \frac{\sqrt[3]{2^3}}{\sqrt[3]{35}} = \frac{2}{\sqrt[3]{35}} \cdot \frac{\sqrt[3]{35^2}}{\sqrt[3]{35^2}} = \frac{2\sqrt[3]{1225}}{35}$

In the red step, we rationalized the denominator by multiplying the top and bottom by $\sqrt[3]{35^2}$, so that the denominator would be $\sqrt[3]{35^3}$ or just 35. Be careful when rationalizing the denominator with higher roots!

Vocabulary

n^{th} **root**
The n^{th} root of a number, x^n, is x, $\sqrt[n]{x^n} = x$.

Problem Set

Reduce the following radical expressions.

1. $\sqrt[3]{81}$
2. $\sqrt[5]{128}$
3. $\sqrt{\frac{25}{8}}^{4}$
4. $\sqrt[6]{64}^{5}$
5. $\sqrt[3]{\frac{8}{81}}^{2}$
6. $\sqrt[4]{\frac{243}{16}}$
7. $\sqrt[3]{24x^5}$
8. $\sqrt[4]{48x^7y^{13}}$
9. $\sqrt[5]{\frac{160x^8}{y^7}}$
10. $\sqrt[3]{1000x^6}^{2}$
11. $\sqrt[4]{\frac{162x^5}{y^3z^{10}}}$
12. $\sqrt{40x^3y^4}^{3}$

Rational Exponents and Roots

Objective

To introduce rational exponents and relate them to n^{th} roots.

Guidance

Now that you are familiar with nth roots, we will convert them into exponents. Let's look at the square root and see if we can use the properties of exponents to determine what exponential number it is equivalent to.

Investigation: Writing the Square Root as an Exponent

TABLE 3.8:

1. Evaluate $\left(\sqrt{x}\right)^2$. What happens?	The $\sqrt{\ }$ and the ^{2} cancel each other out, $\left(\sqrt{x^2}\right) = x$.
2. Recall that when a power is raised to another power, we multiply the exponents. Therefore, we can rewrite the exponents and root as an equation, $n \cdot 2 = 1$. Solve for n.	$\frac{n \cdot \cancel{2}}{\cancel{2}} = \frac{1}{2}$ $n = \frac{1}{2}$
3. From #2, we can conclude that $\sqrt{\ } = \frac{1}{2}$.	$\left(\sqrt{x}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 = x^{\left(\frac{1}{2}\right) \cdot 2} = x^1 = x$

From this investigation, we see that $\sqrt{x} = x^{\frac{1}{2}}$. We can extend this idea to the other roots as well; $\sqrt[3]{x} = x^{\frac{1}{3}} = \sqrt[4]{x} = x^{\frac{1}{4}}, \ldots \sqrt[n]{x} = x^{\frac{1}{n}}$.

Example A

Find $256^{\frac{1}{4}}$.

Solution: Rewrite this expression in terms of roots. A number to the one-fourth power is the same as the fourth root.

$256^{\frac{1}{4}} = \sqrt[4]{256} = \sqrt[4]{4^4} = 4$

Therefore, $256^{\frac{1}{4}} = 4$.

Example B

Find $49^{\frac{3}{2}}$.

Solution: This problem is the same as the ones in the previous concept. However, now, the root is written in the exponent. Rewrite the problem.

$49^{\frac{3}{2}} = \left(49^3\right)^{\frac{1}{2}} = \sqrt{49^3}$ or $\left(\sqrt{49}\right)^3$

From the previous concept, we know that it is easier to evaluate the second option above. $\left(\sqrt{49}\right)^3 = 7^3 = 343$.

The Rational Exponent Theorem: For any real number a, root n, and exponent m, the following is always true: $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

Example C

Find $5^{\frac{2}{3}}$ using a calculator. Round your answer to the nearest hundredth.

Solution: To type this into a calculator, the keystrokes would probably look like: $5^{\frac{2}{3}}$. The "^" symbol is used to indicate a power. Anything in parenthesis after the "^" would be in the exponent. Evaluating this, we have 2.924017738..., or just 2.92.

Other calculators might have a x^y button. This button has the same purpose as the ^ and would be used in the exact same way.

Guided Practice

1. Rewrite $\sqrt[7]{12}$ using rational exponents. Then, use a calculator to find the answer.

2. Rewrite $845^{\frac{4}{9}}$ using roots. Then, use a calculator to find the answer.

Evaluate without a calculator.

3. $125^{\frac{4}{3}}$

4. $256^{\frac{5}{8}}$

5. $\sqrt{81^{\frac{1}{2}}}$

Answers

1. Using rational exponents, the 7^{th} root becomes the $\frac{1}{7}$ power;$12^{\frac{1}{7}} = 1.426$.

2. Using roots, the 9 in the denominator of the exponent is the root; $\sqrt[9]{845^4} = 19.99$. To enter this into a calculator, you can use the rational exponents. If you have a TI-83 or 84, press **MATH** and select **5**: $\sqrt[x]{\ }$. On the screen, you should type 9 $\sqrt[x]{\ }$ 845^4 to get the correct answer. You can also enter 845^$\left(\frac{4}{9}\right)$ and get the exact same answer

3. $125^{\frac{4}{3}} = \left(\sqrt[3]{125}\right)^4 = 5^4 = 625$

4. $256^{\frac{5}{8}} = \left(\sqrt[8]{256}\right)^5 = 2^5 = 32$

5. $\sqrt{81^{\frac{1}{2}}} = \sqrt{\sqrt{81}} = \sqrt{9} = 3$

Vocabulary

Rational Exponent
An exponent that can be written as a fraction. For any n^{th} root, the n of the root can be written in the denominator of a rational exponent. $\sqrt[n]{x} = x^{\frac{1}{n}}$.

Problem Set

Write the following expressions using rational exponents and then evaluate using a calculator. Answers should be rounded to the nearest hundredth.

1. $\sqrt[5]{45}$
2. $\sqrt[9]{140}$
3. $\sqrt[8]{50^3}$

Write the following expressions using roots and then evaluate using a calculator. Answers should be rounded to the nearest hundredth.

4. $72^{\frac{5}{3}}$
5. $95^{\frac{2}{3}}$
6. $125^{\frac{3}{4}}$

Evaluate the following without a calculator.

7. $64^{\frac{2}{3}}$
8. $27^{\frac{4}{3}}$
9. $16^{\frac{5}{4}}$

10. $\sqrt{25^3}$
11. $\sqrt[2]{9}^5$
12. $\sqrt[5]{32^2}$

For the following problems, rewrite the expressions with rational exponents and then simplify the exponent and evaluate without a calculator.

13. $\sqrt[4]{\left(\frac{2}{3}\right)^8}$
14. $\sqrt[3]{\frac{7}{2}}^6$
15. $\sqrt{(16)^{\frac{1}{2}}}^6$

Applying the Laws of Exponents to Rational Exponents

Objective

To use the laws of exponents with rational exponents.

Guidance

When simplifying expressions with rational exponents, all the laws of exponents that were learned in the *Polynomial Functions* chapter are still valid. On top of that, all the rules of fractions still apply.

Example A

Simplify $x^{\frac{1}{2}} \cdot x^{\frac{3}{4}}$.

Solution: Recall from the Product Property of Exponents, that when two numbers with the same base are multiplied we *add* the exponents. Here, the exponents do not have the same base, so we need to find a common denominator and then add the numerators.

$x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} = x^{\frac{2}{4}} \cdot x^{\frac{3}{4}} = x^{\frac{5}{4}}$

This rational exponent does not reduce, so we are done.

Example B

Simplify $\frac{4x^{\frac{2}{3}}y^4}{16x^3y^{\frac{5}{6}}}$

Solution: This problem utilizes the Quotient Property of Exponents. Subtract the exponents with the same base and reduce $\frac{4}{16}$.

$\frac{4x^{\frac{2}{3}}y^4}{16x^3y^{\frac{5}{6}}} = \frac{1}{4}x^{\left(\frac{2}{3}\right)-3}y^{\frac{4-5}{6}} = \frac{1}{4}x^{\frac{-7}{3}}y^{\frac{19}{6}}$

If you are writing your answer in terms of positive exponents, your answer would be $\frac{y^{\frac{19}{6}}}{4x^{\frac{7}{3}}}$. Notice, that when a rational exponent is improper we do not change it to a mixed number.

If we were to write the answer using roots, then we would take out the whole numbers. For example, $y = \frac{19}{6}$ can be written as $y^{\frac{19}{6}} = y^3 y^{\frac{1}{6}} = y^3 \sqrt[6]{y}$ because 6 goes into 19, 3 times with a remainder of 1.

Example C

Simplify $\left(9x^{\frac{6}{10}}\right)^{\frac{5}{2}}$.

Solution: This example uses the Powers Property of Exponents. When a power is raised to another power, we multiply the exponents.

$\left(9x^{\frac{6}{15}}\right)^{\frac{5}{2}} = 9^{\frac{5}{2}} \cdot x^{\left(\frac{6}{10}\right)\cdot\left(\frac{5}{2}\right)} = \sqrt{9}^5 x^{\frac{30}{20}} = 243x^{\frac{3}{2}}$

Example D

Simplify $\frac{\left(2x^{\frac{1}{2}}y^6\right)^{\frac{2}{3}}}{4x^{\frac{5}{4}}y^{\frac{9}{4}}}$.

Solution: On the numerator, the entire expression is raised to the $\frac{2}{3}$ power. Distribute this power to everything inside the parenthesis. Then, use the Powers Property of Exponents and rewrite 4 as 2^2.

$\frac{\left(2x^{\frac{1}{2}}y^6\right)^{\frac{2}{3}}}{4x^{\frac{5}{4}}y^{\frac{9}{4}}} = \frac{2^{\frac{2}{3}}x^{\frac{1}{3}}y^4}{2^2x^{\frac{5}{4}}y^{\frac{9}{4}}}$

Combine like terms by subtracting the exponents.

$\frac{2^{\frac{2}{3}}x^{\frac{1}{3}}y^4}{2^2x^{\frac{5}{4}}y^{\frac{9}{4}}} = 2^{\left(\frac{2}{3}\right)-2}x^{\left(\frac{1}{3}\right)-\left(\frac{5}{4}\right)}y^{4-\left(\frac{9}{4}\right)} = 2^{\frac{-4}{3}}x^{\frac{-11}{12}}y^{\frac{7}{4}}$

Finally, rewrite the answer with positive exponents by moving the 2 and x into the denominator. $\frac{y^{\frac{7}{4}}}{2^{\frac{4}{3}}x^{\frac{11}{12}}}$

Guided Practice

Simplify each expression. Reduce all rational exponents and write final answers using positive exponents.

1. $4d^{\frac{3}{5}} \cdot 8^{\frac{1}{3}}d^{\frac{2}{5}}$
2. $\frac{w^{\frac{7}{4}}}{w^{\frac{1}{2}}}$
3. $\left(3^{\frac{3}{2}}x^4y^{\frac{6}{5}}\right)^{\frac{4}{3}}$

Answers

1. Change 4 and 8 so that they are powers of 2 and then add exponents with the same base.

$4d^{\frac{3}{5}} \cdot 8^{\frac{1}{3}}d^{\frac{2}{5}} = 2^2d^{\frac{3}{5}} \cdot \left(2^3\right)^{\frac{1}{3}}d^{\frac{2}{5}} = 2^3d^{\frac{5}{5}} = 8d$

2. Subtract the exponents. Change the $\frac{1}{2}$ power to $\frac{2}{4}$.

$\frac{w^{\frac{7}{4}}}{w^{\frac{1}{2}}} = \frac{w^{\frac{7}{4}}}{w^{\frac{2}{4}}} = w^{\frac{5}{4}}$

3. Distribute the $\frac{4}{3}$ power to everything inside the parenthesis and reduce.

$\left(3^{\frac{3}{2}}x^4y^{\frac{6}{5}}\right)^{\frac{4}{3}} = 3^{\frac{12}{6}}x^{\frac{16}{3}}y^{\frac{24}{15}} = 3^2x^{\frac{16}{3}}y^{\frac{8}{5}} = 9x^{\frac{16}{3}}y^{\frac{8}{5}}$

Problem Set

Simplify each expression. Reduce all rational exponents and write final answer using positive exponents.

1. $\frac{1}{5}a^{\frac{4}{5}}25^{\frac{3}{2}}a^{\frac{3}{5}}$
2. $7b^{\frac{4}{3}}49^{\frac{1}{2}}b^{-\frac{2}{3}}$
3. $\frac{m^{\frac{8}{9}}}{m^{\frac{2}{3}}}$
4. $\frac{x^{\frac{4}{7}}y^{\frac{11}{6}}}{x^{\frac{1}{14}}y^{\frac{5}{3}}}$
5. $\frac{8^{\frac{5}{3}}r^5s^{\frac{3}{4}}t^{\frac{1}{3}}}{2^4r^{\frac{21}{5}}s^2t^{\frac{7}{9}}}$

6. $\left(a^{\frac{3}{2}}b^{\frac{4}{5}}\right)^{\frac{10}{3}}$
7. $\left(5x^{\frac{5}{7}}y^{4}\right)^{\frac{3}{2}}$
8. $\left(\frac{4x^{\frac{2}{5}}}{9y^{\frac{4}{5}}}\right)^{\frac{5}{2}}$
9. $\left(\frac{75d^{\frac{18}{5}}}{3d^{\frac{3}{5}}}\right)^{\frac{5}{2}}$
10. $\left(\frac{81^{\frac{3}{2}}a^{3}}{8a^{\frac{9}{2}}}\right)^{\frac{1}{3}}$
11. $27^{\frac{2}{3}}m^{\frac{4}{5}}n^{-\frac{3}{2}}4^{\frac{1}{2}}m^{-\frac{2}{3}}n^{\frac{8}{5}}$
12. $\left(\frac{3x^{\frac{3}{8}}y^{\frac{2}{5}}}{5x^{\frac{1}{4}}y^{-\frac{3}{10}}}\right)^{2}$

3.21 Graphing Square Root and Cubed Root Functions

Objective

To graph square root and cubed root functions by hand and using a graphing calculator.

Review Queue

Graph the following functions without using a graphing calculator.

1. $y = -2x + 5$
2. $y = x^2 + 4x - 5$
3. $y = -(x-1)(x+7)$
4. Using a graphing calculator, graph $f(x) = x^3 - 6x^2 - 9x + 54$. Find any minimums or maximum and all solutions.

Graphing Square Root Functions

Objective

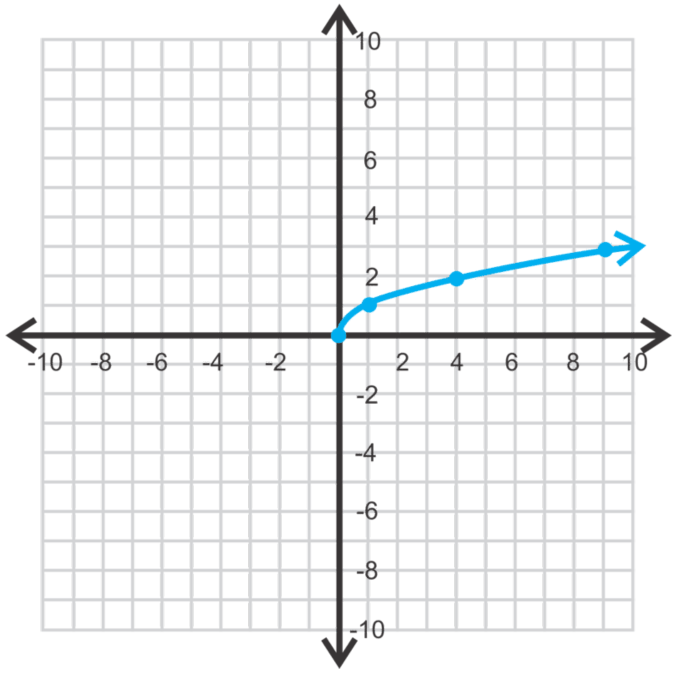
To graph a square root function with and without a calculator.

Guidance

A square root function has the form $y = a\sqrt{x-h} + k$, where $y = \sqrt{x}$ is the parent graph. Graphing the parent graph, we have:

TABLE 3.9:

x	y
16	4
9	3
4	2
1	1
0	0
-1	not a real number



Notice that this shape is half of a parabola, lying on its side. For $y = \sqrt{x}$, the output is the same as the input of $y = x^2$. The domain and range of $y = \sqrt{x}$ are all positive real numbers, including zero. x cannot be negative because you cannot take the square root of a negative number.

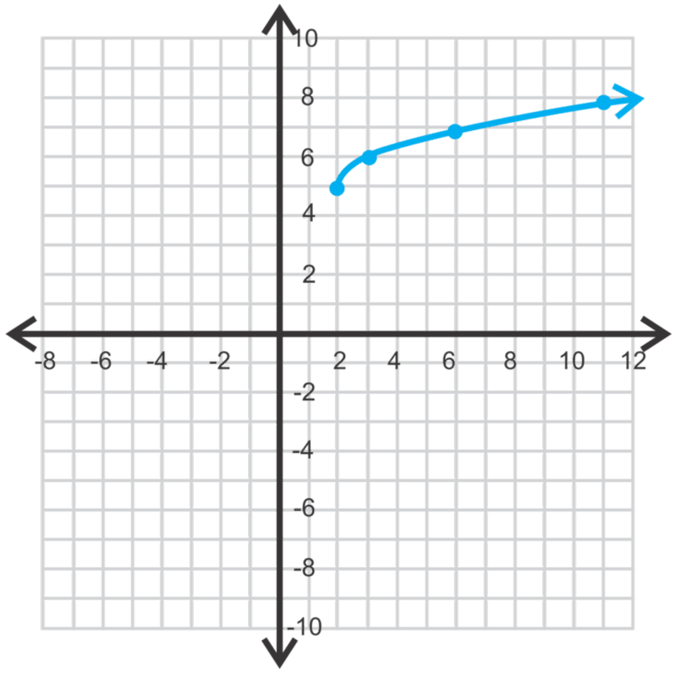
Example A

Graph $y = \sqrt{x-2}+5$ without a calculator.

Solution: To graph this function, draw a table. $x = 2$ is a critical value because it makes the radical zero.

TABLE 3.10:

x	y
2	5
3	6
6	7
11	8



After plotting the points, we see that the shape is exactly the same as the parent graph. It is just shifted up 5 and to the right 2. Therefore, we can conclude that h is the **horizontal shift** and k is the **vertical shift**.

The domain is all real numbers such that $x \geq 2$ and the range is all real numbers such that $y \geq 5$.

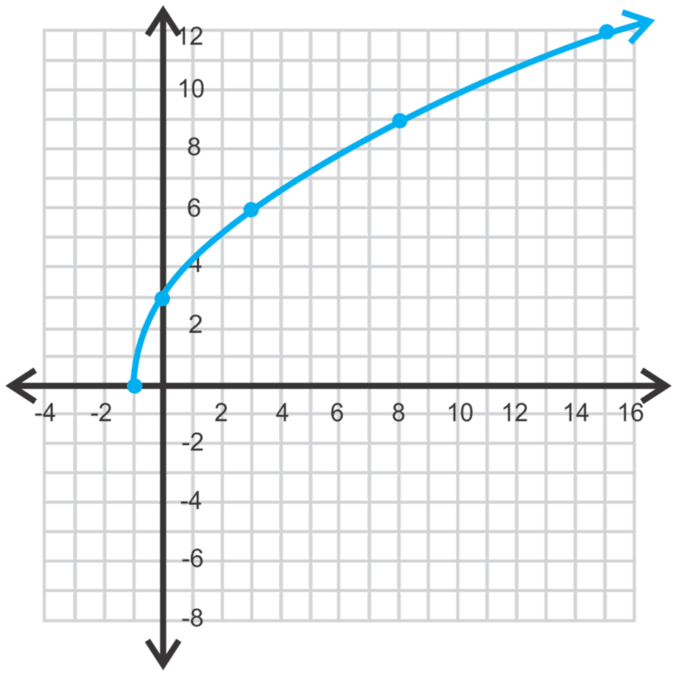
Example B

Graph $y = 3\sqrt{x+1}$. Find the domain and range.

Solution: From the previous example, we already know that there is going to be a horizontal shift to the left one unit. The 3 in front of the radical changes the width of the function. Let's make a table.

TABLE 3.11:

x	y
-1	0
0	3
3	6
8	9
15	12



Notice that this graph grows much faster than the parent graph. Extracting (h,k) from the equation, the starting point is $(-1,0)$ and then rather than increase at a "slope" of 1, it is three times larger than that.

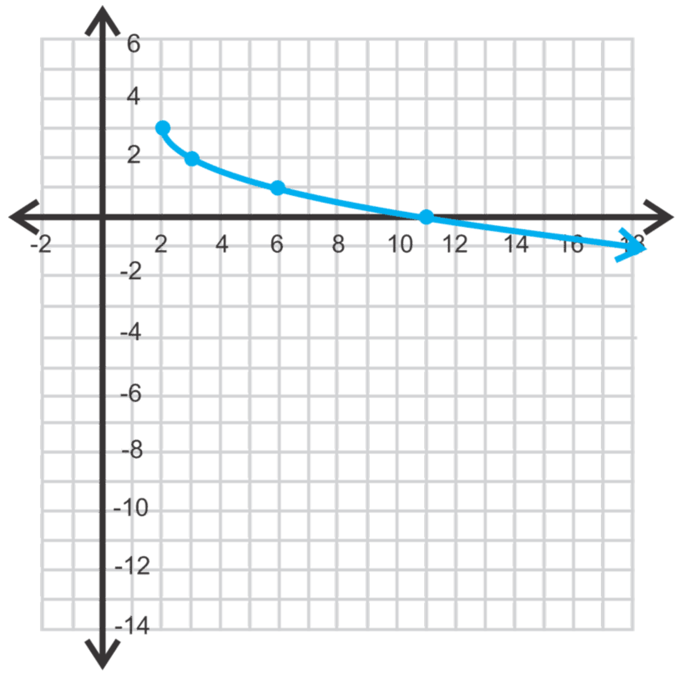
Example C

Graph $f(x) = -\sqrt{x-2}+3$.

Solution: Extracting (h,k) from the equation, we find that the starting point is $(2,3)$. The negative sign in front of the radical indicates a reflection. Let's make a table. Because the starting point is $(2,3)$, we should only pick x-values after $x=2$.

TABLE 3.12:

x	y
2	3
3	2
6	1
11	0
18	-1



The negative sign in front of the radical, we now see, results in a reflection over x-axis.

Using the graphing calculator: If you wanted to graph this function using the TI-83 or 84, press $Y =$ and clear out any functions. Then, press the negative sign, (-) and **2nd** x^2, which is $\sqrt{}$. Then, type in the rest of the function, so that $Y = -\sqrt{}(X-2)+3$. Press **GRAPH** and adjust the window.

Guided Practice

1. Evaluate $y = -2\sqrt{x-5}+8$ when $x = 9$.

Graph the following square root functions. Describe the relationship to the parent graph and find the domain and range. Use a graphing calculator for #3.

2. $y = \sqrt{-x}$

3. $f(x) = \frac{1}{2}\sqrt{x+3}$

4. $f(x) = -4\sqrt{x-5}+1$

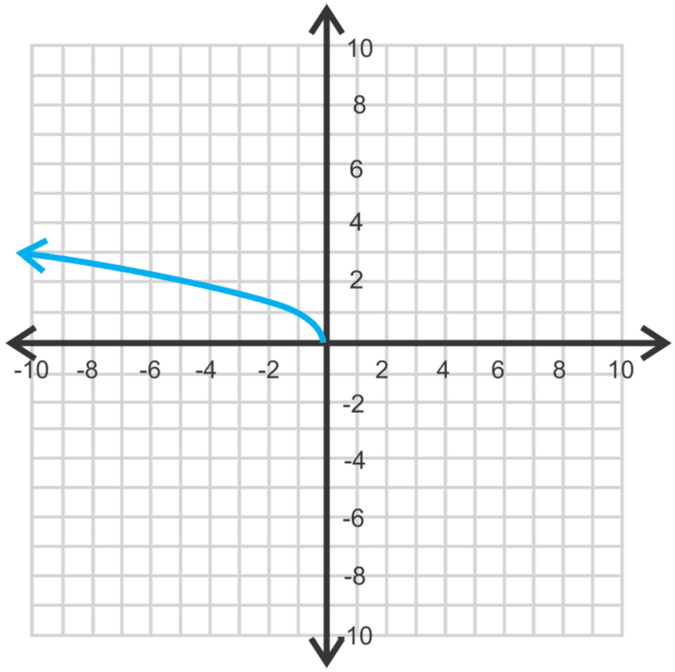
Answers

1. Plug in $x = 9$ into the equation and solve for y.

$y = -2\sqrt{9-5}+8 = -2\sqrt{4}+8 = -2(2)+8 = -4+8 = -4$

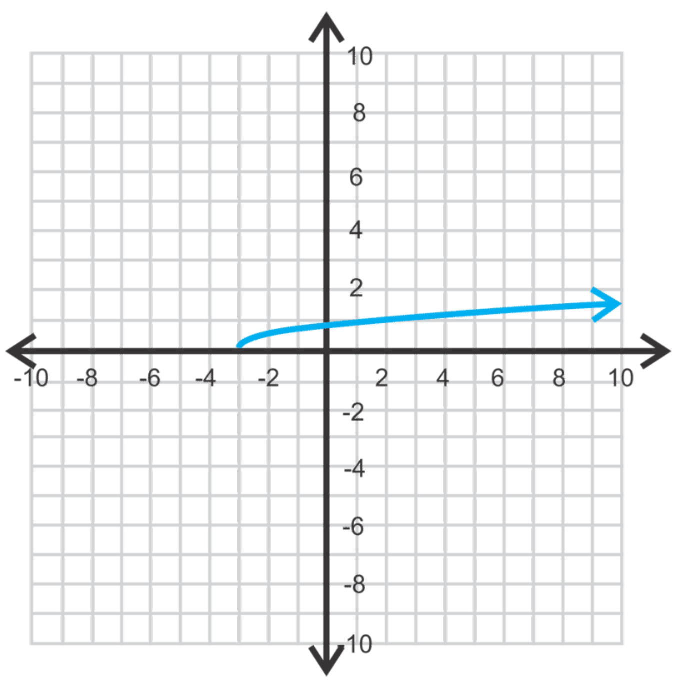
2. Here, the negative is under the radical. This graph is a reflection of the parent graph over the y-axis.

The domain is all real numbers less than or equal to zero. The range is all real numbers greater than or equal to zero.

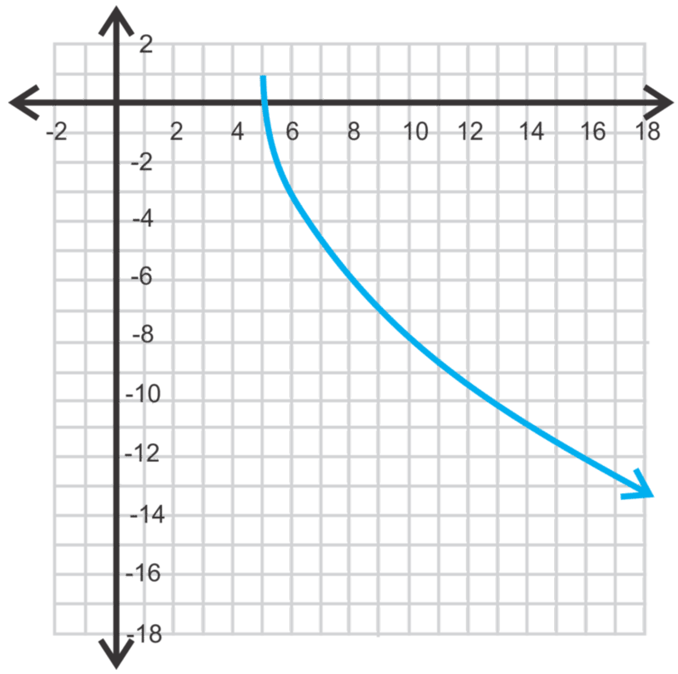


3. The starting point of this function is $(-3,0)$ and it is going to "grow" half as fast as the parent graph.

The domain is all real numbers greater than or equal to -3. The range is all real numbers greater than or equal to zero.



4. Using the graphing calculator, the function should be typed in as: $Y = -4\sqrt{}(X-5)+1$. It will be a reflection over the x-axis, have a starting point of $(5,1)$ and grow four times as fast as the parent graph.



Vocabulary

General Equation for a Square Root Function

$f(x) = a\sqrt{x-h}+k$ where h is the horizontal shift and k is the vertical shift.

Starting point

The initial point of a square root function, (h,k).

Problem Set

Graph the following square root functions. Use your calculator to check your answers.

1. $\sqrt{x+2}$
2. $\sqrt{x-5}-2$
3. $-2\sqrt{x+1}$
4. $1+\sqrt{x-3}$
5. $\frac{1}{2}\sqrt{x+8}$
6. $3\sqrt{x+6}$
7. $2\sqrt{1-x}$
8. $\sqrt{x+3}-5$
9. $4\sqrt{x+9}-8$
10. $-\frac{3}{2}\sqrt{x-3}+6$
11. $-3\sqrt{5-x}+7$
12. $2\sqrt{3-x}-9$

Graphing Cubed Root Functions

Objective

To graph a cubed root function with and without a calculator.

Guidance

A cubed root function is different from that of a square root. Their general forms look very similar, $y = a\sqrt[3]{x-h}+k$ and the parent graph is $y=\sqrt[3]{x}$. However, we can take the cubed root of a negative number, therefore, it will be defined for all values of x. Graphing the parent graph, we have:

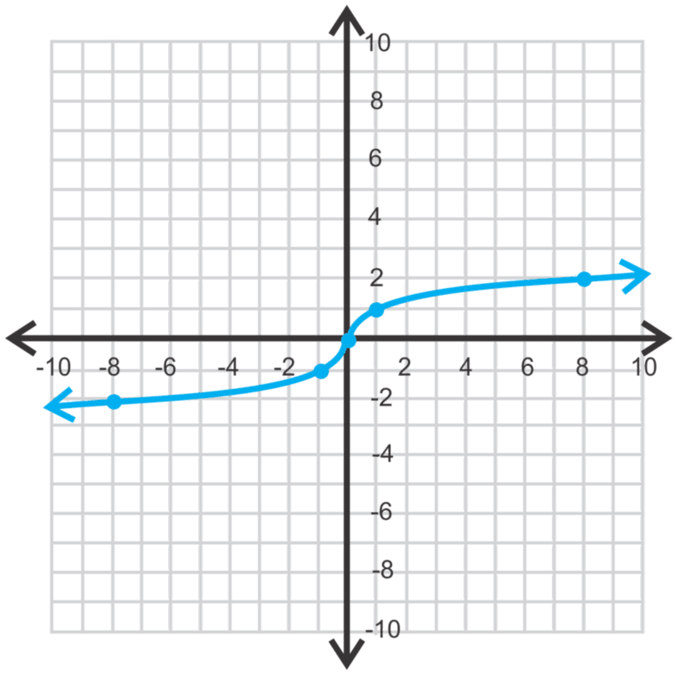


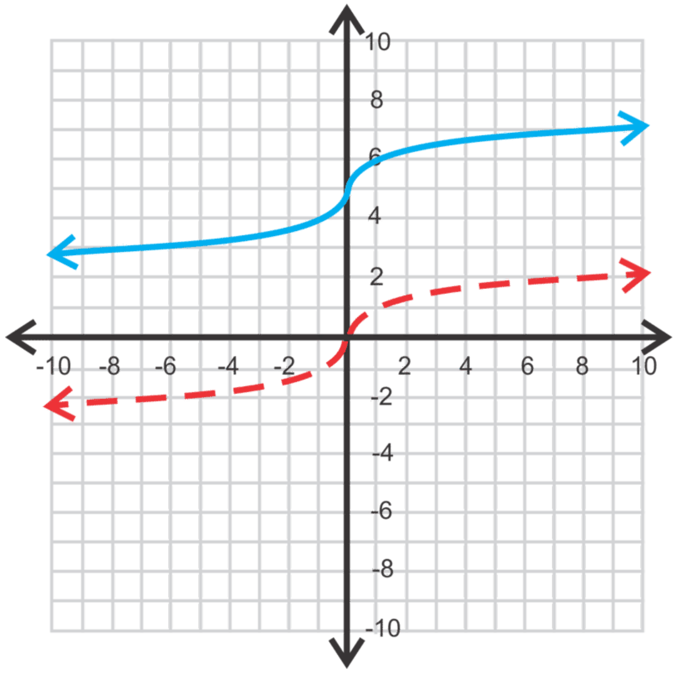
TABLE 3.13:

x	y
-27	-3
-8	-2
-1	-1
0	0
1	1
8	2
27	3

For $y=\sqrt[3]{x}$, the output is the same as the input of $y = x^3$. The domain and range of $y=\sqrt[3]{x}$ are all real numbers. Notice there is no "starting point" like the square root functions, the (h,k) now refers to the point where the function bends.

Example A

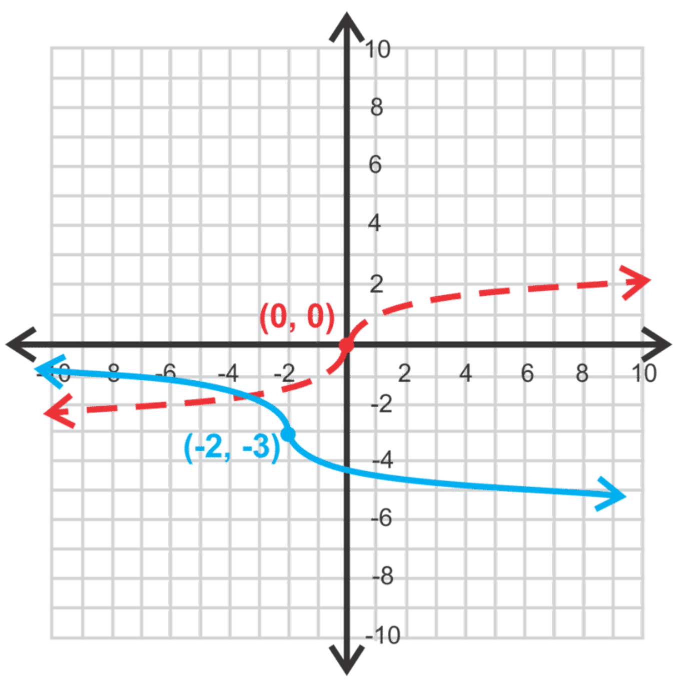
Describe how to obtain the graph of $y=\sqrt[3]{x}+5$ from $y=\sqrt[3]{x}$.



Solution: From the previous concept, we know that the +5 indicates a vertical shift of 5 units up. Therefore, this graph will look exactly the same as the parent graph, shifted up five units.

Example B

Graph $y = -\sqrt[3]{x+2} - 3$. Find the domain and range.



Solution: From the previous example, we know that from the parent graph, this function is going to shift to the left two units and down three units. The negative sign will result in a reflection.

Alternate Method: If you want to use a table (like in the previous concept), that will also work. Here is a table, then plot the points. (h, k) should always be the middle point in your table.

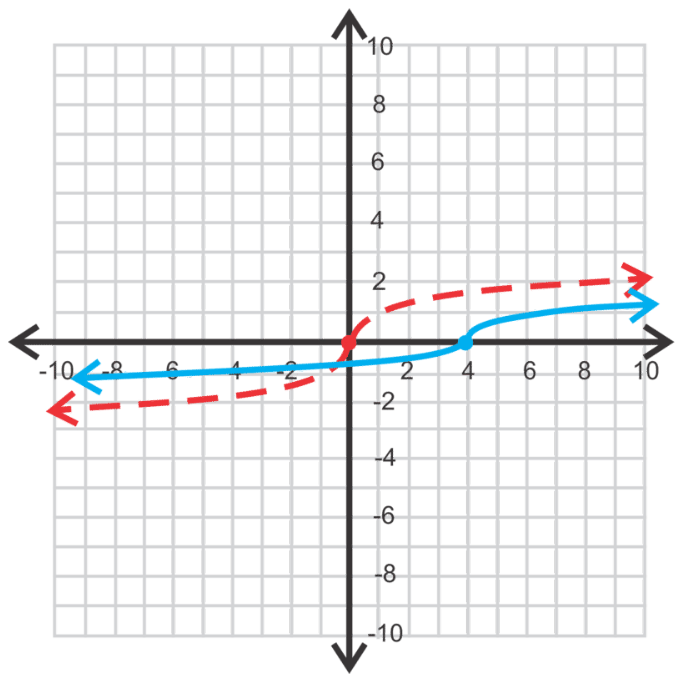
TABLE 3.14:

x	y
6	-5
-1	-4
-2	-3
-3	-2
-10	-1

Example C

Graph $f(x) = \frac{1}{2}\sqrt[3]{x-4}$.

Solution: The -4 tells us that, from the parent graph, the function will shift to the right four units. The $\frac{1}{2}$ effects how quickly the function will "grow". Because it is less than one, it will grow slower than the parent graph.



Using the graphing calculator: If you wanted to graph this function using the TI-83 or 84, press $Y =$ and clear out any functions. Then, press $(1 \div 2)$, **MATH** and scroll down to **4**: $\sqrt[3]{\ }$ and press **ENTER**. Then, type in the rest of the function, so that $Y = \left(\frac{1}{2}\right)\sqrt[3]{\ }(X-4)$. Press **GRAPH** and adjust the window.

Important Note: The domain and range of all cubed root functions are both all real numbers.

Guided Practice

1. Evaluate $y = \sqrt[3]{x+4} - 11$ when $x = -12$.

2. Describe how to obtain the graph of $y = \sqrt[3]{x+4} - 11$ from $y = \sqrt[3]{x}$.

Graph the following cubed root functions. Check your graphs on the graphing calculator.

3. $y = \sqrt[3]{x-2} - 4$

4. $f(x) = -3\sqrt{x} - 1$

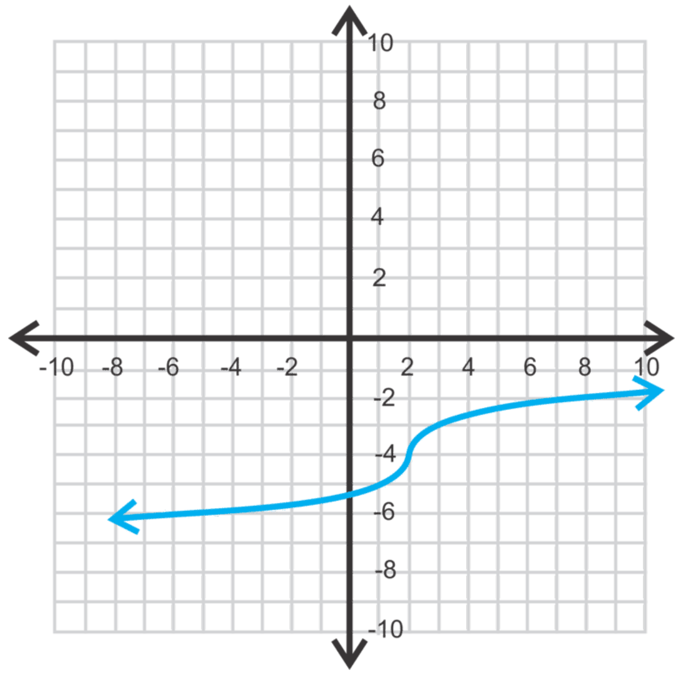
Answers

1. Plug in $x = -12$ and solve for y.

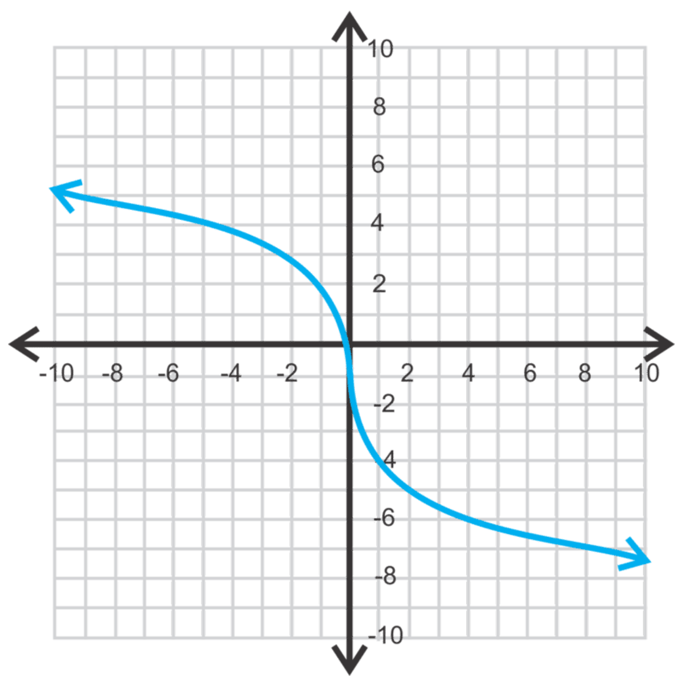
$y = \sqrt[3]{-12+4} - 11 = \sqrt[3]{-8} + 4 = -2 + 4 = 2$

2. Starting with $y=\sqrt[3]{x}$, you would obtain $y=\sqrt[3]{x+4}-11$ by shifting the function to the left four units and down 11 units.

3. This function is a horizontal shift to the right two units and down four units.



4. This function is a reflection of $y=\sqrt[3]{x}$ and stretched to be three times as large. Lastly, it is shifted down one unit.



Vocabulary

General Equation for a Cubed Root Function

$f(x)=a\sqrt[3]{x-h}+k$, where h is the horizontal shift and k is the vertical shift.

Problem Set

Graph the following cubed root functions. Use your calculator to check your answers.

1. $\sqrt[3]{x}+4$
2. $\sqrt[3]{x-3}$
3. $\sqrt[3]{x+2}-1$
4. $-\sqrt[3]{x}-6$
5. $2\sqrt[3]{x+1}$
6. $-3\sqrt[3]{x}+5$
7. $\frac{1}{2}\sqrt[3]{1-x}$
8. $2\sqrt[3]{x+4}-3$
9. $-\frac{1}{3}\sqrt[3]{x-5}+2$
10. $\sqrt[3]{6-x}+7$
11. $-5\sqrt[3]{x-1}+3$
12. $4\sqrt[3]{7-x}-8$

Extracting the Equation from a Graph

Objective

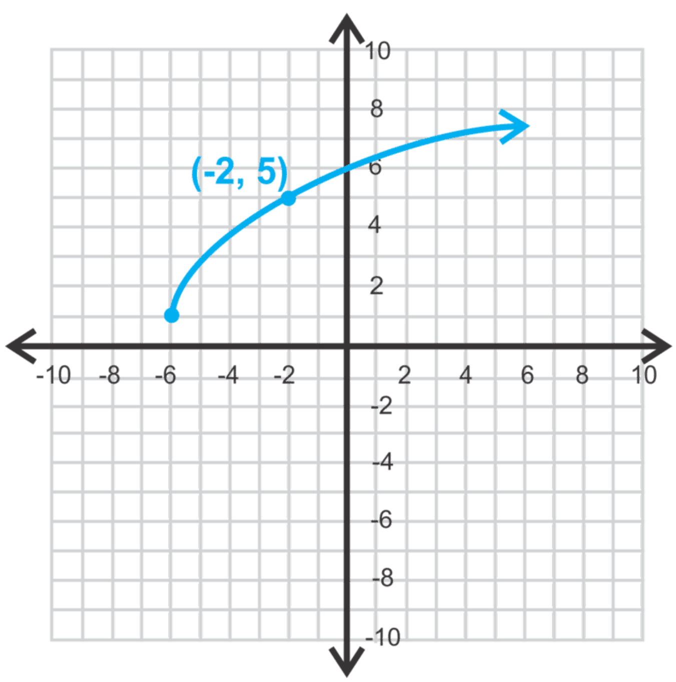
To look at the graph of a square root or cubed root function and determine the equation.

Guidance

This concept is the opposite of the previous two. Instead of graphing from the equation, we will now find the equation, given the graph.

Example A

Determine the equation of the graph below.



Solution: From the previous two concepts, we know this is a square root function, so the general form is $y =$

$a\sqrt{x-h}+k$. The starting point is $(-6,1)$. Plugging this in for h and k, we have $y=a\sqrt{x+6}+1$. Now, find a, using the given point, $(-2,5)$. Let's substitute it in for x and y and solve for a.

$$\begin{aligned}5 &= a\sqrt{-2+6}+1\\4 &= a\sqrt{4}\\4 &= 2a\\2 &= a\end{aligned}$$

The equation is $y=2\sqrt{x+6}+1$.

Example B

Find the equation of the cubed root function where $h=-1$ and $k=-4$ and passes through $(-28,-3)$.

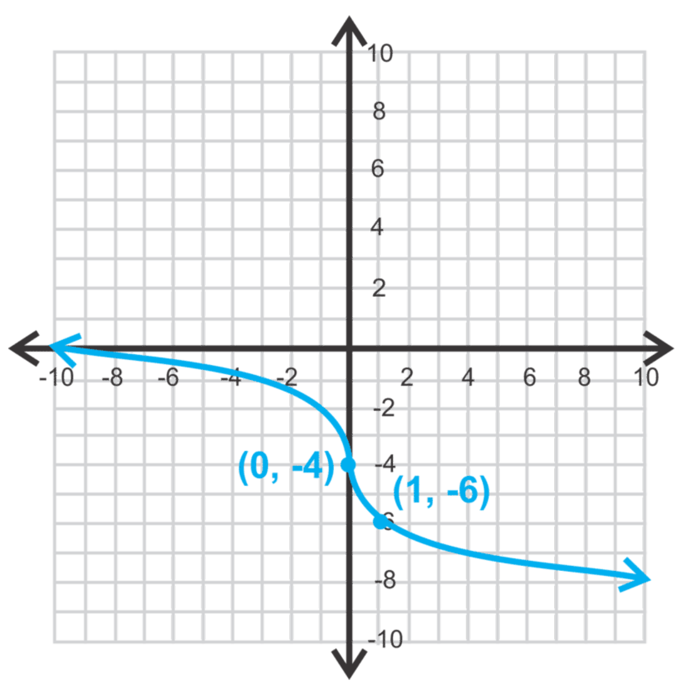
Solution: First, plug in what we know to the general equation; $y=\sqrt[3]{x-h}+k \Rightarrow y=a\sqrt[3]{x+1}-4$. Now, substitute $x=-28$ and $y=-3$ and solve for a.

$$\begin{aligned}-3 &= a\sqrt[3]{-28+1}-4\\1 &= -3a\\-\frac{1}{3} &= a\end{aligned}$$

The equation of the function is $y=-\frac{1}{3}\sqrt[3]{x+1}-4$.

Example C

Find the equation of the function below.



Solution: It looks like $(0,-4)$ is (h,k). Plug this in for h and k and then use the second point to find a.

$$
\begin{aligned}
-6 &= a\sqrt[3]{1-0}-4 \\
-2 &= a\sqrt[3]{1} \\
-2 &= a
\end{aligned}
$$

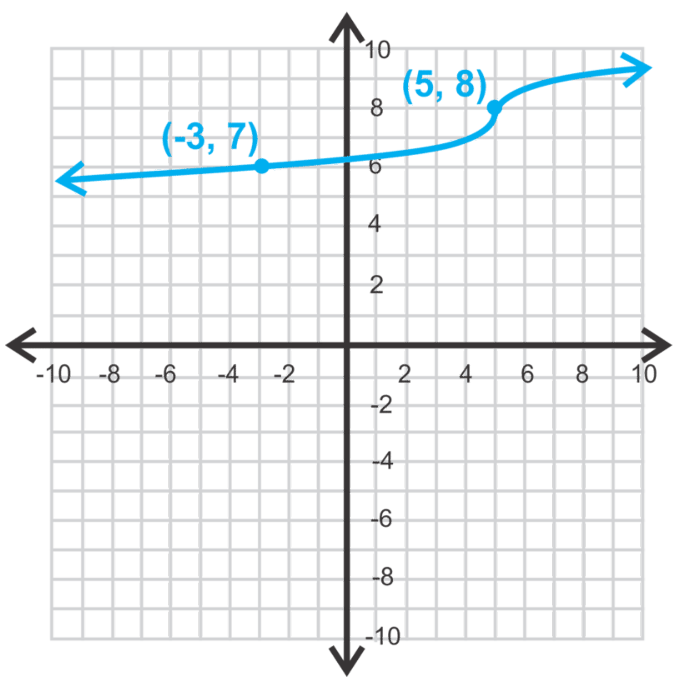
The equation of this function is $y = -2\sqrt[3]{x} - 4$.

When finding the equation of a cubed root function, you may assume that one of the given points is (h,k). Whichever point is on the "bend" is (h,k) for the purposes of this text.

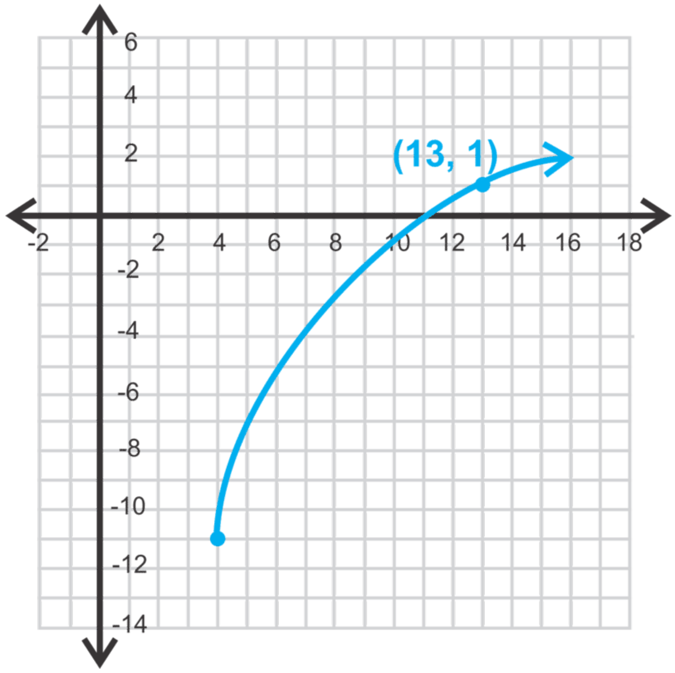
Guided Practice

Find the equation of the functions below.

1.



2.



3. Find the equation of a square root equation with a starting point of $(-5,-3)$ and passes through $(4,-6)$.

Answers

1. Substitute what you know into the general equation to solve for a. From Example C, you may assume that $(5,8)$ is (h,k) and $(-3,7)$ is (x,y).

$$\begin{aligned} y &= a\sqrt[3]{x-5}+8 \\ 7 &= a\sqrt[3]{-3-5}+8 \\ -1 &= -2a \\ \frac{1}{2} &= a \end{aligned}$$

The equation of this function is $y = \frac{1}{2}\sqrt[3]{x-5}+8$.

2. Substitute what you know into the general equation to solve for a. From the graph, the starting point, or (h,k) is $(4,-11)$ and $(13,1)$ are a point on the graph.

$$\begin{aligned} y &= a\sqrt{x-4}-11 \\ 1 &= a\sqrt{13-4}-11 \\ 12 &= 3a \\ 4 &= a \end{aligned}$$

The equation of this function is $y = 4\sqrt{x-4}-11$.

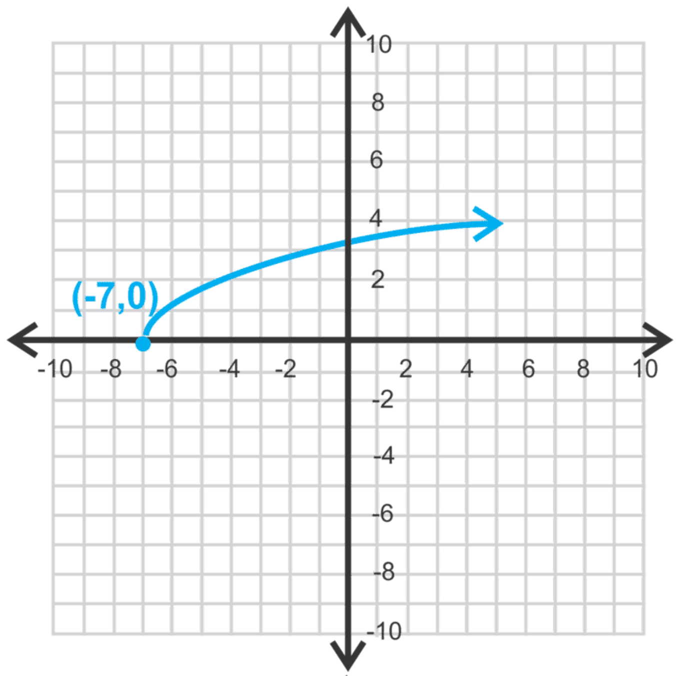
3. Substitute what you know into the general equation to solve for a. From the graph, the starting point, or (h,k) is $(-5,-3)$ and $(4,-6)$ are a point on the graph.

$$
\begin{aligned}
y &= a\sqrt{x+5}-3 \\
-6 &= a\sqrt{4+5}-3 \\
-3 &= 3a \\
-1 &= a
\end{aligned}
$$

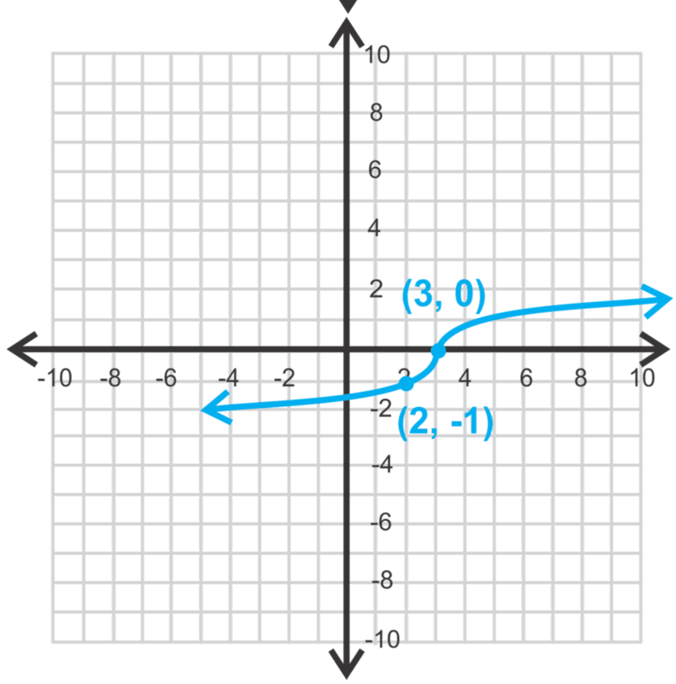
The equation of this function is $y = -\sqrt{x+5}-3$.

Problem Set

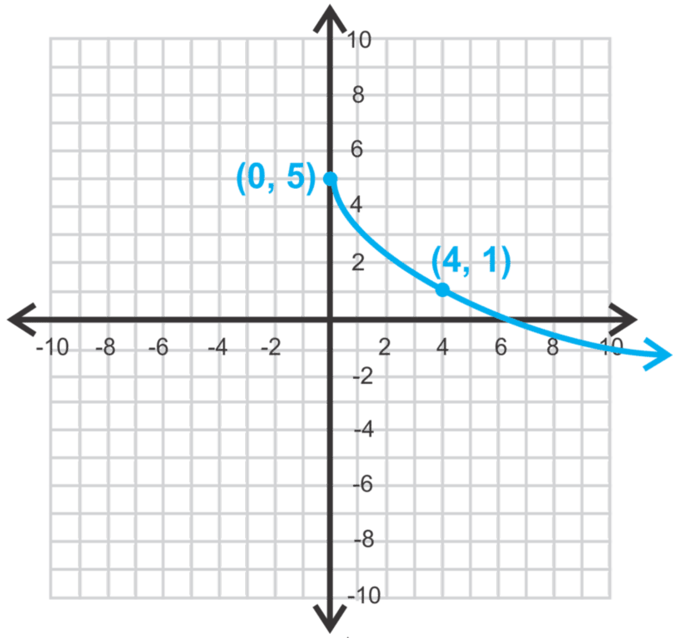
Write the equation for each function graphed below.



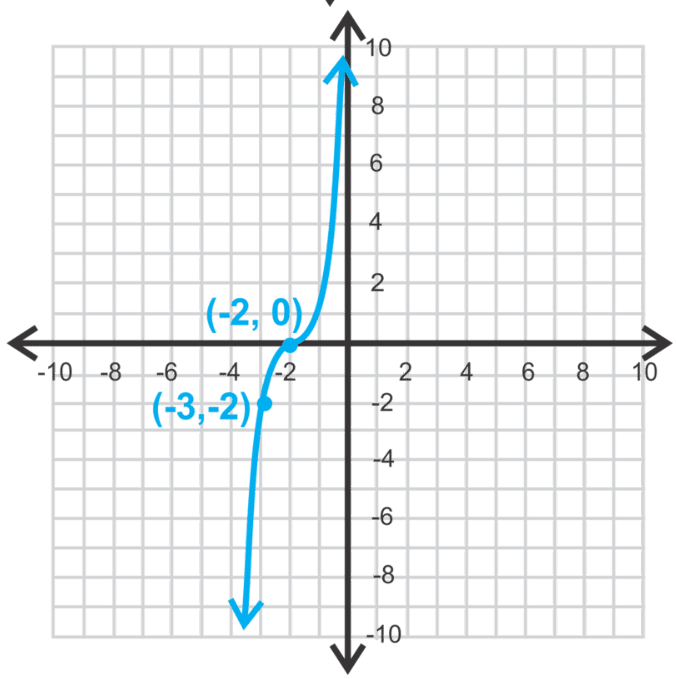
1.



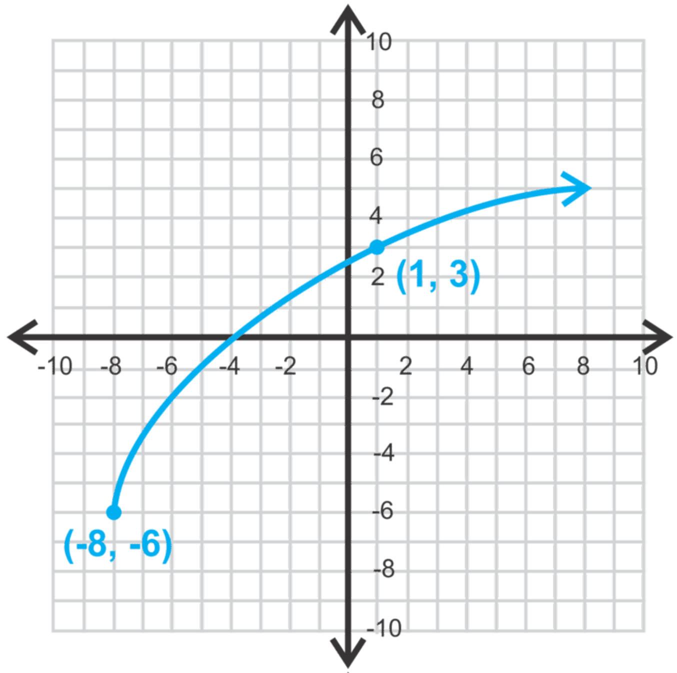
2.



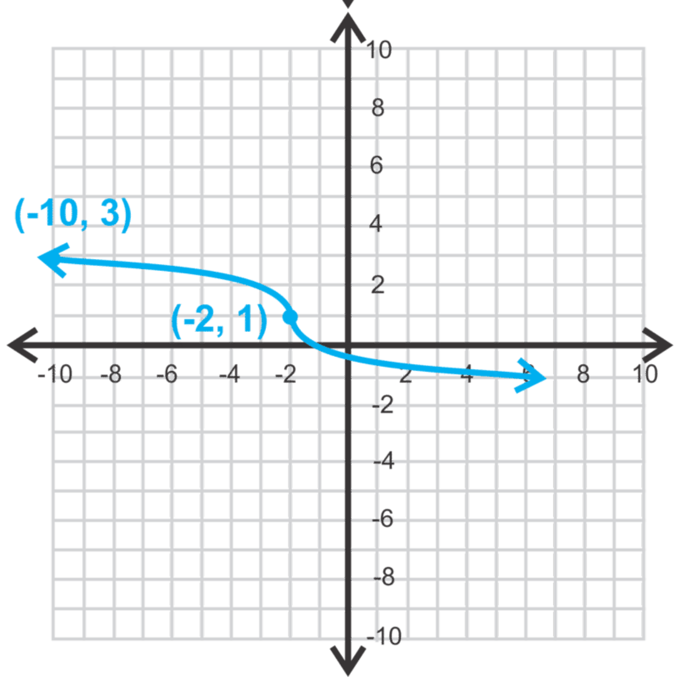
3.



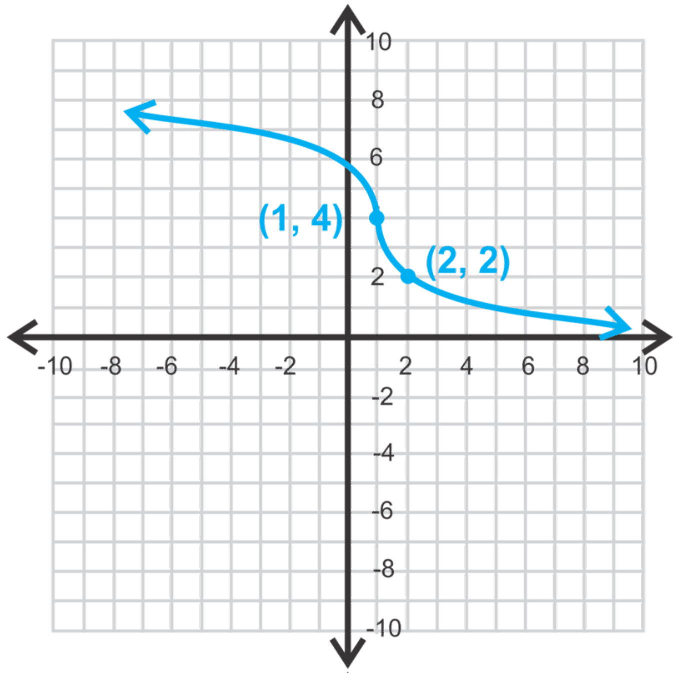
4.


10
8
6
4
2
(1, 3)
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4
-6
-8
-10
(-8, -6)

5.


10
8
6
4
2
(-10, 3)
(-2, 1)
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4
-6
-8
-10

6.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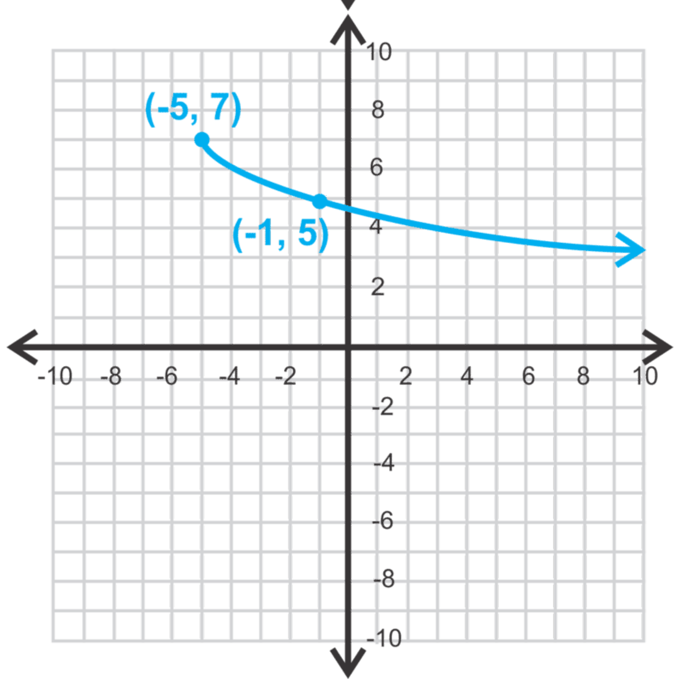
10
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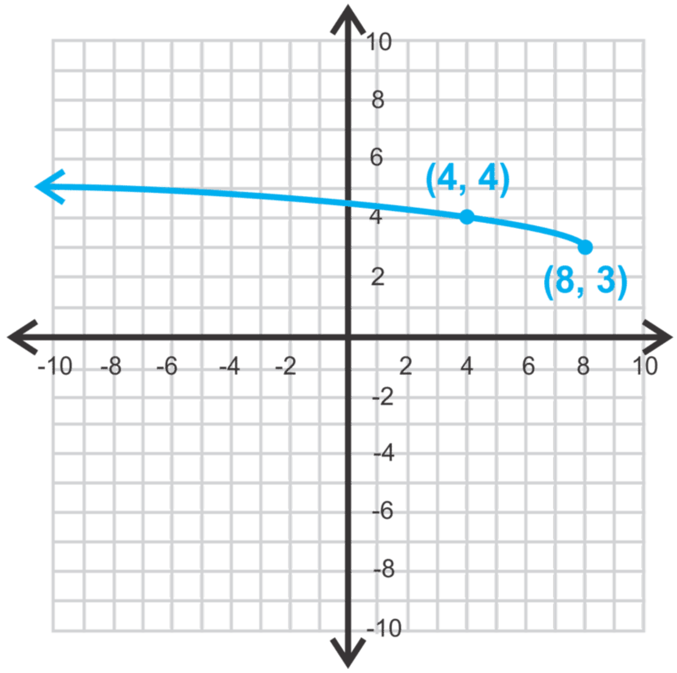
9. 

10. Write the equation of a square root function with starting point $(-1,6)$ passing through $(3,16)$.
11. Write the equation of a square root function with starting point $(-6,-3)$ passing through $(10,-15)$.
12. Write the equation of a cube root function with $(h,k) = (2,7)$ passing through $(10,11)$.

3.22 Solving Radical Equations

Objective

To solve radical equations.

Review Queue

Solve for x.

1. $x^2 - 9x + 14 = 0$
2. $3x^2 - 11x - 20 = 0$
3. $\sqrt{x} = 4$

Solving Simple Radical Equations

Objective

To solve basic radical equations.

Guidance

Solving radical equations are very similar to solving other types of equations. The objective is to get x by itself. However, now there are radicals within the equations. Recall that the opposite of the square root of something is to square it.

Example A

Is $x = 5$ the solution to $\sqrt{2x+15} = 8$?

Solution: Plug in 5 for x to see if the equation holds true. If it does, then 5 is the solution.

$$\begin{aligned}\sqrt{2(5)+15} &= 8\\ \sqrt{10+15} &= 9\\ \sqrt{25} &\neq 8\end{aligned}$$

We know that $\sqrt{25} = 5$, so $x = 5$ is not the solution.

Example B

Solve $\sqrt{2x-5} + 7 = 16$.

Solution: To solve for x, we need to isolate the radical. Subtract 7 from both sides.

$$\begin{aligned}\sqrt{2x-5} + 7 &= 16\\ \sqrt{2x-5} &= 9\end{aligned}$$

Now, we can square both sides to eliminate the radical. Only square both sides when the radical is alone on one side of the equals sign.

$$\sqrt{2x-5}^2 = 9^2$$
$$2x-5 = 81$$
$$2x = 86$$
$$x = 43$$

Check: $\sqrt{2(43)-5}+7 = \sqrt{86-5}+7 = \sqrt{81}+7 = 9+7 = 16 \boxed{\checkmark}$

ALWAYS check your answers when solving radical equations. Sometimes, you will solve an equation, get a solution, and then plug it back in and it will not work. These types of solutions are called **extraneous solutions** and are not actually considered solutions to the equation.

Example C

Solve $3\sqrt[3]{x-8}-2 = -14$.

Solution: Again, isolate the radical first. Add 2 to both sides and divide by 3.

$$3\sqrt[3]{x-8}-2 = -14$$
$$3\sqrt[3]{x-8} = -12$$
$$\sqrt[3]{x-8} = -4$$

Now, cube both sides to eliminate the radical.

$$\sqrt[3]{x-8}^3 = (-4)^3$$
$$x-8 = -64$$
$$x = -56$$

Check: $3\sqrt[3]{-56-8}-2 = 3\sqrt[3]{-64}-2 = 3\cdot -4-2 = -12-2 = -14 \boxed{\checkmark}$

Guided Practice

Solve the equations and check your answers.

1. $\sqrt{x+5} = 6$
2. $5\sqrt{2x-1}+1 = 26$
3. $\sqrt[4]{3x+11}-2 = 3$

Answers

1. The radical is already isolated here. Square both sides and solve for x.

$$\sqrt{x+5}^2 = 6^2$$
$$x+5 = 36$$
$$x = 31$$

Check: $\sqrt{31+5} = \sqrt{36} = 6 \boxed{\checkmark}$

2. Isolate the radical by subtracting 1 and then dividing by 5.

$$\begin{aligned} 5\sqrt{2x-1}+1 &= 26 \\ 5\sqrt{2x-1} &= 25 \\ \sqrt{2x-1} &= 5 \end{aligned}$$

Square both sides and continue to solve for x.

$$\begin{aligned} \sqrt{2x-1}^2 &= 5^2 \\ 2x-1 &= 25 \\ 2x &= 26 \\ x &= 13 \end{aligned}$$

Check: $5\sqrt{2(13)-1}+1 = 5\sqrt{26-1} = 5\sqrt{25}+1 = 5\cdot 5+1 = 25+1 = 26\ \boxed{\checkmark}$

3. In this problem, we have a fourth root. That means, once we isolate the radical, we must raise both sides to the fourth power to eliminate it.

$$\begin{aligned} \sqrt[4]{3x+11}-2 &= 3 \\ \sqrt[4]{3x-11}^4 &= 5^4 \\ 3x-11 &= 625 \\ 3x &= 636 \\ x &= 212 \end{aligned}$$

Check: $\sqrt[4]{3(212)+11}-2 = \sqrt[4]{636-11}-2 = \sqrt[4]{625}-2 = 5-2 = 3\ \boxed{\checkmark}$

Vocabulary

Extraneous Solution
A solved-for value of x, that when checked, is not actually a solution.

Problem Set

Solve the equations and check your answers.

1. $\sqrt{x+5} = 6$
2. $2-\sqrt{x+1} = 0$
3. $4\sqrt{5-x} = 12$
4. $\sqrt{x+9}+7 = 11$
5. $\frac{1}{2}\sqrt[3]{x-2} = 1$
6. $\sqrt[3]{x+3}+5 = 9$
7. $5\sqrt{15-x}+2 = 17$
8. $-5 = \sqrt[5]{x-5}-7$
9. $\sqrt[4]{x-6}+10 = 13$
10. $\frac{8}{5}\sqrt[3]{x+5} = 8$
11. $3\sqrt{x+7}-2 = 25$
12. $\sqrt[4]{235+x}+9 = 14$

Solving Radical Equations with Variables on Both Sides

Objective

To solve more complicated radical equations.

Guidance

In this concept, we will continue solving radical equations. Here, we will address variables and radicals on both sides of the equation.

Example A

Solve $\sqrt{4x+1}-x=-1$

Solution: Now we have an x that is not under the radical. We will still isolate the radical.

$$\begin{aligned}\sqrt{4x+1}-x&=-1\\ \sqrt{4x-1}&=x-1\end{aligned}$$

Now, we can square both sides. Be careful when squaring $x-1$, the answer is not x^2-1.

$$\begin{aligned}\sqrt{4x+1}^2&=(x-1)^2\\ 4x+1&=x^2-2x+1\end{aligned}$$

This problem is now a quadratic. To solve quadratic equations, we must either factor, when possible or use the Quadratic Formula. Combine like terms and set one side equal to zero.

$$\begin{aligned}4x+1&=x^2-2x+1\\ 0&=x^2-6x\\ 0&=x(x-6)\\ x&=0\ or\ 6\end{aligned}$$

Check both solutions: $\sqrt{4(0)+1}-1=\sqrt{0+1}-1=1-1=0\neq-1$. 0 is an extraneous solution. $\sqrt{4(6)+1}-6=\sqrt{24+1}-6=5-6=-1$ $\boxed{\checkmark}$ Therefore, 6 is the only solution.

Example B

Solve $\sqrt{8x-11}-\sqrt{3x+19}=0$.

Solution: In this example, you need to isolate both radicals. To do this, subtract the second radical from both sides. Then, square both sides to eliminate the variable.

$$\begin{aligned}\sqrt{8x-11}-\sqrt{3x+19}&=0\\ \sqrt{8x-11}^2&=\sqrt{3x+19}^2\\ 8x-11&=3x+19\\ 5x&=30\\ x&=6\end{aligned}$$

Check: $\sqrt{8(6)-11}-\sqrt{3(6)+19}=\sqrt{48-11}-\sqrt{18+19}=\sqrt{37}-\sqrt{37}=0\ \boxed{\checkmark}$

Example C

Solve $\sqrt[4]{4x+1}=x$

Solution: The radical is isolated. To eliminate it, we must raise both sides to the fourth power.

$$\begin{aligned}\sqrt[4]{2x^2-1}^4&=x^4\\2x^2-1&=x^4\\0&=x^4-2x^2+1\\0&=(x^2-1)(x^2-1)\\0&=(x-1)(x+1)(x-1)(x+1)\\x&=1\ or\ -1\end{aligned}$$

Check: $\sqrt[4]{2(1)^2-1}=\sqrt[4]{2-1}=\sqrt[4]{1}=1\ \boxed{\checkmark}$ and $\sqrt[4]{2(-1)^2-1}=\sqrt[4]{2-1}=\sqrt[4]{1}=1\ \boxed{\checkmark}$

Guided Practice

Solve the following radical equations. Check for extraneous solutions.

1. $\sqrt[3]{4x^3-24}=x$
2. $\sqrt{5x-3}=\sqrt{3x+19}$
3. $\sqrt{6x-5}-x=-10$

Answers

1. The radical is isolated. Cube both sides to eliminate the cubed root.

$$\begin{aligned}\sqrt[3]{4x^3-24}^3&=x^3\\4x^3-24&=x^3\\-24&=-3x^3\\8&=x^3\\2&=x\end{aligned}$$

Check: $\sqrt[3]{4(2)^3-24}=\sqrt[3]{32-24}=\sqrt[3]{8}=2\ \boxed{\checkmark}$

2. Square both sides to solve for x.

$$\begin{aligned}\sqrt{5x-3}^2&=\sqrt{3x+19}^2\\5x-3&=3x+19\\2x&=22\\x&=11\end{aligned}$$

Check:

$$\begin{aligned}\sqrt{5(11)-3}&=\sqrt{3(11)+19}\\\sqrt{55-3}&=\sqrt{33+19}\quad\boxed{\checkmark}\\\sqrt{52}&=\sqrt{52}\end{aligned}$$

3. Add x to both sides and square to eliminate the radical.

$$\begin{aligned}\sqrt{6x-5}^2 &= (x-10)^2 \\ 6x-5 &= x^2-20x+100 \\ 0 &= x^2-26x+105 \\ 0 &= (x-21)(x-5) \\ x &= 21 \text{ or } 5\end{aligned}$$

Check both solutions:

$$\begin{aligned}x=21:\ \sqrt{6(21)-5}-21 &= \sqrt{126-5}-21 = \sqrt{121}-21 = 11-21 = -10 \boxed{\checkmark} \\ x=5:\ \sqrt{6(5)-5}-21 &= \sqrt{30-5}-21 = \sqrt{25}-21 = 5-21 \neq -10\end{aligned}$$

5 is an extraneous solution.

Problem Set

Solve the following radical equations. Be sure to check for extraneous solutions.

1. $\sqrt{x-3} = x-5$
2. $\sqrt{x+3+15} = x-12$
3. $\sqrt[4]{3x^2+54} = x$
4. $\sqrt{x^2+60} = 4\sqrt{x}$
5. $\sqrt{x^4+5x^3} = 2\sqrt{2x+10}$
6. $x = \sqrt{5x-6}$
7. $\sqrt{3x+4} = x-2$
8. $\sqrt{x^3+8x-}\sqrt{9x^2-60} = 0$
9. $x = \sqrt[3]{4x+4-x^2}$
10. $\sqrt[4]{x^3+3} = 2\sqrt[4]{x+3}$
11. $x^2 - \sqrt{42x^2+343} = 0$
12. $x\sqrt{x^2-21} = 2\sqrt{x^3-25x+25}$

For questions 13 and 14, you will need to use the method illustrated in the example below.

$$\begin{aligned}\sqrt{x-15} &= \sqrt{x}-3 \\ \left(\sqrt{x-15}\right)^2 &= \left(\sqrt{x}-3\right)^2 \\ x-15 &= x-6\sqrt{x}+9 \\ -24 &= -6\sqrt{x} \\ (4)^2 &= \left(\sqrt{x}\right)^2 \\ 16 &= x\end{aligned}$$

1. Square both sides
2. Combine like terms to isolate the remaining radical
3. Square both sides again to solve

Check:

$$\begin{aligned}\sqrt{16-15} &= \sqrt{16}-3\\ \sqrt{1} &= 4-3\\ 1 &= 1\end{aligned}$$

13. $\sqrt{x+11}-2=\sqrt{x-21}$
14. $\sqrt{x-6}=\sqrt{7x}-22$

Solving Rational Exponent Equations

Objective

To solve equations where the variable has a rational exponent.

Guidance

This concept is very similar to the previous two. When solving a rational exponent equation, isolate the variable. Then, to eliminate the exponent, you will need to raise everything to the reciprocal power.

Example A

Solve $3x^{\frac{5}{2}}=96$.

Solution: First, divide both sides by 3 to isolate x.

$$\begin{aligned}3x^{\frac{5}{2}} &= 96\\ x^{\frac{5}{2}} &= 32\end{aligned}$$

x is raised to the five-halves power. To cancel out this exponent, we need to raise everything to the two-fifths power.

$$\begin{aligned}\left(x^{\frac{5}{2}}\right)^{\frac{2}{5}} &= 32^{\frac{2}{5}}\\ x &= 32^{\frac{2}{5}}\\ x &= \sqrt[5]{32}^2 = 2^2 = 4\end{aligned}$$

Check: $3(4)^{\frac{5}{2}}=3\cdot 2^5=3\cdot 32=96\ \boxed{\checkmark}$

Example B

Solve $-2(x-5)^{\frac{3}{4}}+48=-202$.

Solution: Isolate $(x-5)^{\frac{3}{4}}$ by subtracting 48 and dividing by -2.

$$\begin{aligned}-2(x-5)^{\frac{3}{4}}+48 &= -202\\ -2(x-5)^{\frac{3}{4}} &= -250\\ (x-5)^{\frac{3}{4}} &= -125\end{aligned}$$

To undo the three-fourths power, raise everything to the four-thirds power.

$$\left[(x-5)^{\frac{3}{4}}\right]^{\frac{4}{3}} = (-125)^{\frac{4}{3}}$$
$$x-5 = 625$$
$$x = 630$$

Check: $-2(630-5)^{\frac{3}{4}}+48 = -2 \cdot 625^{\frac{3}{4}}+48 = -2 \cdot 125+48 = -250+48 = -202 \boxed{\checkmark}$

Guided Practice

Solve the following rational exponent equations and check for extraneous solutions.

1. $8(3x-1)^{\frac{2}{3}} = 200$

2. $6x^{\frac{3}{2}} - 141 = 1917$

Answers

1. Divide both sides by 8 and raise everything to the three-halves power.

$$8(3x-1)^{\frac{2}{3}} = 200$$
$$\left[(3x-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = (25)^{\frac{3}{2}}$$
$$3x-1 = 125$$
$$3x = 126$$
$$x = 42$$

Check: $8(3(42)-1)^{\frac{2}{3}} = 8(126-1)^{\frac{2}{3}} = 8(125)^{\frac{2}{3}} = 8 \cdot 25 = 200 \boxed{\checkmark}$

2. Here, only the x is raised to the three-halves power. Subtract 141 from both sides and divide by 6. Then, eliminate the exponent by raising both sides to the two-thirds power.

$$6x^{\frac{3}{2}} - 141 = 1917$$
$$6x^{\frac{3}{2}} = 2058$$
$$x^{\frac{3}{2}} = 343$$
$$x = 343^{\frac{2}{3}} = 7^2 = 49$$

Check: $6(49)^{\frac{3}{2}} - 141 = 6 \cdot 343 - 141 = 2058 - 141 = 1917 \boxed{\checkmark}$

Problem Set

1. $2x^{\frac{3}{2}} = 54$
2. $3x^{\frac{1}{3}} + 5 = 17$
3. $(7x-3)^{\frac{2}{5}} = 4$
4. $(4x+5)^{\frac{1}{2}} = x-4$
5. $x^{\frac{5}{2}} = 16x^{\frac{1}{2}}$
6. $(5x+7)^{\frac{3}{5}} = 8$
7. $5x^{\frac{2}{3}} = 45$

8. $(7x-8)^{\frac{2}{3}} = 4(x-5)^{\frac{2}{3}}$
9. $7x^{\frac{3}{7}} + 9 = 65$
10. $4997 = 5x^{\frac{3}{2}} - 3$
11. $2x^{\frac{3}{4}} = 686$
12. $x^3 = (4x-3)^{\frac{3}{2}}$

3.23 Graphing Rational Functions

Objective

To graph several different types of rational functions and identify the critical values.

Review Queue

Graph the following functions.

1. $y = 2x - 3$

2. $y = x^2 - 2x - 15$

3. $y = -2x^2 - x + 15$

Graphing

Objective

To graph basic rational functions.

Guidance

A **rational function** is in the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The parent graph for rational functions is $y = \frac{1}{x}$, and the shape is called a **hyperbola**.

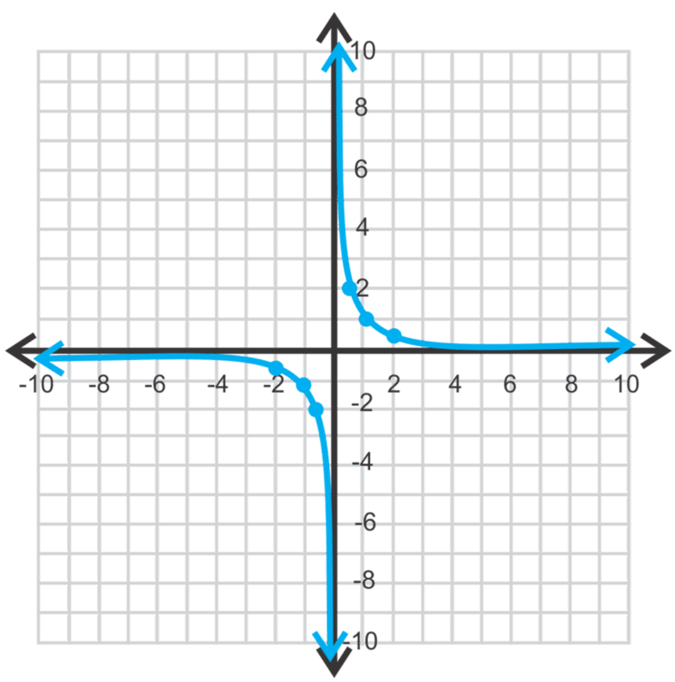


TABLE 3.15:

x	y
-4	$-\frac{1}{4}$

TABLE 3.15: (continued)

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2

TABLE 3.16:

x	y
4	$\frac{1}{4}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2

Notice the following properties of this hyperbola: the x-axis is a horizontal asymptote, the y-axis is a vertical asymptote, and the domain and range are all real numbers except where the asymptotes are. Recall that the vertical asymptote is the value that makes the denominator zero because we cannot divide by zero. For the horizontal asymptote, it is the value where the range is not defined.

The two parts of the graph are called **branches**. In the case with a hyperbola, the branches are always symmetrical about the point where the asymptotes intersect. In this example, they are symmetrical about the origin.

In this lesson, all the rational functions will have the form $f(x) = \frac{a}{x-h} + k$.

Example A

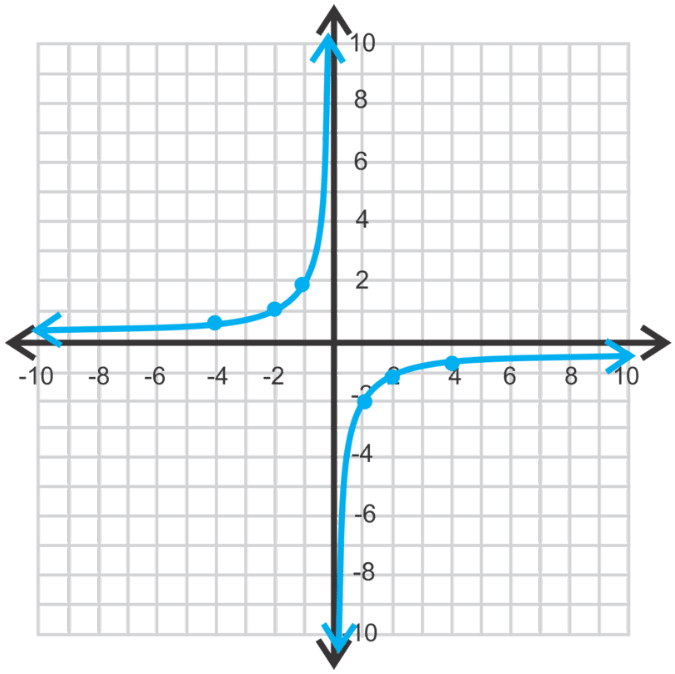
Graph $f(x) = \frac{-2}{x}$. Find any asymptotes, the domain, range, and any zeros.

Solution: Let's make a table of values.

TABLE 3.17:

x	y
1	-2
2	-1
4	$-\frac{1}{2}$

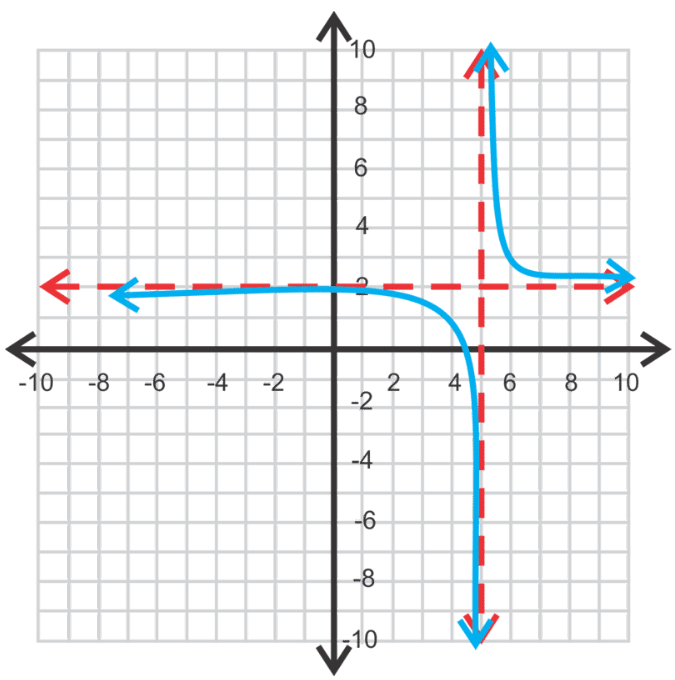
Notice that these branches are in the second and fourth quadrants. This is because of the negative sign in front of the 2, or a. The horizontal and vertical asymptotes are still the x and y-axes. There are no zeros, or x-intercepts, because the x-axis is an asymptote. The domain and range are all non-zero real numbers (all real numbers except zero).



Example B

Graph $y = \frac{1}{x-5} + 2$. Find all asymptotes, zeros, the domain and range.

Solution: For $y = \frac{1}{x-5} + 2$, the vertical asymptote is $x = 5$ because that would make the denominator zero and we cannot divide by zero. When $x = 5$, the value of the function would be $y = \frac{1}{0} + 2$, making the range undefined at $y = 2$. The shape and location of the branches are the same as the parent graph, just shifted to the right 5 units and up 2 units.



Therefore, for the general form of a rational function, $y = \frac{a}{x-h} + k, x = h$ is the vertical asymptote and $y = k$ is the horizontal asymptote.

The domain is all real numbers; $x \neq 5$ and the range is all real numbers; $y \neq 2$. To find the zero, set the function equal to zero and solve for x.

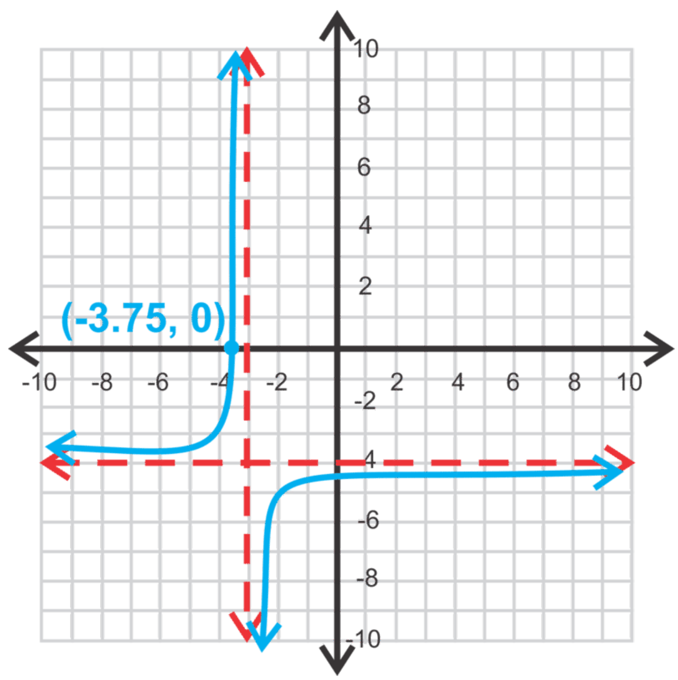
$$\begin{aligned} 0 &= \frac{1}{x-5}+2 \\ -2 &= \frac{1}{x-5} \\ -2x+10 &= 1 \\ -2x &= -9 \\ x &= \frac{9}{2} = 4.5 \end{aligned}$$

To find the y-intercept, set $x=0$, and solve for y. $y = \frac{1}{0-5}+2 = -\frac{1}{5}+2 = 1\frac{4}{5}$.

Example C

Find the equation of the hyperbola below.

Solution: We know that the numerator will be negative because the branches of this hyperbola are in the second and fourth quadrants. The asymptotes are $x=-3$ and $y=-4$. So far, we know $y=\frac{a}{x+3}-4$. In order to determine a, we can use the given x-intercept.



$$\begin{aligned} 0 &= \frac{a}{-3.75+3}-4 \\ 4 &= \frac{a}{-0.75} \\ -3 &= a \end{aligned}$$

The equation is $y=\frac{-3}{x+3}-4$

Guided Practice

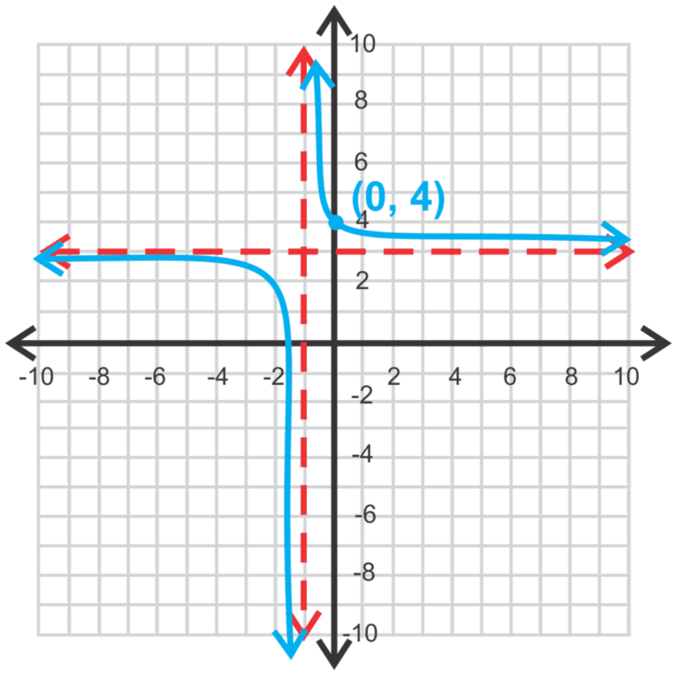
1. What are the asymptotes for $f(x)=\frac{-1}{x+6}+9$? Is $(-5,-8)$ on the graph?

Graph the following rational functions. Find the zero, y-intercept, asymptotes, domain and range.

2. $y=\frac{4}{x}-2$

3. $y = \frac{2}{x-1} + 3$

4. Determine the equation of the hyperbola.

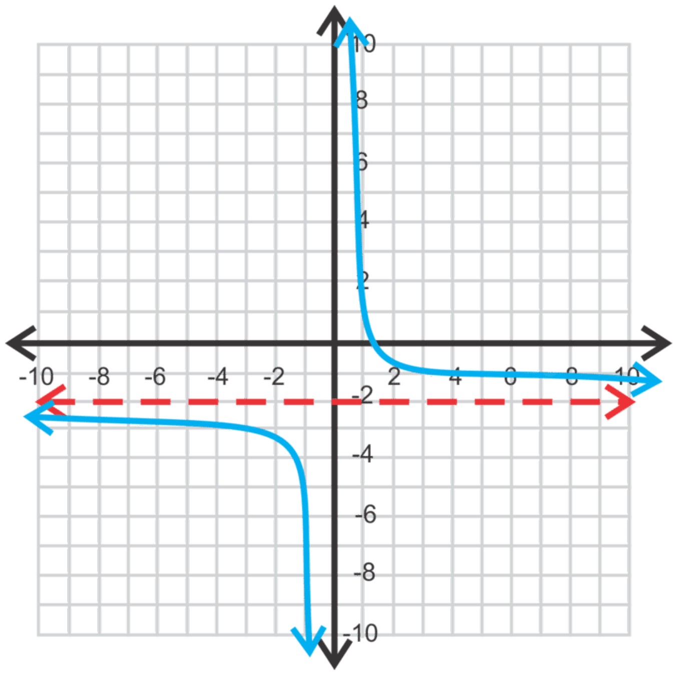


Answers

1. The asymptotes are $x = -6$ and $y = 9$. To see if the point $(-5, -8)$ is on the graph, substitute it in for x and y.

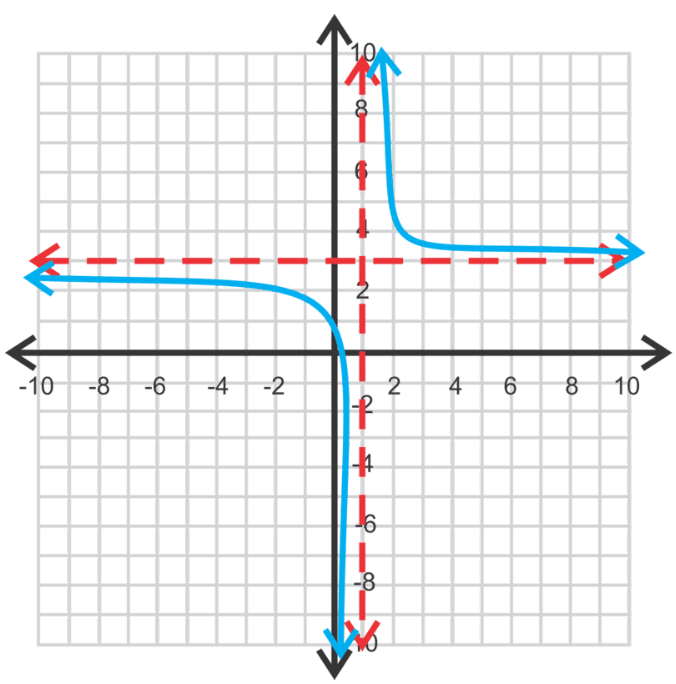
$$\begin{aligned}-8 &= \frac{-1}{-5+6} + 9 \\ -8 &= -1 + 9\end{aligned} \qquad -8 \neq 8, \text{ therefore, the point } (-5,-8) \text{ is not on the graph.}$$

2. There is no y-intercept because the y-axis is an asymptote. The other asymptote is $y = -2$. The domain is all real numbers; $x \neq 0$. The range is all real numbers; $y \neq -2$. The zero is:



$$\begin{aligned}
0 &= \frac{4}{x} - 2 \\
2 &= \frac{4}{x} \\
2x &= 4 \\
x &= 2
\end{aligned}$$

3. The asymptotes are $x = 1$ and $y = 3$. Therefore, the domain is all real numbers except 1 and the range is all real numbers except 3. The y-intercept is $y = \frac{2}{0-1} + 3 = -2 + 3 = 1$ and the zero is:



$$\begin{aligned}
0 &= \frac{2}{x-1}+3 \\
-3 &= \frac{2}{x-1} \\
-3x+3 &= 2 \\
-3x &= -1 \rightarrow x = \frac{1}{3}
\end{aligned}$$

4. The asymptotes are $x=-1, y=3$, making the equation $y=\frac{a}{x+1}+3$. Taking the y-intercept, we can solve for a.

$$\begin{aligned}
4 &= \frac{a}{0+1}+3 \\
1 &= a
\end{aligned}$$

The equation is $y=\frac{1}{x+1}+3$.

Vocabulary

Rational Function
A function in the form $\frac{p(x)}{q(x)}$, where p and q are both functions and $q \neq 0$.

Hyperbola
The shape of a rational function.

Branches
The two pieces of a hyperbola.

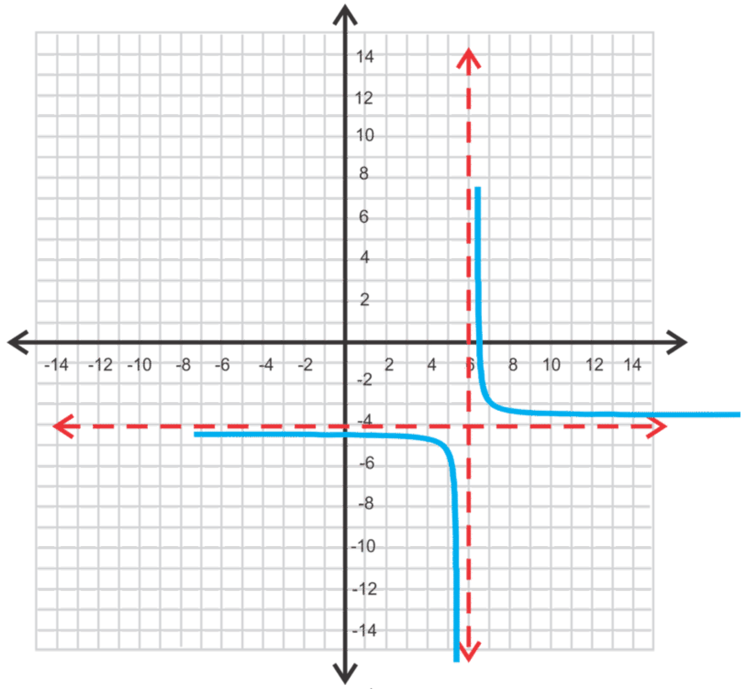
Problem Set

1. What are the asymptotes for $y=\frac{2}{x+8}-3$? Is $(-6,-2)$ a point on the graph?
2. What are the asymptotes for $y=6-\frac{1}{x-4}$? Is $(5,4)$ a point on the graph?

For problems 3-8, graph each rational function, state the equations of the asymptotes, the domain and range and the intercepts.

3. $y=\frac{3}{x}$
4. $y=\frac{2}{x+5}$
5. $y=\frac{1}{x-3}-4$
6. $y=\frac{2}{x+4}-3$
7. $y=\frac{5}{x}+2$
8. $y=3-\frac{1}{x+2}$

Write the equations of the hyperbolas.



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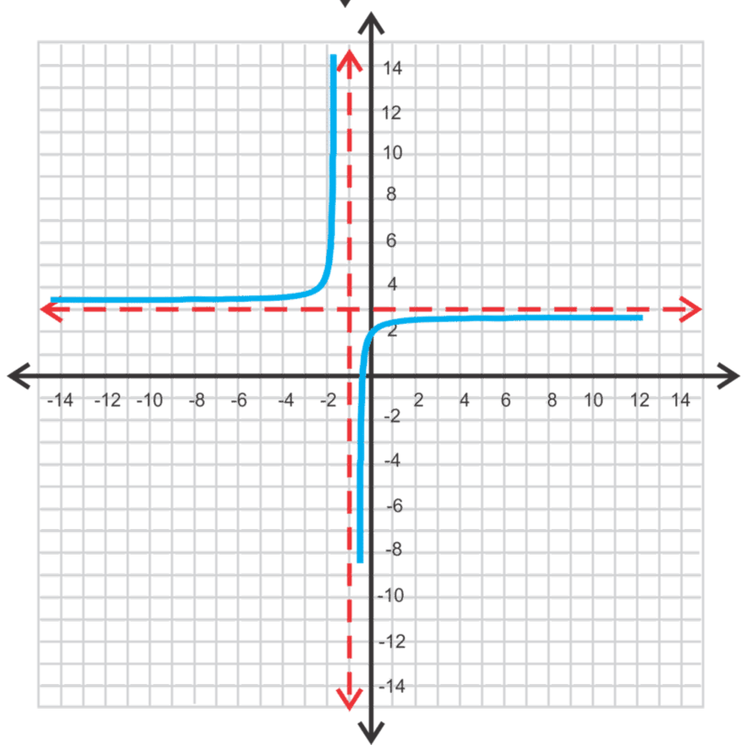


10.

Graphing

Objective

To graph rational functions when the numerator and denominator have the same degree.

Guidance

In the previous concept, we graphed functions in the form $y = \frac{1}{x-h} + k$, where $x = h$ and $y = k$ are the asymptotes. In this concept, we will extend graphing rational functions when both the denominator and numerator are linear or both quadratic. So, there will be no "k" term in this concept. Let's go through an example to determine any patterns in graphing this type of rational function.

Example A

Graph $f(x) = \frac{2x-1}{x+4}$. Find asymptotes, x and y intercepts, domain and range.

Solution: To find the vertical asymptote, it is the same as before, the value that makes the denominator zero. In this case, $x = -4$. Also the same is how to find the x and y intercepts.

y-intercept (when $x = 0$): $y = \frac{2 \cdot 0 - 1}{0+4} = -\frac{1}{4}$

x-intercept (when $y = 0$):

$$\begin{aligned} 0 &= \frac{2x-1}{x+4} \\ 0 &= 2x-1 \\ 1 &= 2x \\ \frac{1}{2} &= x \end{aligned}$$

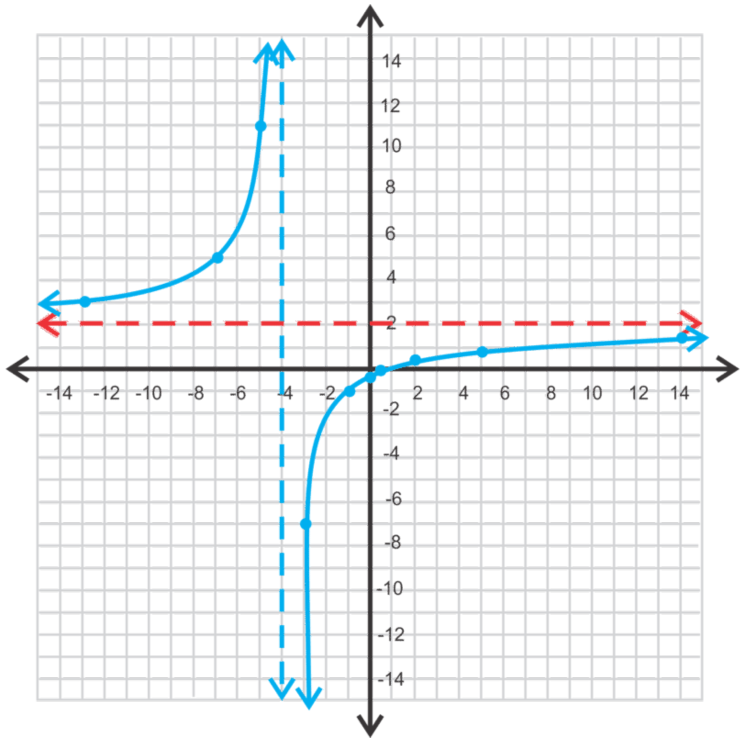
When solving for the x-intercept, to get the denominator out, we multiplied both sides by $x+4$. But, when we multiply anything by 0, it remains 0. Therefore, to find the x-intercept, we only need to set the numerator equal to zero and solve for x.

The last thing to find is the horizontal asymptote. We know that the function is positive, so the branches will be in the first and third quadrants. Let's make a table.

TABLE 3.18:

x	y
-13	3
-7	5
-5	11
-3	-7
-1	-1
0	-0.25
2	0.5
5	1
14	1.5

It looks like the horizontal asymptote is $y = 2$ because both branches seem to approach 2 as x gets larger, both positive and negative. If we plug in $x = 86, y = 1.9$ and when $x = -94, y = 2.1$. As you can see, even when x is very large, the function is still approaching 2.



Looking back at the original equation, $f(x) = \frac{2x-1}{x+4}$, extract the leading coefficients and leave them numerator over denominator, $\frac{2}{1}$. This is the horizontal asymptote. We can generalize this pattern for all rational functions. *When the degree of the numerator is <u>equal</u> to the degree of the denominator, the horizontal asymptote is the ratio of the leading coefficients.*

Finally, the domain is all real numbers; $x \neq -4$ and the range is all real numbers; $y \neq 2$.

Example B

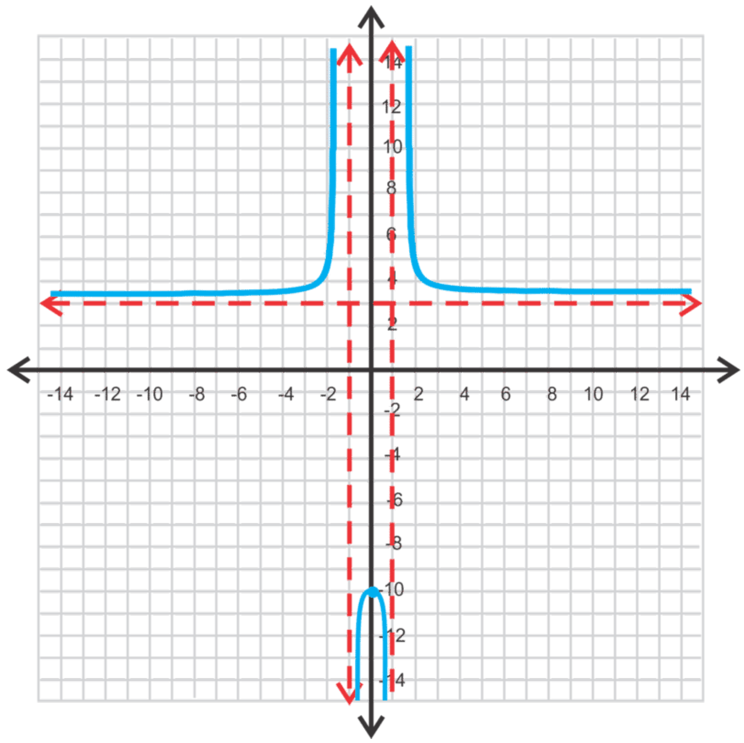
Graph $y = \frac{3x^2+10}{x^2-1}$. Find the asymptotes, intercepts, domain, and range.

Solution: From the previous example, we can conclude that the horizontal asymptote is at $y = 3$. Because the denominator is squared, there will be two vertical asymptotes because $x^2 - 1$ factors to $(x-1)(x+1)$. Therefore, the vertical asymptotes are $x = 1$ and $x = -1$. As for the intercepts, there are no x-intercepts because there is no real solution for $3x^2 + 10 = 0$. Solving for the y-intercept, we have $y = \frac{10}{-1} = -10$.

At this point, put the equation in your calculator to see the general shape. To graph this function using a TI-83 or 84, enter the function into $Y =$ like this: $\frac{(3x\hat{}2+10)}{(x\hat{}2-1)}$ and press GRAPH. You will need to expand the window to include the bottom portion of the graph. The final graph is to the left.

The domain is still all real numbers except the vertical asymptotes. For this function, that would be all real numbers; $x \neq -1, x \neq 1$.

The range is a bit harder to find. Notice the gap in the range from the horizontal asymptote and the y-intercept. Therefore, the range is $(-\infty, -10] \cup (3, \infty)$.



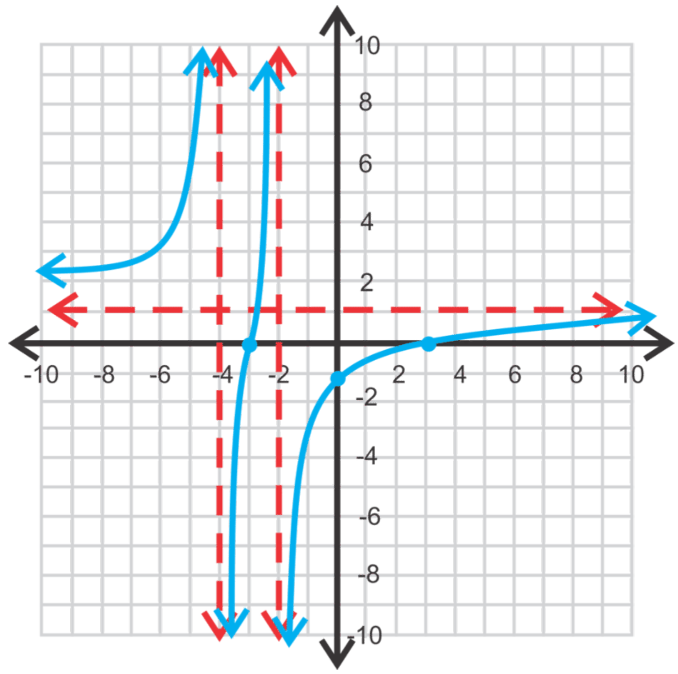
The notation above is one way to write a range of numbers called **interval notation** and was introduced in the *Finding the Domain and Range of Functions* concept. The $\cup$ symbol means "union." Notice that $-\infty$ and ∞ are not included in the range.

In general, rational functions with quadratics in the denominator are split into six regions and have branches in three of them, like the example above. However, there are cases when there are no zeros or vertical asymptotes and those look very different. You should always graph the function in a graphing calculator after you find the critical values and make as accurate a sketch as you can.

Example C

Graph $y = \frac{x^2-9}{x^2+6x+8}$ in your graphing calculator. Find all asymptotes, intercepts, the domain and range.

Solution: The y-intercept is $y = -\frac{9}{8}$ and the x-intercepts are $0 = x^2 - 9 \rightarrow x = 3, -3$. The horizontal asymptote is $y = 1$ and when we factor and solve the denominator we get the vertical asymptotes; $x^2 + 6x + 8 = 0 \rightarrow x = -4, -2$.



At this point, use the graphing calculator to determine the orientation of the function. Notice that the middle section curves up like an x^3 function. This is because the zero is between the two vertical asymptotes.

Notice that the middle section passes through the horizontal asymptote. Let's see what happens at $y = 1$.

$$\begin{aligned}
1 &= \frac{x^2-9}{x^2+6x+8} \\
x^2+6x+8 &= x^2-9 \\
6x &= -17 \\
x &= -\frac{17}{6} \approx -2.83
\end{aligned}$$

Because we found a value for x, this means the range is all real numbers. The domain is $x \in \mathbb{R}; x \neq 3, -3$.

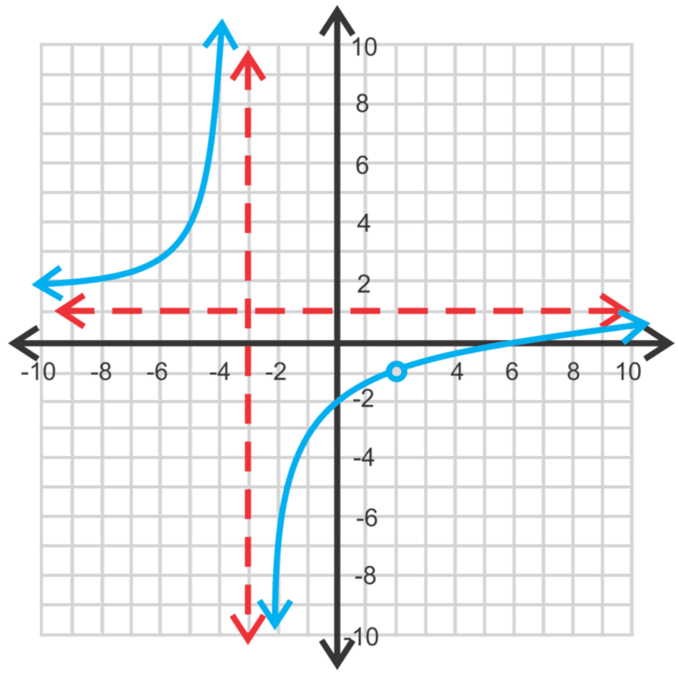
Example D

Graph $f(x) = \frac{x^2-8x+12}{x^2-x-6}$. Find the intercepts, asymptotes, domain and range.

Solution: Let's factor the numerator and denominator to find the intercepts and vertical asymptotes.

$f(x) = \frac{x^2-8x+12}{x^2+x-6} = \frac{(x-6)(x-2)}{(x+3)(x-2)}$

Notice that the numerator and denominator both have a factor of $(x-2)$. When this happens, a **hole** is created because $x = 2$ is both a zero and an asymptote. Therefore, $x = 2$ is a hole and neither a zero nor an asymptote.



There is a vertical asymptote at $x = -3$ and a zero at $x = 6$. The horizontal asymptote is at $y = 1$. The graph of $f(x) = \frac{x^2-8x+12}{x^2-x-6}$ will look like the graph of $f(x) = \frac{x-6}{x+3}$, but with a hole at $x = 2$. A hole is not part of the domain. And, the output value that corresponds with the hole is not part of the range. In this example, $f(2) = \frac{2-6}{2+3} = \frac{-4}{6} = -\frac{2}{3}$ is not part of the range.

The domain is $x \in \mathbb{R}; x \neq 2, -3$ and the range is $y \in \mathbb{R}; y \neq 1, -\frac{2}{3}$.

Guided Practice

Graph the following functions. Find all intercepts, asymptotes, the domain and range. Double-check your answers with a graphing calculator.

1. $y = \frac{4x-5}{2x+7}$

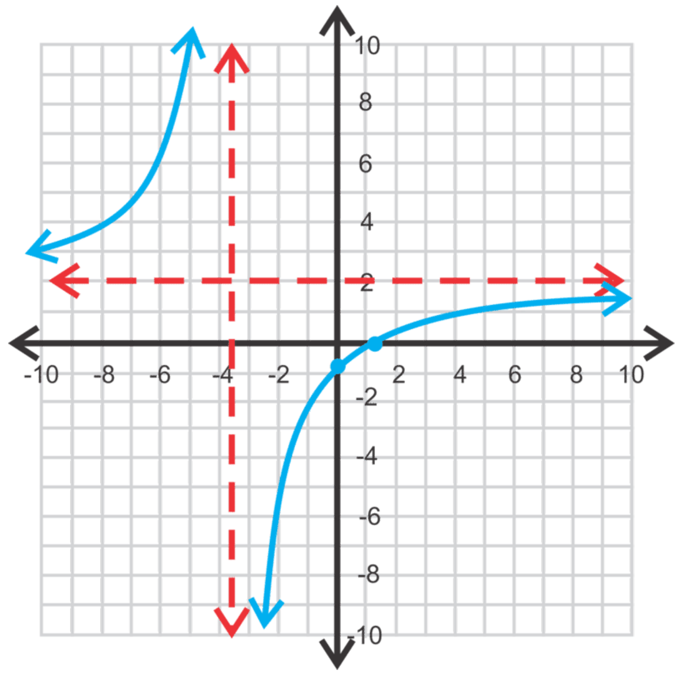
2. $f(x) = \frac{x^2-9}{x^2+1}$

3. $y = \frac{2x^2+7x+3}{x^2+3x+2}$

4. $y = \frac{x^2-4}{2x^2-5x+2}$

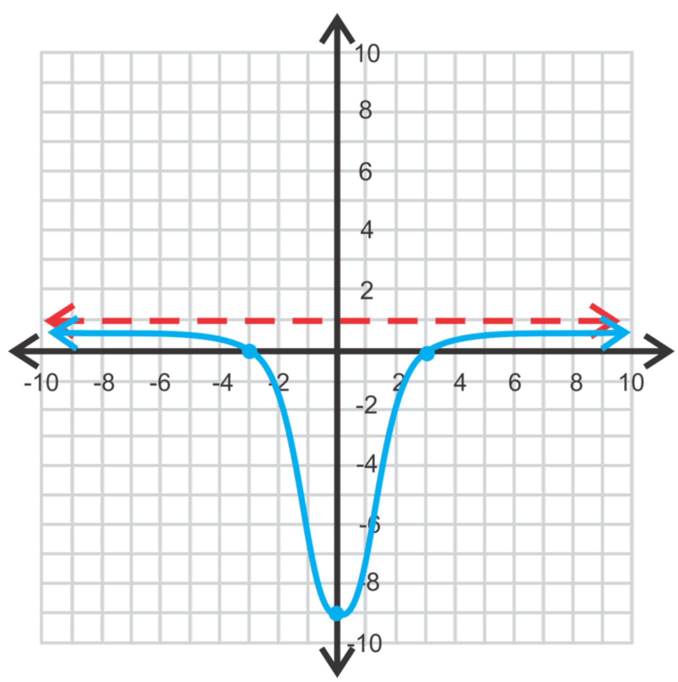
Answers

1. y-intercept: $y = \frac{-5}{7} = -\frac{5}{7}$, x-intercept: $0 = 4x-5 \rightarrow x = \frac{5}{4}$, horizontal asymptote: $y = \frac{4}{2} = 2$, vertical asymptote: $2x+7 = 0 \rightarrow x = -\frac{7}{2}$, domain: $\mathbb{R}; x \neq -\frac{7}{2}$, range: $\mathbb{R}; y \neq 2$



2. y-intercept: $y = \frac{-9}{1} = -9$, x-intercepts: $0 = x^2 - 9 \rightarrow x = \pm 3$, horizontal asymptote: $y = 1$, vertical asymptote: none, domain: $\mathbb{R}$, range: $\mathbb{R}; y \neq 1$

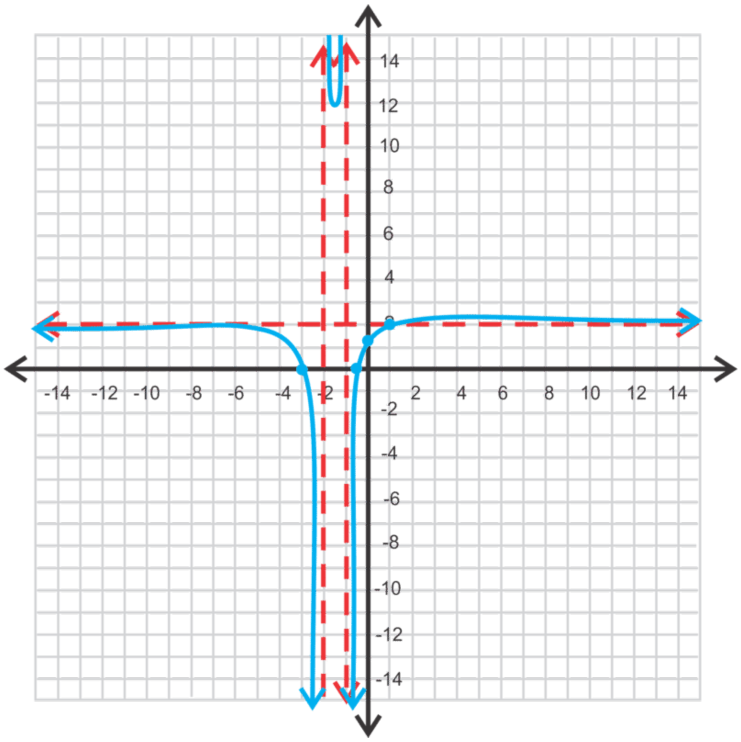
* Special Note: When there are no vertical asymptotes and the numerator and denominator are both quadratics, this is the general shape. It could also be reflected over the horizontal asymptote.



3. y-intercept: $\left(0, \frac{3}{2}\right)$, x-intercepts: $(-3,0)$ and $\left(-\frac{1}{2}, 0\right)$, horizontal asymptote: $y = 2$, vertical asymptotes: $x = -2, x = -1$.

domain: $\mathbb{R}; x \neq -1, -2$

range: $y \in (-\infty, 2.1] \cup [12, \infty)$



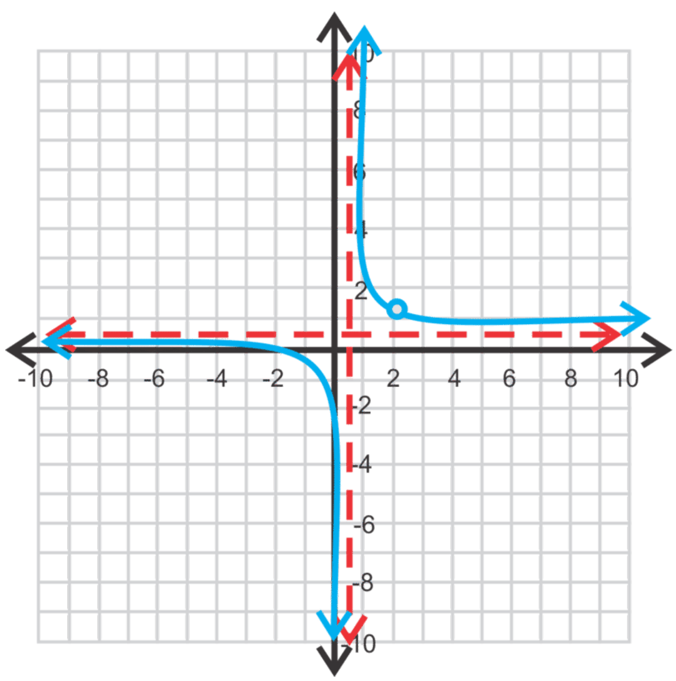
4. horizontal asymptote: $y = \frac{1}{2}$, y-intercept: $(0, -2)$

vertical asymptotes: $x = \frac{1}{2}$, x-intercept: $(-2, 0)$

hole: $x = 2, f(2) = \frac{4}{3}$

domain: $\mathbb{R}; x \neq \frac{1}{2}, 2$

range: $\mathbb{R}; y \neq \frac{1}{2}, \frac{4}{3}$



Vocabulary

Degree

The largest exponent in a polynomial.

Interval Notation

One way to write the domain or range of a function. [and] include the endpoint(s) of the interval and (and) do not. The $\cup$ symbol is used to join two intervals of a domain or range.

Hole

An input value that is a vertical asymptote and a zero. It is not considered part of the domain. An important note, the graphing calculator will not show a hole in the picture.

Problem Set

Graph the following rational functions. Write down the equations of the asymptotes, the domain and range, x and y intercepts and identify any holes.

1. $y = \frac{x+3}{x-5}$
2. $y = \frac{5x+2}{x-4}$
3. $y = \frac{3-x}{2x+10}$
4. $y = \frac{x^2+5x+6}{x^2-8x+12}$
5. $y = \frac{x^2+4}{2x^2+x-3}$
6. $y = \frac{2x^2-x-10}{3x^2+10x+8}$
7. $y = \frac{x^2-4}{x^2+3x-10}$
8. $y = \frac{6x^2-7x-3}{4x^2-1}$
9. $y = \frac{x^3-8}{x^3+x^2-4x-4}$
10. Graph $y = \frac{1}{x-2} + 3$ and $y = \frac{3x-5}{x-2}$ on the same set of axes. Compare the two. What do you notice? Explain your results.

Graphing

Objective

To learn how to graph rational functions where the degrees of the numerator and denominator are not the same.

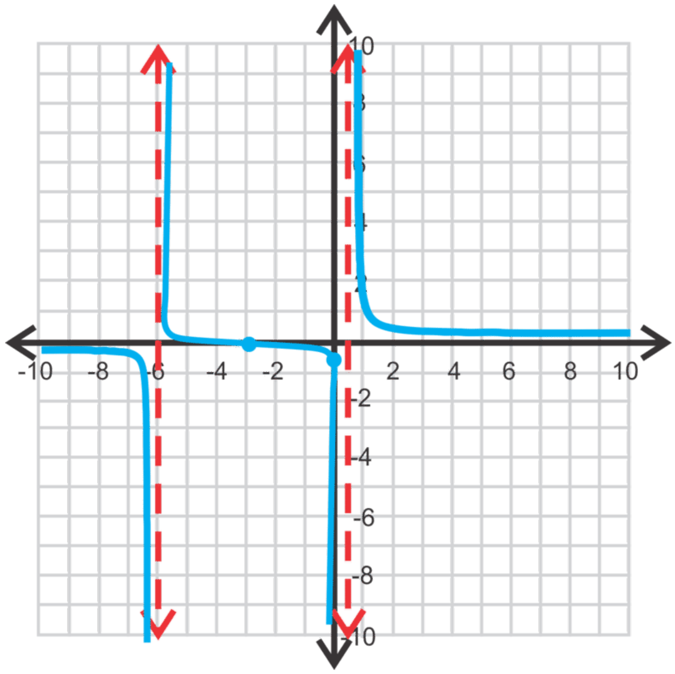
Guidance

In this concept we will touch on the different possibilities for the remaining types of rational functions. You will need to use your graphing calculator throughout this concept to ensure your sketches are correct.

Example A

Graph $y = \frac{x+3}{2x^2+11x-6}$. Find all asymptotes, intercepts, the domain and range.

Solution: In this example *the degree of the numerator is less than the degree of the denominator.* Whenever this happens the horizontal asymptote will be $y = 0$, or the x-axis. Now, even though the x-axis is the horizontal asymptote, there will still be a zero at $x = -3$ (solving the numerator for x and setting it equal to zero). The vertical asymptotes will be the solutions to $2x^2 + 11x - 6 = 0$. Factoring this quadratic, we have $(2x-1)(x+6) = 0$ and the solutions are $x = \frac{1}{2}$ and -6. The y-intercept is $\left(0, -\frac{1}{2}\right)$. At this point, we can plug our function into the graphing calculator to get the general shape.



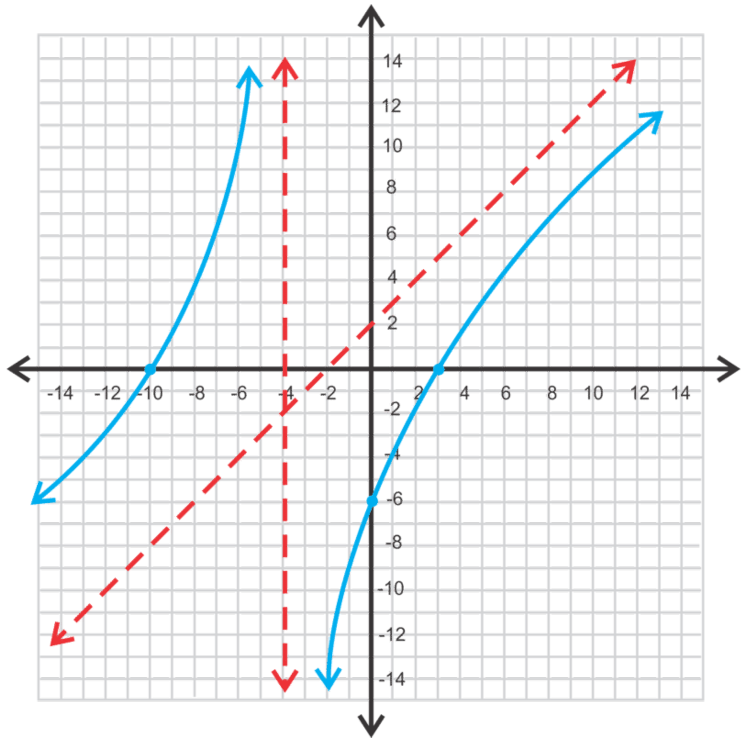
Because the middle portion crosses over the horizontal asymptote, the range will be all real numbers. The domain is $x \in \mathbb{R}; x = -6; x \neq \frac{1}{2}$.

Be careful when graphing any rational function. This function does not look like the graph to the left in a TI-83/84. This is because the calculator does not have the ability to draw the asymptotes separately and wants to make the function continuous. Make sure to double-check the table ($2^{\text{nd}} \rightarrow$ GRAPH) to find where the function is undefined.

Example B

Graph $f(x) = \frac{x^2+7x-30}{x+5}$. Find all asymptotes, intercepts, the domain and range.

Solution: In this example *the degree of the numerator is greater than the degree of the denominator*. When this happens, there is no horizontal asymptote. Instead there is a **slant asymptote**. Recall that this function represents division. If we were to divide $x^2+7x-30$ by $x+5$, the answer would be $x+2-\frac{20}{x+5}$. The slant asymptote would be the answer, minus the remainder. Therefore,for this problem the slant asymptote is $y = x+2$. Everything else is the same. The y-intercept is $\frac{-30}{5} \rightarrow (0,-6)$ and the x-intercepts are the solutions to the numerator, $x^2+7x-30 = 0 \rightarrow (x+10)(x-3) \rightarrow x = -10, 3$. There is a vertical asymptote at $x = -5$. At this point, you can either test a few points to see where the branches are or use your graphing calculator.

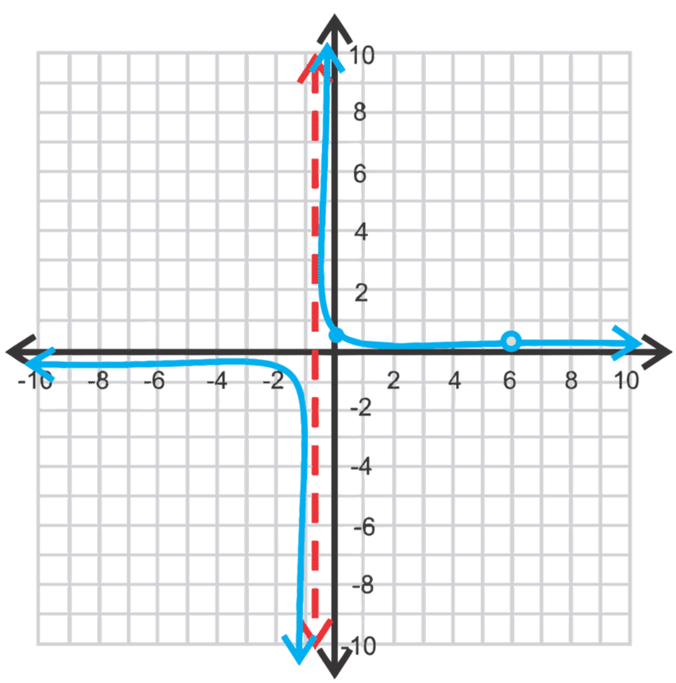


The domain would be all real numbers; $x \neq -5$. Because of the slant asymptote, there are no restrictions on the range. It is all real numbers.

Example C

Graph $y = \frac{x-6}{3x^2-16x-12}$. Find the asymptotes and intercepts.

Solution: Because the degree of the numerator is less than the degree of the denominator, there will be a horizontal asymptote along the x-axis. Next, let's find the vertical asymptotes by factoring the denominator; $(x-6)(3x+2)$. Notice that the denominator has a factor of $(x-6)$, which is the entirety of the numerator. That means there will be a *hole* at $x = 6$.



Therefore, the graph of $y = \frac{x-6}{3x^2-16x-12}$ will be the same as $y = \frac{1}{3x+2}$ except with a hole at $x = 6$. There is no x-intercept, the vertical asymptote is at $x = -\frac{2}{3}$ and the y-intercept is $\left(0, \frac{1}{2}\right)$.

Recap

For a rational function; $f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + \ldots + a_0}{b_n x^n + \ldots + b_0}$

1. If $m < n$, then there is a horizontal asymptote at $y = 0$.
2. If $m = n$, then there is a horizontal asymptote at $y = \frac{a_m}{b_n}$ (ratio of the leading coefficients).
3. If $m > n$, then there is a slant asymptote at $y = (a_m x^m + \ldots + a_0) \div (b_n x^n + \ldots + b_0)$ without the remainder. In this concept, we will only have functions where m is one greater than n.

Guided Practice

Graph the following functions. Find any intercepts and asymptotes.

1. $y = \frac{3x+5}{2x^2+9x+20}$

2. $f(x) = \frac{x^2+4x+4}{x^2-3x-4}$

3. $g(x) = \frac{x^2-16}{x+3}$

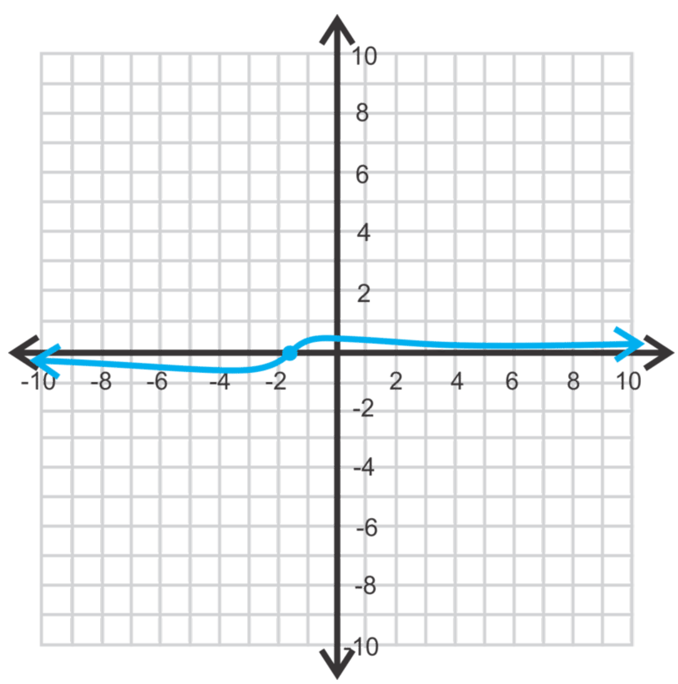
4. $y = \frac{2x+3}{6x^2-x-15}$

Answers

1. x-intercept: $\left(-\frac{5}{3}, 0\right)$, y-intercept: $\left(0, \frac{1}{4}\right)$

horizontal asymptote: $y = 0$

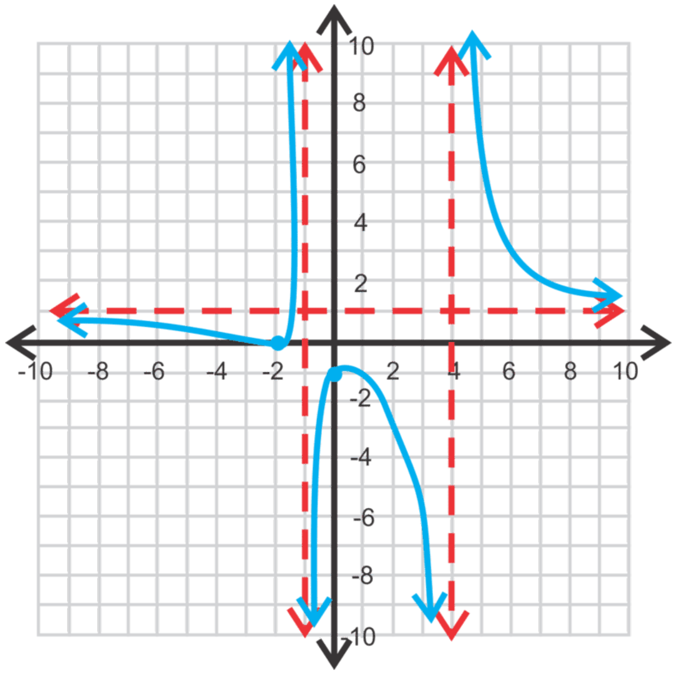
vertical asymptotes: none



2. x-intercept: $(-2, 0)$, y-intercept: $(0, -1)$

horizontal asymptote: $y = 1$

vertical asymptotes: $x = 4$ and $x = -1$

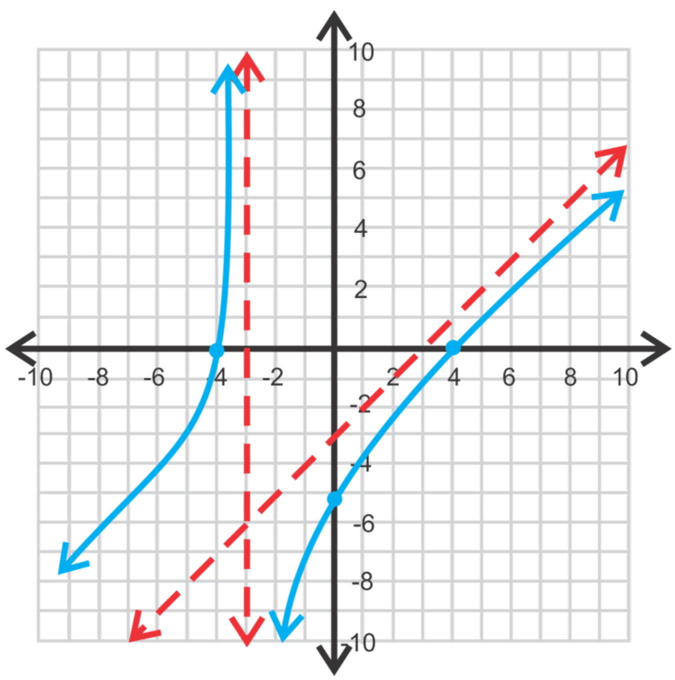


3. x-intercepts: $(-4,0)$ and $(4,0)$

y-intercept: $\left(0,-\frac{16}{3}\right)$

slant asymptote: $y=x-3$

vertical asymptotes: $x=-3$

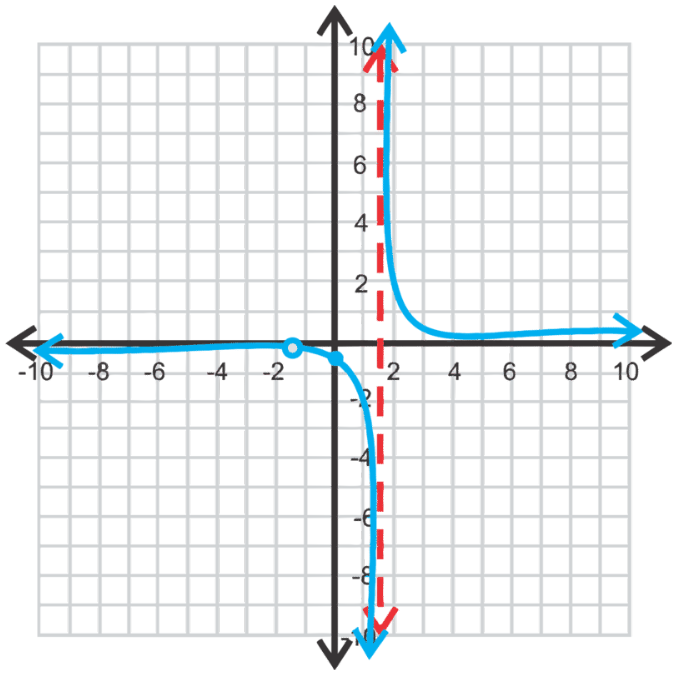


4. x-intercepts: none, hole at $x=-\frac{3}{2}$

y-intercept: $\left(0,-\frac{1}{5}\right)$

horizontal asymptote: $y=0$

vertical asymptote: $x=\frac{5}{3}$



Vocabulary

Slant Asymptote
In a rational function, when the degree of the numerator is greater than the degree of the denominator a slant asymptote is produced instead of a horizontal one. It is the result of long division of the function, without the remainder.

Problem Set

Graph the following functions. Find any intercepts, asymptote and holes.

1. $y = \frac{x+1}{x^2-x-12}$
2. $f(x) = \frac{x^2+3x-10}{x-3}$
3. $y = \frac{x-7}{2x^2-11x-21}$
4. $g(x) = \frac{2x^2-2}{3x+5}$
5. $y = \frac{x^2+x-30}{x+6}$
6. $f(x) = \frac{x^2+x-30}{2x^3-5x^2-4x+3}$
7. $y = \frac{x^3-2x^2-3x}{x^2-5x+6}$
8. $f(x) = \frac{2x+5}{x^2+5x-6}$
9. $g(x) = \frac{-x^2+3x+4}{2x-6}$
10. Determine the slant asymptote of $y = \frac{3x^2-x-10}{3x+5}$. Now, graph this function. Is there really a slant asymptote? Can you explain your results?

3.24 Simplifying, Multiplying, and Dividing Rational Expressions

Objective

To simplify, multiply, and divide rational expressions.

Review Queue

Simplify the following fractions.

1. $\frac{8}{20}$
2. $\frac{6x^3y^2}{9xy^5}$
3. $\frac{7a^5bc^2}{35ab^4c^9}$

Multiply or divide the following fractions.

4. $\frac{4}{5} \cdot \frac{10}{18}$
5. $\frac{2}{3} \div \frac{1}{4}$
6. $\frac{12}{5} \div \frac{3}{10}$

Simplifying Rational Expressions

Objective

To simplify rational expressions involving factorable polynomials.

Guidance

Recall that a rational function is a function, $f(x)$, such that $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials. A **rational expression**, is just $\frac{p(x)}{q(x)}$. Like any fraction, a rational expression can be simplified. To simplify a rational expression, you will need to factor the polynomials, determine if any factors are the same, and then cancel out any like factors.

Fraction: $\frac{9}{15} = \frac{\cancel{3} \cdot 3}{\cancel{3} \cdot 5} = \frac{3}{5}$

Rational Expression: $\frac{x^2+6x+9}{x^2+8x+15} = \frac{\cancel{(x+3)}(x+3)}{\cancel{(x+3)}(x+5)} = \frac{x+3}{x+5}$

With both fractions, we broke apart the numerator and denominator into the prime factorization. Then, we canceled the common factors.

Important Note: $\frac{x+3}{x+5}$ is completely factored. ***Do not*** cancel out the x's! $\frac{3x}{5x}$ reduces to $\frac{3}{5}$, but $\frac{x+3}{x+5}$ does not because of the addition sign. To prove this, we will plug in a number for x to and show that the fraction does not reduce to $\frac{3}{5}$. If $x = 2$, then $\frac{2+3}{2+5} = \frac{5}{7} \neq \frac{3}{5}$.

Example A

Simplify $\frac{2x^3}{4x^2-6x}$.

Solution: The numerator factors to be $2x^3 = 2 \cdot x \cdot x \cdot x$ and the denominator is $4x^2 - 6x = 2x(2x-3)$.

$\frac{2x^3}{4x^2-6x} = \frac{\cancel{2} \cdot \cancel{x} \cdot x \cdot x}{\cancel{2} \cdot \cancel{x} \cdot (2x-3)} = \frac{x^2}{2x-3}$

Example B

Simplify $\frac{6x^2-7x-3}{2x^3-3x^2}$.

Solution: If you need to review factoring, see the *Factoring Quadratics when the Leading Coefficient is 1* concept and the *Factoring Quadratics when the Leading Coefficient is not 1* concept. Otherwise, factor the numerator and find the GCF of the denominator and cancel out the like terms.

$\frac{6x^2-7x-3}{2x^3-3x^2} = \frac{\cancel{(2x-3)}(3x+1)}{x^2\cancel{(2x-3)}} = \frac{3x+1}{x^2}$

Example C

Simplify $\frac{x^2-6x+27}{2x^2-19x+9}$.

Solution: Factor both the top and bottom and see if there are any common factors.

$\frac{x^2-6x+27}{2x^2-19x+9} = \frac{\cancel{(x-9)}(x+3)}{\cancel{(x-9)}(2x-1)} = \frac{x+3}{2x-1}$

Special Note: Not every polynomial in a rational function will be factorable. Sometimes there are no common factors. When this happens, write "not factorable."

Guided Practice

If possible, simplify the following rational functions.

1. $\frac{3x^2-x}{3x^2}$
2. $\frac{x^2+6x+8}{x^2+6x+9}$
3. $\frac{2x^2+x-10}{6x^2+17x+5}$
4. $\frac{x^3-4x}{x^5+4x^3-32x}$

Answers

1. $\frac{3x^2-x}{3x^2} = \frac{\cancel{x}(3x-1)}{3\cdot\cancel{x}\cdot x} = \frac{3x-1}{3x}$
2. $\frac{x^2+6x+8}{x^2+6x+9} = \frac{(x+4)(x+2)}{(x+3)(x+3)}$ There are no common factors, so this is reduced.
3. $\frac{2x^2+x-10}{6x^2+17x+5} = \frac{\cancel{(2x+5)}(x-2)}{\cancel{(2x+5)}(3x+1)} = \frac{x-2}{3x+1}$
4. In this problem, the denominator will factor like a quadratic once an x is pulled out of each term.

$\frac{x^3-4x}{x^5+4x^3-32x} = \frac{x(x^2-4)}{x(x^4+4x^2-32)} = \frac{x(x-2)(x+2)}{x(x^2-4)(x^2+8)} = \frac{\cancel{x(x-2)(x+2)}}{\cancel{x(x-2)(x+2)}(x^2+8)} = \frac{1}{x^2+8}$

Vocabulary

Rational Expression
A fraction with polynomials in the numerator and denominator.

Problem Set

Simplify the following Rational Expressions.

1. $\frac{4x^3}{2x^2+3x}$
2. $\frac{x^3+x^2-2x}{x^4+4x^3-5x^2}$
3. $\frac{2x^2-5x-3}{2x^2-7x-4}$
4. $\frac{5x^2+37x+14}{5x^3-33x^2-14x}$
5. $\frac{8x^2-60x-32}{-4x^2+26x+48}$
6. $\frac{6x^3-24x^2+30x-120}{9x^4+36x^2-45}$

7. $\frac{6x^2+5x-4}{6x^2-x-1}$
8. $\frac{x^4+8x}{x^4-2x^3+4x^2}$
9. $\frac{6x^4-3x^3-63x^2}{12x^2-84x}$
10. $\frac{x^5-3x^3-4x}{x^4+2x^3+x^2+2x}$
11. $\frac{-3x^2+25x-8}{x^3-8x^2+x-8}$
12. $\frac{-x^3+3x^2+13x-15}{-2x^3+7x^2+20x-25}$

Multiplying Rational Expressions

Objective

To multiply together two or more rational expressions and simplify.

Guidance

We take the previous concept one step further in this one and multiply two rational expressions together. When multiplying rational expressions, it is just like multiplying fractions. However, it is much, much easier to factor the rational expressions before multiplying because factors could cancel out.

Example A

Multiply $\frac{x^2-4x}{x^3-9x} \cdot \frac{x^2+8x+15}{x^2-2x-8}$

Solution: Rather than multiply together each numerator and denominator to get very complicated polynomials, it is much easier to first factor and then cancel out any common factors.

$\frac{x^2-4x}{x^3-9x} \cdot \frac{x^2+8x+15}{x^2-2x-8} = \frac{x(x-4)}{x(x-3)(x+3)} \cdot \frac{(x+3)(x+5)}{(x+2)(x-4)}$

At this point, we see there are common factors between the fractions.

$\frac{\cancel{x}\cancel{(x-4)}}{\cancel{x}(x-3)\cancel{(x+3)}} \cdot \frac{\cancel{(x+3)}(x+5)}{(x+2)\cancel{(x-4)}} = \frac{x+5}{(x-3)(x+2)}$

At this point, the answer is in **factored form** and simplified. You do not need to multiply out the base.

Example B

Multiply $\frac{4x^2y^5z}{6xyz^6} \cdot \frac{15y^4}{35x^4}$

Solution: These rational expressions are monomials with more than one variable. Here, we need to remember the laws of exponents from earlier concepts. Remember to add the exponents when multiplying and subtract the exponents when dividing. The easiest way to solve this type of problem is to multiply the two fractions together first and then subtract common exponents.

$\frac{4x^2y^5z}{6xyz^6} \cdot \frac{15y^4}{35x^4} = \frac{60x^2y^9z}{210x^5yz^6} = \frac{2y^8}{7x^3z^5}$

You can reverse the order and cancel any common exponents first and then multiply, but sometimes that can get confusing.

Example C

Multiply $\frac{4x^2+4x+1}{2x^2-9x-5} \cdot (3x-2) \cdot \frac{x^2-25}{6x^2-x-2}$

Solution: Because the middle term is a linear expression, rewrite it over 1 to make it a fraction.

$\frac{4x^2+4x+1}{2x^2-9x-5} \cdot (3x-2) \cdot \frac{x^2-25}{6x^2-x-2} = \frac{\cancel{(2x+1)}\cancel{(2x+1)}}{\cancel{(2x+1)}\cancel{(x-5)}} \cdot \frac{\cancel{3x-2}}{1} \cdot \frac{\cancel{(x-5)}(x+5)}{\cancel{(3x-2)}\cancel{(2x+1)}} = x+5$

Guided Practice

Multiply the following expressions.

1. $\frac{4x^2-8x}{10x^3} \cdot \frac{15x^2-5x}{x-2}$
2. $\frac{x^2+6x-7}{x^2-36} \cdot \frac{x^2-2x-24}{2x^2+8x-42}$
3. $\frac{4x^2y^7}{32x^4y^3} \cdot \frac{16x^2}{8y^6}$

Answers

1. $\frac{4x^2-8x}{10x^3} \cdot \frac{15x^2-5x}{x-2} = \frac{\cancel{2} \cdot 2\cancel{x}\cancel{(x-2)}}{\cancel{2} \cdot \cancel{5x} \cdot \cancel{x} \cdot x} \cdot \frac{\cancel{5x}(3x-1)}{\cancel{x-2}} = \frac{2(3x-1)}{x}$
2. $\frac{x^2+6x-7}{x^2-36} \cdot \frac{x^2-2x-24}{2x^2+8x-42} = \frac{\cancel{(x+7)}(x-1)}{\cancel{(x-6)}(x+6)} \cdot \frac{\cancel{(x-6)}(x+4)}{2\cancel{(x+7)}(x-3)} = \frac{(x-1)(x+4)}{2(x-3)(x+6)}$
3. $\frac{4x^2y^7}{32x^4y^3} \cdot \frac{16x^2}{8y^6} = \frac{64x^4y^7}{256x^4y^9} = \frac{1}{4y^2}$

Problem Set

Multiply the following expressions. Simplify your answers.

1. $\frac{8x^2y^3}{5x^3y} \cdot \frac{15xy^8}{2x^3y^5}$
2. $\frac{11x^3y^9}{2x^4} \cdot \frac{6x^7y^2}{33xy^3}$
3. $\frac{18x^3y^6}{13x^8y^2} \cdot \frac{39x^{12}y^5}{9x^2y^9}$
4. $\frac{3x+3}{y-3} \cdot \frac{y^2-y-6}{2x+2}$
5. $\frac{6}{2x+3} \cdot \frac{4x^2+4x-3}{3x+3}$
6. $\frac{6+x}{2x-1} \cdot \frac{x^2+5x-3}{x^2+5x-6}$
7. $\frac{3x-21}{x-3} \cdot \frac{-x^2+x+6}{x^2-5x-14}$
8. $\frac{6x^2+5x+1}{8x^2-2x-3} \cdot \frac{4x^2+28x-30}{6x^2-7x-3}$
9. $\frac{x^2+9x-36}{x^2-9} \cdot \frac{x^2+8x+15}{-x^2+11x+12}$
10. $\frac{2x^2+x-21}{x^2+2x-48} \cdot (4-x) \cdot \frac{2x^2-9x-18}{2x^2-x-28}$
11. $\frac{8x^2-10x-3}{4x^3+x^2-36x-9} \cdot \frac{5x+3}{x-1} \cdot \frac{x^3+3x^2-x-3}{5x^2+8x+3}$

Dividing Rational Expressions

Objective

To divide two or more rational expressions.

Guidance

Dividing rational expressions has one additional step than multiply them. Recall that when you divide fractions, you need to flip the second fraction and change the problem to multiplication. The same rule applies to dividing rational expressions.

Example A

Divide $\frac{5a^3b^4}{12ab^8} \div \frac{15b^6}{8a^6}$.

Solution: Flip the second fraction, change the $\div$ sign to multiplication and solve.

$\frac{5a^3b^4}{12ab^8} \div \frac{15b^6}{8a^6} = \frac{5a^3b^4}{12ab^8} \cdot \frac{8a^6}{15b^6} = \frac{40a^9b^4}{180ab^{14}} = \frac{2a^8}{9b^{10}}$

Example B

Divide $\frac{x^4-3x^2-4}{2x^2+x-10} \div \frac{x^3-3x^2+x-3}{x-2}$

Solution: Flip the second fraction, change the $\div$ sign to multiplication and solve.

$$\begin{aligned}
\frac{x^4-3x^2-4}{2x^2+x-10} \div \frac{x^3-3x^2+x-3}{x-2} &= \frac{x^4-3x^2-4}{2x^2+x-10} \cdot \frac{x-2}{x^3-3x^2+x-3} \\
&= \frac{(x^2-4)(x^2+1)}{(2x-5)(x+2)} \cdot \frac{x-2}{(x^2+1)(x-3)} \\
&= \frac{(x-2)\cancel{(x+2)}\cancel{(x^2+1)}}{(2x-5)\cancel{(x+2)}} \cdot \frac{x-2}{\cancel{(x^2+1)}(x-3)} \\
&= \frac{(x-2)^2}{(2x-5)(x-3)}
\end{aligned}$$

Review the *Factoring by Grouping* concept to factor the blue polynomial and the *Factoring in Quadratic Form* concept to factor the red polynomial.

Example C

Perform the indicated operations: $\frac{x^3-8}{x^2-6x+9} \div (x^2+3x-10) \cdot \frac{x^2+x-12}{x^2+11x+30}$

Solution: Flip the second term, factor, and cancel. The blue polynomial is a difference of cubes. Review the *Sum and Difference of Cubes* concept for how to factor this polynomial.

$$\begin{aligned}
\frac{x^3-8}{x^2-6x+9} \div (x^2+3x-10) \cdot \frac{x^2+x-12}{x^2+11x+30} &= \frac{x^3-8}{x^2-6x+9} \cdot \frac{1}{x^2+3x-10} \cdot \frac{x^2+2x-15}{x^2+11x+30} \\
&= \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-3)}(x-3)} \cdot \frac{1}{\cancel{(x-2)}\cancel{(x+5)}} \cdot \frac{\cancel{(x+5)}\cancel{(x-3)}}{(x+5)(x+6)} \\
&= \frac{x^2+2x+4}{(x-3)(x+5)(x+6)}
\end{aligned}$$

Guided Practice

Perform the indicated operations.

1. $\frac{a^5b^3c}{6a^2c^9} \div \frac{2a^7b^{11}}{24c^2}$
2. $\frac{x^2+12x-45}{x^2-5x+6} \div \frac{x^2+17x+30}{x^4-16}$
3. $(x^3+2x^2-9x-18) \div \frac{x^2+11x+24}{x^2-11x-24} \div \frac{x^2-6x-16}{x^2+5x-24}$

Answers

1. $\frac{a^5b^3c}{6a^2c^9} \div \frac{2a^7b^{11}}{24c^2} = \frac{a^5b^3c}{6a^2c^9} \cdot \frac{24c^2}{2a^7b^{11}} = \frac{24a^5b^3c^3}{12a^9b^{11}c^9} = \frac{2}{a^4b^8c^6}$

2.

$$\begin{aligned}
\frac{x^2+12x-45}{x^2-5x+6} \div \frac{x^2+17x+30}{x^4-16} &= \frac{x^2+12x-45}{x^2-5x+6} \cdot \frac{x^4-16}{x^2+17x+30} \\
&= \frac{\cancel{(x+15)}\cancel{(x-3)}}{\cancel{(x-3)}\cancel{(x-2)}} \cdot \frac{(x^2+4)\cancel{(x-2)}\cancel{(x+2)}}{\cancel{(x+15)}\cancel{(x+2)}} \\
&= x^2+4
\end{aligned}$$

3.

$$\begin{aligned}
(x^3+2x^2-9x-18) \div \frac{x^2+11x+24}{x^2-11x+24} \div \frac{x^2-6x-16}{x^2+5x-24} &= \frac{x^3+2x^2-9x-18}{1} \cdot \frac{x^2-11x+24}{x^2+11x+24} \cdot \frac{x^2+5x-24}{x^2-6x-16} \\
&= \frac{(x-3)\cancel{(x+3)}\cancel{(x+2)}}{1} \cdot \frac{\cancel{(x-8)}(x-3)}{\cancel{(x+8)}\cancel{(x+3)}} \cdot \frac{\cancel{(x+8)}(x-3)}{\cancel{(x-8)}\cancel{(x+2)}} \\
&= (x-3)^2
\end{aligned}$$

Problem Set

Divide the following expressions. Simplify your answer.

1. $\frac{6a^4b^3}{8a^3b^6} \div \frac{3a^5}{4a^3b^4}$
2. $\frac{12x^5y}{xy^4} \div \frac{18x^3y^6}{3x^2y^3}$
3. $\frac{16x^3y^9z^3}{15x^5y^2z} \div \frac{42xy^7z^2}{45x^2yz^5}$
4. $\frac{x^2+2x-3}{x^2-3x+2} \div \frac{x^2+3x}{4x-8}$
5. $\frac{x^2-2x-3}{x^2+6x+5} \div \frac{4x-12}{x^2+8x+15}$
6. $\frac{x^2+6x+2}{12-3x} \div \frac{6x^2-13x-5}{x^2-4x}$
7. $\frac{x^2-5x}{x^2+x-6} \div \frac{x^2-2x-15}{x^3+3x^2-4x-12}$
8. $\frac{3x^3-3x^2-6x}{2x^2+15x-8} \div \frac{6x^2+18x-60}{2x^2+9x-5}$
9. $\frac{x^3+27}{x^2+5x-14} \div \frac{x^2-x-12}{2x^2+2x-40} \div \frac{1}{x-2}$
10. $\frac{x^2+2x-15}{2x^3+7x^2-4x} \div (5x+3) \div \frac{21-10x+x^2}{5x^3+23x^2+12x}$

3.25 Adding & Subtracting Rational Expressions and Complex Fractions

Objective

To add and subtract two or more rational expressions.

Review Queue

Add or subtract the following fractions.

1. $\frac{1}{2}+\frac{3}{4}$
2. $\frac{7}{8}-\frac{1}{3}$
3. $\frac{4}{5}-\frac{4}{15}$
4. $\frac{7}{18}+\frac{11}{24}$

Adding and Subtracting Rational Expressions with Like Denominators

Objective

To add and subtract rational expressions with like denominators.

Guidance

Recall, that when you add or subtract fractions, the denominators must be the same. The same is true of adding and subtracting rational expressions. The denominators must be the same expression and then you can add or subtract the numerators.

Example A

Add $\frac{x}{x-6}+\frac{7}{x-6}$.

Solution: In this concept, the denominators will always be the same. Therefore, all you will need to do is add the numerators and simplify if needed.

$\frac{x}{x-6}+\frac{7}{x-6}=\frac{x+7}{x-6}$

Example B

Subtract $\frac{x^2-4}{x-3}-\frac{2x-1}{x-3}$.

Solution: You need to be a little more careful with subtraction. The entire expression in the second numerator is being subtracted. Think of the minus sign like distributing -1 to that numerator.

$$\begin{aligned}\frac{x^2-4}{x-3}-\frac{2x-1}{x-3}&=\frac{x^2-4-(2x+1)}{x-3}\\&=\frac{x^2-4-2x-1}{x-3}\\&=\frac{x^2-2x-3}{x-3}\end{aligned}$$

At this point, factor the numerator if possible.

$\frac{x^2-2x-3}{x-3} = \frac{\cancel{(x-3)}(x+1)}{\cancel{x-3}} = x+1$

Example C

Add $\frac{x+7}{2x^2+14x+20} + \frac{x+1}{2x^2+14x+20}$.

Solution: Add the numerators and simplify the denominator.

$$\begin{aligned}\frac{x+7}{2x^2+14x+20} + \frac{x+1}{2x^2+14x+20} &= \frac{2x+8}{2x^2+14x+20}\\ &= \frac{\cancel{2}(x+4)}{\cancel{2}(x+5)(x+2)}\\ &= \frac{(x+4)}{(x+5)(x+2)}\end{aligned}$$

Guided Practice

Add or subtract the following rational expressions.

1. $\frac{3}{x^2-9} - \frac{x+7}{x^2-9}$
2. $\frac{5x-6}{2x+3} + \frac{x-12}{2x+3}$
3. $\frac{x^2+2}{4x^2-4x-3} - \frac{x^2-2x+1}{4x^2-4x-3}$

Answers

1. $\frac{3}{x^2-9} - \frac{x+7}{x^2-9} = \frac{3-(x+7)}{x^2-9} = \frac{3-x-7}{x^2-9} = \frac{-x-4}{x^2-9}$

We did not bother to factor the denominator because we know that the factors of -9 are 3 and -3 and will not cancel with $-x-4$,.

2. $\frac{5x-6}{2x+3} + \frac{x-12}{2x+3} = \frac{6x-18}{2x+3} = \frac{6(x-3)}{2x+3}$

3.

$$\begin{aligned}\frac{x^2+2}{4x^2-4x-3} - \frac{x^2-2x+1}{4x^2-4x-3} &= \frac{x^2+2-(x^2-2x+1)}{4x^2 14x-3}\\ &= \frac{x^2+2-x^2+2x-1}{4x^2-4x-3}\\ &= \frac{2x+1}{4x^2-4x-3}\end{aligned}$$

At this point, we will factor the denominator to see if any factors cancel with the numerator.

$\frac{2x+1}{4x^2-4x-3} = \frac{\cancel{2x+1}}{\cancel{(2x+1)}(2x-3)} = \frac{1}{2x-3}$

Problem Set

Add or subtract the following rational expressions.

1. $\frac{3}{x} + \frac{x+1}{x}$
2. $\frac{5}{x+1} + \frac{x-4}{x+1}$
3. $\frac{x+15}{x-2} - \frac{10}{x-2}$
4. $\frac{4x-3}{x+3} + \frac{15}{x+3}$
5. $\frac{3x+8}{x^2-4x-5} + \frac{2x+3}{x^2-4x-5}$
6. $\frac{5x+3}{x^2-4} - \frac{2x+9}{x^2-4}$
7. $\frac{3x^2+x}{x^3-8} + \frac{4}{x^3-8} - \frac{2x^2-x}{x^3-8}$

8. $\frac{4x+3}{x^2+1}-\frac{x+2}{x^2+1}+\frac{1-x}{x^2+1}$
9. $\frac{18x^2-7x+2}{8x^3+4x^2-18x-9}-\frac{3x^2+13x-4}{8x^3+4x^2-18x-9}+\frac{5x^2-13}{8x^3+4x^2-18x-9}$
10. $\frac{2x^2+3x}{x^3+2x^2-16x-32}+\frac{5x^2-13}{x^3+2x^2-16x-32}-\frac{4x^2+9x+11}{x^3+2x^2-16x-32}$

Adding and Subtracting Rational Expressions where One Denominator is the LCD

Objective

To add and subtract rational expressions where one denominator is the Lowest Common Denominator (LCD).

Guidance

Recall when two fractions do not have the same denominator. You have to multiply one or both fractions by a number to create equivalent fractions in order to combine them.

$\frac{1}{2}+\frac{3}{4}$

Here, 2 goes into 4 twice. So, we will multiply the first fraction by $\frac{2}{2}$ to get a denominator of 4. Then, the two fractions can be added.

$\frac{2}{2}\cdot\frac{1}{2}+\frac{3}{4}=\frac{2}{4}+\frac{3}{4}=\frac{5}{4}$

Once the denominators are the same, the fractions can be combined. We will apply this idea to rational expressions in order to add or subtract ones without like denominators.

Example A

Subtract $\frac{3x-5}{2x+8}-\frac{x^2-6}{x+4}$.

Solution: Factoring the denominator of the first fraction, we have $2(x+4)$. The second fraction needs to be multiplied by $\frac{2}{2}$ in order to make the denominators the same.

$$\begin{aligned}\frac{3x-5}{2x+8}-\frac{x^2-6}{x+4}&=\frac{3x-5}{2(x+4)}-\frac{x^2-6}{x+4}\cdot\frac{2}{2}\\&=\frac{3x-5}{2(x+4)}-\frac{2x^2-12}{2(x+4)}\end{aligned}$$

Now that the denominators are the same, subtract the second rational expression just like in the previous concept.

$$\begin{aligned}&=\frac{3x-5-(2x^2-12)}{2(x+4)}\\&=\frac{3x-5-2x^2+12}{2(x+4)}\\&=\frac{-2x^2+3x+7}{2(x+4)}\end{aligned}$$

The numerator is not factorable, so we are done.

Example B

Add $\frac{2x-3}{x+5}+\frac{x^2+1}{x^2-2x-35}$

Solution: Factoring the second denominator, we have $x^2 - 2x - 35 = (x+5)(x-7)$. So, we need to multiply the first fraction by $\frac{x-7}{x-7}$.

$$\overbrace{\frac{(x-7)}{(x-7)} \cdot \frac{(2x-3)}{(x+5)}}^{FOIL} + \frac{x^2+1}{(x-7)(x+5)} = \frac{2x^2-17x+21}{(x-7)(x+5)} + \frac{x^2+1}{(x-7)(x+5)}$$
$$= \frac{3x^2-17x+22}{(x-7)(x+5)}$$

Example C

Subtract $\frac{7x+2}{2x^2+18x+40} - \frac{6}{x+5}$.

Solution: Factoring the first denominator, we have $2x^2+18x+40 = 2(x^2+9x+20) = 2(x+4)(x+5)$. This is the Lowest Common Denominator, or LCD. The second fraction needs the 2 and the $(x+4)$.

$$\frac{7x+2}{2x^2+18x+40} - \frac{6-x}{x+5} = \frac{7x+2}{2(x+5)(x+4)} - \frac{6-x}{x+5} \cdot \frac{2(x+4)}{2(x+4)}$$
$$= \frac{7x+2}{2(x+5)(x+4)} - \frac{2(6-x)(x+4)}{2(x+5)(x+4)}$$
$$= \frac{7x+2}{2(x+5)(x+4)} - \frac{48+4x-2x^2}{2(x+5)(x+4)}$$
$$= \frac{7x+2-(48+4x-2x^2)}{2(x+5)(x+4)}$$
$$= \frac{7x+2-48-4x+2x^2}{2(x+5)(x+4)}$$
$$= \frac{2x^2+3x-46}{2(x+5)(x+4)}$$

Guided Practice

Perform the indicated operation.

1. $\frac{2}{x+1} - \frac{x}{3x+3}$
2. $\frac{x-10}{x^2+4x-24} + \frac{x+3}{x+6}$
3. $\frac{3x^2-5}{3x^2-12} + \frac{x+8}{3x+6}$

Answers

1. The LCD is $3x+3$ or $3(x+1)$. Multiply the first fraction by $\frac{3}{3}$.

$$\frac{2}{x+1} - \frac{x}{3x+3} = \frac{3}{3} \cdot \frac{2}{x+1} - \frac{x}{3(x+1)}$$
$$= \frac{6}{3(x+1)} - \frac{x}{3(x+1)}$$
$$= \frac{6-x}{3(x+1)}$$

2. Here, the LCD $x^2+4x-24$ or $(x+6)(x-4)$. Multiply the second fraction by $\frac{x-4}{x-4}$.

$$\begin{aligned}\frac{x-10}{x^2+4x-24}+\frac{x+3}{x+6} &= \frac{x-10}{(x+6)(x-4)}+\frac{x+3}{x+6}\cdot\frac{x-4}{x-4}\\ &= \frac{x-10}{(x+6)(x-4)}+\frac{x^2-x-12}{(x+6)(x-4)}\\ &= \frac{x-10+x^2-x-12}{(x+6)(x-4)}\\ &= \frac{x^2-22}{(x+6)(x-4)}\end{aligned}$$

3. The LCD is $3x^2-12=3(x-2)(x+2)$. The second fraction's denominator factors to be $3x+6=3(x+2)$, so it needs to be multiplied by $\frac{x-2}{x-2}$.

$$\begin{aligned}\frac{3x^2-5}{3x^2-12}+\frac{x+8}{3x+6} &= \frac{3x^2-5}{3(x-2)(x+2)}+\frac{x+8}{3(x+2)}\cdot\frac{x-2}{x-2}\\ &= \frac{3x^2-5}{3(x-2)(x+2)}+\frac{x^2+6x-16}{3(x-2)(x+2)}\\ &= \frac{3x^2-5+x^2+6x-16}{3(x-2)(x+2)}\\ &= \frac{4x^2+6x-21}{3(x-2)(x+2)}\end{aligned}$$

Problem Set

Perform the indicated operations.

1. $\frac{3}{x}-\frac{5}{4x}$
2. $\frac{x+2}{x+3}+\frac{x-1}{x^2+3x}$
3. $\frac{x}{x-7}-\frac{2x+7}{3x-21}$
4. $\frac{x^2+3x-10}{x^2-4}-\frac{x}{x+2}$
5. $\frac{5x+14}{2x^2-7x-15}-\frac{3}{x-5}$
6. $\frac{x-3}{3x^2+x-10}+\frac{3}{x+2}$
7. $\frac{x+1}{6x+2}+\frac{x^2-7x}{12x^2-14x-6}$
8. $\frac{-3x^2-10x+15}{10x^2-x-3}+\frac{x+4}{2x+1}$
9. $\frac{8}{2x-5}-\frac{x+5}{2x^2+x-15}$
10. $\frac{2}{x+2}+\frac{3x+16}{x^2-x-6}-\frac{2}{x-3}$
11. $\frac{6x^2+4x+8}{x^3+3x^2-x-3}+\frac{x-4}{x^2-1}-\frac{3x}{x^2+2x-3}$

Adding and Subtracting Rational Expressions with Unlike Denominators

Objective

To add and subtract rational expressions with unlike denominators.

Guidance

In the previous two concepts we have eased our way up to this one. Now we will add two rational expressions where were you will have to multiply both fractions by a constant in order to get the Lowest Common Denominator or LCD. Recall how to add fractions where the denominators are not the same.

$\frac{4}{15}+\frac{5}{18}$

Find the LCD. $15 = 3 \cdot 5$ and $18 = 3 \cdot 6$. So, they have a common factor of 3. Anytime two denominators have a common factor, it only needs to be listed once in the LCD. The LCD is therefore $3 \cdot 5 \cdot 6 = 90$.

$$\begin{aligned}
\frac{4}{15}+\frac{5}{18} &= \frac{4}{3 \cdot 5}+\frac{5}{3 \cdot 6} \\
&= \frac{6}{6} \cdot \frac{4}{3 \cdot 5}+\frac{5}{3 \cdot 6} \cdot \frac{5}{5} \\
&= \frac{24}{90}+\frac{25}{90} \\
&= \frac{49}{90}
\end{aligned}$$

We multiplied the first fraction by $\frac{6}{6}$ to obtain 90 in the denominator. Recall that a number over itself is $6 \div 6 = 1$. Therefore, we haven't changed the value of the fraction. We multiplied the second fraction by $\frac{5}{5}$. We will now apply this idea to rational expressions.

Example A

Add $\frac{x+5}{x^2-3x}+\frac{3}{x^2+2x}$.

Solution: First factor each denominator to find the LCD. The first denominator, factored, is $x^2-3x = x(x-3)$. The second denominator is $x^2+2x = x(x+2)$. Both denominators have and x, so we only need to list it once. The LCD is $x(x-3)(x+2)$.

$\frac{x+5}{x^2-3x}+\frac{3}{x^2+2x} = \frac{x+5}{x(x-3)}+\frac{3}{x(x+2)}$

Looking at the two denominators factored, we see that the first fraction needs to be multiplied by $\frac{x+2}{x+2}$ and the second fraction needs to be multiplied by $\frac{x+3}{x+3}$.

$$\begin{aligned}
&= \frac{x+2}{x+2} \cdot \frac{x+5}{x(x-3)}+\frac{3}{x(x+2)} \cdot \frac{x-3}{x-3} \\
&= \frac{(x+2)(x+5)+3(x-3)}{x(x+2)(x-3)}
\end{aligned}$$

At this point, we need to FOIL the first expression and distribute the 3 to the second. Lastly we need to combine like terms.

$$\begin{aligned}
&= \frac{x^2+7x+10+3x-9}{x(x+2)(x-3)} \\
&= \frac{x^2+10x+1}{x(x+2)(x-3)}
\end{aligned}$$

The quadratic in the numerator is not factorable, so we are done.

Example B

Add $\frac{4}{x+6}+\frac{x-2}{3x+1}$.

Solution: The denominators have no common factors, so the LCD will be $(x+6)(3x+1)$.

$$\begin{aligned}
\frac{4}{x+6}+\frac{x-2}{3x+1} &= \frac{3x+1}{3x+1}\cdot\frac{4}{x+6}+\frac{x-2}{3x+1}\cdot\frac{x+6}{x+6}\\
&= \frac{4(3x+1)}{(3x+1)(x+6)}+\frac{(x-2)(x+6)}{(3x+1)(x+6)}\\
&= \frac{12x+4+x^2+4x-12}{(3x+1)(x+6)}\\
&= \frac{x^2+16x-8}{(3x+1)(x+6)}
\end{aligned}$$

Example C

Subtract $\frac{x-1}{x^2+5x+4}-\frac{x+2}{2x^2+13x+20}$.

Solution: To find the LCD, we need to factor the denominators.

$$\begin{aligned}
x^2+5x+4 &= (x+1)(x+4)\\
2x^2+13x+20 &= (2x+5)(x+4)\\
LCD &= (x+1)(2x+5)(x+4)
\end{aligned}$$

$$\begin{aligned}
\frac{x-1}{x^2+5x+4}-\frac{x+2}{2x^2+13x+20} &= \frac{x-1}{(x+1)(x+4)}-\frac{x+2}{(2x+5)(x+4)}\\
&= \frac{2x+5}{2x+5}\cdot\frac{x-1}{(x+1)(x+4)}-\frac{x+2}{(2x+5)(x+4)}\cdot\frac{x+1}{x+1}\\
&= \frac{(2x+5)(x-1)-(x+2)(x+1)}{(x+1)(2x+5)(x+4)}\\
&= \frac{2x^2+3x-5-(x^2+3x+2)}{(x+1)(2x+5)(x+4)}\\
&= \frac{2x^2+3x-5-x^2-3x-2}{(x+1)(2x+5)(x+4)}\\
&= \frac{x^2-7}{(x+1)(2x+5)(x+4)}
\end{aligned}$$

Guided Practice

Perform the indicated operation.

1. $\frac{3}{x^2-6x}+\frac{5-x}{2x-12}$

2. $\frac{x}{x^2+4x+4}-\frac{x-5}{x^2+5x+6}$

3. $\frac{2x}{x^2-x-20}+\frac{x^2-9}{x^2-1}$

Answers

1. The LCD is $3x(x-6)$.

$$
\begin{aligned}
\frac{3}{x^2-6x}+\frac{5-x}{2x-12} &= \frac{2}{2}\cdot\frac{3}{x(x-6)}+\frac{5-x}{2(x-6)}\cdot\frac{x}{x} \\
&= \frac{6+x(5-x)}{2x(x-6)} \\
&= \frac{6+5x-x^2}{2x(x-6)} \\
&= \frac{-1(x^2-5x-6)}{2x(x-6)}
\end{aligned}
$$

We pulled a -1 out of the numerator so we can factor it.

$$
\begin{aligned}
&= \frac{-1\cancel{(x-6)}(x+1)}{2x\cancel{(x-6)}} \\
&= \frac{-x-1}{2x}
\end{aligned}
$$

2. The LCD is $(x+2)(x+2)(x+3)$.

$$
\begin{aligned}
\frac{x}{x^2+4x+4}-\frac{x-5}{x^2+5x+6} &= \frac{x+3}{x+3}\cdot\frac{x}{(x+2)(x+2)}-\frac{x-5}{(x+2)(x+3)}\cdot\frac{x+2}{x+2} \\
&= \frac{x(x+3)-(x-5)(x+2)}{(x+2)(x+2)(x+3)} \\
&= \frac{x^2+3x-(x^2-3x-10)}{(x+2)^2(x+3)} \\
&= \frac{x^2+3x-x^2+3x+10}{(x+2)^2(x+3)} \\
&= \frac{6x+10}{(x+2)^2(x+3)} \\
&= \frac{2(3x+5)}{(x+2)^2(x+3)}
\end{aligned}
$$

3. The LCD is $(x-5)(x+4)(x+1)(x-1)$.

$$
\begin{aligned}
\frac{2x}{x^2-x-20}+\frac{x^2-9}{x^2-1} &= \frac{(x+1)(x-1)}{(x+1)(x-1)}\cdot\frac{2x}{(x-5)(x+4)}+\frac{x^2-9}{(x+1)(x-1)}\cdot\frac{(x-5)(x+4)}{(x-5)(x+4)} \\
&= \frac{2x(x+1)(x-1)+(x^2-9)(x-5)(x+4)}{(x-5)(x+4)(x+1)(x-1)} \\
&= \frac{2x^3-2x+x^4-x^3-29x^2+9x+180}{(x-5)(x+4)(x+1)(x-1)} \\
&= \frac{x^4+x^3-29x^2+7x+180}{(x-5)(x+4)(x+1)(x-1)}
\end{aligned}
$$

Problem Set

Perform the indicated operation.

1. $\frac{5}{3x}+\frac{x}{2}$
2. $\frac{x+1}{x^2}-\frac{5}{7x}$
3. $\frac{x-5}{4x}+\frac{3}{x+2}$
4. $\frac{5}{2x+6}+\frac{x-2}{x^2+2x-3}$
5. $\frac{4x+3}{2x^2+11x-6}-\frac{3x-1}{2x^2-x}$
6. $\frac{x}{3x^2+x-2}-\frac{2}{15x-10}$
7. $\frac{3x}{x^2-3x-10}+\frac{x+1}{x^2-2x-15}-\frac{2}{x^2+5x+6}$
8. $\frac{7+x}{x^2-2x}-\frac{x-6}{3x^2+5x}-\frac{x+4}{3x^2-x-10}$
9. $\frac{3x+2}{x^2-1}-\frac{10x-7}{5x^2+5x}+\frac{3}{x-1}$
10. $\frac{x+6}{2x-1}+\frac{2x}{3x+2}-\frac{5}{x}$

Complex Fractions

Objective

To simplify complex fractions.

Guidance

A **complex fraction** is a fraction that has fractions in the numerator and/or denominator. To simplify a complex fraction, you will need to combine all that you have learned in the previous five concepts.

Example A

Simplify $\frac{\frac{9x}{x+2}}{\frac{3}{x^2-4}}$.

Solution: This complex fraction is a fraction divided by another fraction. Rewrite the complex fraction as a division problem.

$$\frac{\frac{9x}{x+2}}{\frac{3}{x^2-4}}=\frac{9x}{x+2}\div\frac{3}{x^2-4}$$

.

Now, this is just like a problem from the *Dividing Rational Expressions* concept. Flip the second fraction, change the problem to multiplication and simplify.

$$\frac{9x}{x+2}\div\frac{3}{x^2-4}=\frac{9x}{x+2}\cdot\frac{x^2-4}{3}=\frac{\overset{3}{\cancel{9}}x}{\cancel{x+2}}\cdot\frac{\cancel{(x+2)}(x-2)}{\cancel{3}}=3x(x-2)$$

Example B

Simplify $\frac{\frac{1}{x}+\frac{1}{x+1}}{4-\frac{1}{x}}$.

Solution: To simplify this complex fraction, we first need to add the fractions in the numerator and subtract the two in the denominator. The LCD of the numerator is $x(x+1)$ and the denominator is just x.

$$\frac{\frac{1}{x}+\frac{1}{x+1}}{4-\frac{1}{x}}=\frac{\frac{x+1}{x+1}\cdot\frac{1}{x}+\frac{1}{x+1}\cdot\frac{x}{x}}{\frac{x}{x}\cdot 4-\frac{1}{x}}=\frac{\frac{x+1}{x(x+1)}+\frac{x}{x(x+1)}}{\frac{4x}{x}-\frac{1}{x}}=\frac{\frac{2x+1}{x(x+1)}}{\frac{4x-1}{x}}$$

This fraction is now just like Example A. Divide and simplify if possible.

$$\frac{\frac{2x+1}{x(x+1)}}{\frac{4x-1}{x}}=\frac{2x+1}{x(x+1)}\div\frac{4x-1}{x}=\frac{2x+1}{\cancel{x}(x+1)}\cdot\frac{\cancel{x}}{4x-1}=\frac{2x+1}{(x+1)(4x-1)}$$

Example C

Simplify $\frac{\frac{5-x}{x^2+6x+8}+\frac{x}{x+4}}{\frac{6}{x+2}-\frac{2x+3}{x^2-3x-10}}$.

Solution: First, add the fractions in the numerator and subtract the ones in the denominator.

$$\frac{\frac{5-x}{x^2+6x+8}+\frac{x}{x+4}}{\frac{6}{x+2}-\frac{2x+3}{x^2-3x-10}}=\frac{\frac{5-x}{(x+4)(x+2)}+\frac{x}{x+4}\cdot\frac{x+2}{x+2}}{\frac{x-5}{x-5}\cdot\frac{6}{x+2}-\frac{2x+3}{(x+2)(x-5)}}=\frac{\frac{5-x+x(x+2)}{(x+4)(x+2)}}{\frac{6(x-5)-(2x+3)}{(x+2)(x-5)}}=\frac{\frac{x^2+x+5}{(x+4)(x+2)}}{\frac{4x-36}{(x+2)(x-5)}}$$

Now, rewrite as a division problem, flip, multiply, and simplify.

$$\frac{\frac{x^2+x+5}{(x+4)(x+2)}}{\frac{4x-36}{(x+2)(x-5)}}=\frac{x^2+x+5}{(x+4)(x+2)}\div\frac{4x-36}{(x+2)(x-5)}=\frac{x^2+x+5}{(x+4)\cancel{(x+2)}}\cdot\frac{\cancel{(x+2)}(x-5)}{4(x-9)}=\frac{(x^2+x+5)(x-5)}{4(x+4)(x-9)}$$

Guided Practice

Simplify the complex fractions.

1. $\frac{\frac{5x-20}{x^2}}{\frac{x-4}{x}}$

2. $\frac{\frac{1-x}{x}-\frac{2}{x-1}}{1+\frac{1}{x}}$

3. $\frac{\frac{3}{2x^2+12x+18}+\frac{x}{x^2-9}}{\frac{6x}{3x-9}-\frac{3}{x-3}}$

Answers

1. Rewrite the fraction as a division problem and simplify.

$\frac{\frac{5x-20}{x^2}}{\frac{x-4}{x}}=\frac{5x-20}{x^2}\div\frac{x-4}{x}=\frac{5\cancel{(x-4)}}{x^{\cancel{2}}}\cdot\frac{\cancel{x}}{\cancel{x-4}}=\frac{5}{x}$

2. Add the fractions in the numerator and denominator together.

$\frac{\frac{1-x}{x}-\frac{2}{x-1}}{1+\frac{1}{x}}=\frac{\frac{x-1}{x-1}\cdot\frac{1-x}{x}-\frac{2}{x-1}\cdot\frac{x}{x}}{\frac{x}{x}\cdot 1+\frac{1}{x}}=\frac{\frac{(x-1)(1-x)-2x}{x(x-1)}}{\frac{x+1}{x}}=\frac{\frac{-x^2+1}{x(x-1)}}{\frac{x+1}{x}}$

Now, rewrite the fraction as a division problem and simplify.

$$\begin{aligned}\frac{-x^2+1}{x(x-1)}\div\frac{x+1}{x}&=\frac{-(x^2-1)}{x(x-1)}\cdot\frac{x}{x+1}\\&=\frac{-\cancel{(x-1)}\cancel{(x+1)}}{\cancel{x}\cancel{(x-1)}}\cdot\frac{\cancel{x}}{\cancel{x+1}}\\&=-1\end{aligned}$$

3. Add the numerator and subtract the denominator of this complex fraction.

$$\begin{aligned}
\frac{\frac{3}{2x^2+12x+18}+\frac{x}{x^2-9}}{\frac{6x}{3x-9}-\frac{3}{x-3}} &= \frac{\frac{x-3}{x-3}\cdot\frac{3}{2(x+3)(x+3)}+\frac{x}{(x-3)(x+3)}\cdot\frac{2(x+3)}{2(x+3)}}{\frac{6x}{3(x-3)}-\frac{3}{x-3}\cdot\frac{3}{3}} \\
&= \frac{\frac{3(x-3)+2x(x+3)}{2(x+3)(x+3)(x-3)}}{\frac{6x-9}{3(x-3)}} \\
&= \frac{\frac{2x^2+3x-9}{2(x+3)(x+3)(x-3)}}{\frac{\cancel{3}(2x-3)}{\cancel{3}(x-3)}}
\end{aligned}$$

Now, flip and multiply.

$$\begin{aligned}
\frac{2x^2+3x-9}{2(x+3)(x+3)(x-3)} \div \frac{2x-3}{x-3} &= \frac{\cancel{(x+3)}\cancel{(2x-3)}}{2\cancel{(x+3)}(x+3)\cancel{(x-3)}}\cdot\frac{\cancel{x-3}}{\cancel{2x-3}} \\
&= \frac{1}{2(x+3)}
\end{aligned}$$

Vocabulary

Complex Fraction

A fraction with rational expression(s) in the numerator and denominator.

Problem Set

Simplify the complex fractions.

1. $\frac{\frac{4}{x^2-9}}{\frac{6x}{x+3}}$
2. $\frac{\frac{7x^3}{x^2+5x+6}}{\frac{35x^2}{x+2}}$
3. $\frac{\frac{24x+3}{3x+1}}{\frac{16x+2}{6x^2-13x-5}}$
4. $\frac{\frac{4}{x-1}+\frac{1}{x}}{\frac{1}{x}-5}$
5. $\frac{\frac{3x}{x+4}-\frac{1}{x}}{\frac{3x-4}{x^2+6x+8}}$
6. $\frac{8-\frac{3x}{x+5}}{\frac{10}{x+5}+\frac{5}{x+1}}$
7. $\frac{\frac{x}{x+3}-\frac{4}{2x+1}}{\frac{3}{2x+1}+\frac{6}{x^2-9}}$
8. $\frac{\frac{x+3}{x}+\frac{2x}{5-x}}{\frac{3}{2x}-\frac{4x}{x-5}}$
9. $\frac{\frac{2x}{5x^2-13x-6}+\frac{1}{x-3}}{\frac{4}{5x+2}-\frac{5x}{5x^2-3x-2}}$
10. $\frac{\frac{3x}{x^2-4}+\frac{x+4}{x^2+3x+2}}{\frac{x+1}{x^2-x-2}-\frac{2x}{x^2+2x+1}}$

3.26 Solving Rational Equations

Objective

To solve equations involving rational expressions.

Review Queue

Solve the following equations.

1. $x^2+9x+14=0$
2. $3^{x+2}=9$
3. $\frac{x}{3}=\frac{10}{15}$
4. $\frac{12}{x}=\frac{30}{75}$
5. $4^{2x-1}=10$
6. $\log(x-3)^2=4$

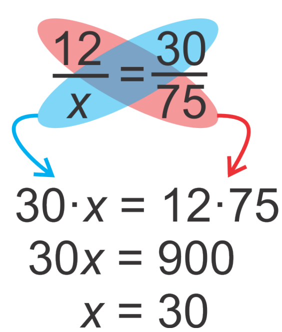
Using Cross-Multiplication

Objective

To use cross-multiplication to solve rational equations.

Guidance

A **rational equation** is an equation where there are rational expressions on both sides of the equal sign. One way to solve rational equations is to use cross-multiplication. #4 from the Review Queue above is an example of a proportion and we can solve it using cross-multiplication.



If you need more of a review of cross-multiplication, see the *Proportion Properties* concept in the Geometry FlexBook® textbook. Otherwise, we will start solving rational equations using cross-multiplication.

Example A

Solve $\frac{x}{2x-3}=\frac{3x}{x+11}$.

Solution: Use cross-multiplication to solve the problem. You can use the example above as a guideline.

$$\frac{x}{2x-3}=\frac{3x}{x+11}$$

$$3x(2x-3)=x(x+11)$$

$$6x^2-9x=x^2+11x$$

$$5x^2-20x=0$$

$$5x(x-4)=0$$

$$x=0 \text{ and } 4$$

Check your answers. It is possible to get extraneous solutions with rational expressions.

$$\begin{aligned}\frac{0}{2\cdot0-3}&=\frac{3\cdot0}{0+11}\\ \frac{0}{-3}&=\frac{0}{11}\ \boxed{\checkmark}\\ 0&=0\end{aligned}$$

$$\begin{aligned}\frac{4}{2\cdot4-3}&=\frac{3\cdot4}{4+11}\\ \frac{4}{5}&=\frac{12}{15}\ \boxed{\checkmark}\\ \frac{4}{5}&=\frac{4}{5}\end{aligned}$$

Example B

Solve $\frac{x+1}{4}=\frac{3}{x-3}$.

Solution: Cross-multiply and solve.

$$\begin{aligned}\frac{x+1}{4}&=\frac{3}{x-3}\\ 12&=x^2-2x-3\\ 0&=x^2-2x-15\\ 0&=(x-5)(x+3)\\ x&=5 \textit{ and } -3\end{aligned}$$

Check your answers. $\frac{5+1}{4}=\frac{3}{5-3}\rightarrow\frac{6}{4}=\frac{3}{2}\ \boxed{\checkmark}$ and $\frac{-3+1}{4}=\frac{3}{-3-3}\rightarrow\frac{-2}{4}=\frac{3}{-6}\ \boxed{\checkmark}$

Example C

Solve $\frac{x^2}{2x-5}=\frac{x+8}{2}$.

Solution: Cross-multiply.

$$\begin{aligned}\frac{x^2}{2x-5}&=\frac{x+8}{2}\\ 2x^2+11x-40&=2x^2\\ 11x-40&=0\\ 11x&=40\\ x&=\frac{40}{11}\end{aligned}$$

Check the answer: $\frac{\left(\frac{40}{11}\right)^2}{\frac{80}{11}-5} = \frac{\frac{40}{11}+8}{2} \rightarrow \frac{1600}{121} \div \frac{25}{11} = \frac{128}{11} \div 2 \rightarrow \frac{64}{11} = \frac{128}{22}$ $\boxed{\checkmark}$

Guided Practice

Solve the following rational equations.

1. $\frac{-x}{x-1} = \frac{x-8}{3}$
2. $\frac{x^2-1}{x+2} = \frac{2x-1}{2}$
3. $\frac{9-x}{x^2} = \frac{4}{3x}$

Answers

1.

$$\begin{aligned}
\frac{-x}{x-1} &= \frac{x-8}{3}\\
x^2-9x+8 &= -3x\\
x^2-6x+8 &= 0\\
(x-4)(x-2) &= 0\\
x &= 4 \textit{ and } 2
\end{aligned}$$

$$\text{Check} : x=4 \rightarrow \frac{-4}{4-1} = \frac{4-8}{3} \boxed{\checkmark} \qquad x=2 \rightarrow \frac{-2}{2-1} = \frac{2-8}{3} \boxed{\checkmark}$$

$$\frac{-4}{3} = \frac{-4}{3} \qquad \frac{-2}{1} = \frac{-6}{3}$$

2.

$$\begin{aligned}
\frac{x^2-1}{x+2} &= \frac{2x-1}{2}\\
2x^2+3x-2 &= 2x^2-2\\
3x &= 0\\
x &= 0
\end{aligned}$$

$$\begin{aligned}
\text{Check} : \frac{0^2-1}{0+2} &= \frac{2(0)-1}{2} \boxed{\checkmark}\\
\frac{-1}{2} &= \frac{-1}{2}
\end{aligned}$$

3.

$$\begin{aligned}
\frac{9-x}{x^2} &= \frac{4}{-3x}\\
4x^2 &= -27x+3x^2\\
x^2+27x &= 0\\
x(x+27) &= 0\\
x &= 0 \textit{ and } -27
\end{aligned}$$

$$\text{Check}: x=0 \rightarrow \frac{9-0}{0^2}=\frac{4}{-3(0)} \qquad x=-27 \rightarrow \frac{9+27}{(-27)^2}=\frac{4}{-3(-27)}$$

$$und = und \qquad \frac{36}{729}=\frac{4}{81} \boxed{\checkmark}$$

$$\frac{4}{81}=\frac{4}{81}$$

$x=0$ is not actually a solution because it is a vertical asymptote for each rational expression, if graphed. Because zero is not part of the domain, it cannot be a solution, and is extraneous.

Vocabulary

Rational Equation
An equation where there are rational expressions on both sides of the equal sign.

Problem Set

Solve the following rational equations.

1. $\frac{2x}{x+3}=\frac{8}{x}$
2. $\frac{4}{x+1}=\frac{x+2}{3}$
3. $\frac{x^2}{x+2}=\frac{x+3}{2}$
4. $\frac{3x}{2x-1}=\frac{2x+1}{x}$
5. $\frac{x+2}{x-3}=\frac{x}{3x-2}$
6. $\frac{x+3}{-3}=\frac{2x+6}{x-3}$
7. $\frac{2x+5}{x-1}=\frac{2}{x-4}$
8. $\frac{6x-1}{4x^2}=\frac{3}{2x+5}$
9. $\frac{5x^2+1}{10}=\frac{x^3-8}{2x}$
10. $\frac{x^2-4}{x+4}=\frac{2x-1}{3}$

Using the LCD

Objective

Using the LCD of the expressions in a rational equation in order to solve for x.

Guidance

In addition to using cross-multiplication to solve a rational equation, we can also use the LCD of all the rational expressions within the equation and eliminate the fraction. To demonstrate, we will walk through a few examples.

Example A

Solve $\frac{5}{2}+\frac{1}{x}=3$.

Solution: The LCD for 2 and x is $2x$. Multiply each term by $2x$, so that the denominators are eliminated. We will put the $2x$ over 1, when multiplying it by the fractions, so that it is easier to line up and cross-cancel.

$$\begin{aligned}\frac{5}{2}+\frac{1}{x}&=3\\ \frac{\cancel{2}x}{1}\cdot\frac{5}{\cancel{2}}+\frac{2\cancel{x}}{1}\cdot\frac{1}{\cancel{x}}&=2x\cdot 3\\ 5x+2&=6x\\ 2&=x\end{aligned}$$

Checking the answer, we have $\frac{5}{2}+\frac{1}{2}=3\rightarrow\frac{6}{2}=3\ \boxed{\checkmark}$

Example B

Solve $\frac{5x}{x-2}=7+\frac{10}{x-2}$.

Solution: Because the denominators are the same, we need to multiply all three terms by $x-2$.

$$\begin{aligned}\frac{5x}{x-2}&=7+\frac{10}{x-2}\\ \cancel{(x-2)}\cdot\frac{5x}{\cancel{x-2}}&=(x-2)\cdot 7+\cancel{(x-2)}\cdot\frac{10}{\cancel{x-2}}\\ 5x&=7x-14+10\\ -2x&=-4\\ x&=2\end{aligned}$$

Checking our answer, we have: $\frac{5\cdot 2}{2-2}=7+\frac{10}{2-2}\rightarrow\frac{10}{0}=7+\frac{10}{0}$. Because the solution is the vertical asymptote of two of the expressions, $x=2$ is an extraneous solution. Therefore, there is no solution to this problem.

Example C

Solve $\frac{3}{x}+\frac{4}{5}=\frac{6}{x-2}$.

Solution: Determine the LCD for 5, x, and $x-2$. It would be the three numbers multiplied together: $5x(x-2)$. Multiply each term by the LCD.

$$\begin{aligned}\frac{3}{x}+\frac{4}{5}&=\frac{6}{x-2}\\ \frac{5\cancel{x}(x-2)}{1}\cdot\frac{3}{\cancel{x}}+\frac{\cancel{5}x(x-2)}{1}\cdot\frac{4}{\cancel{5}}&=\frac{5x\cancel{(x-2)}}{1}\cdot\frac{6}{\cancel{x-2}}\\ 15(x-2)+4x(x-2)&=30x\end{aligned}$$

Multiplying each term by the entire LCD cancels out each denominator, so that we have an equation that we have learned how to solve in previous concepts. Distribute the 15 and $4x$, combine like terms and solve.

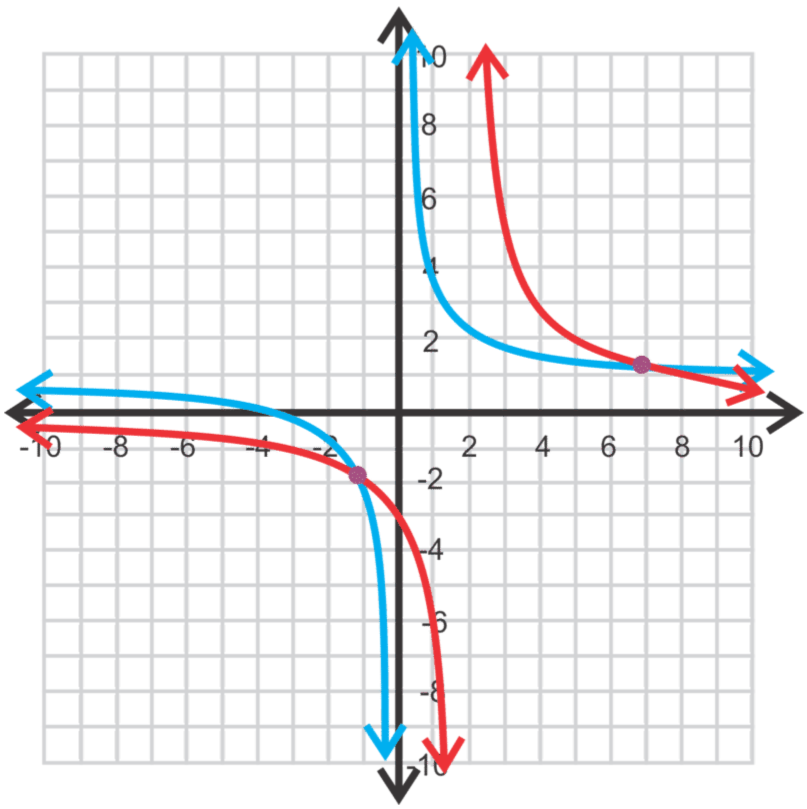
$$\begin{aligned}15x-30+4x^2-8x&=30x\\ 4x^2-23x-30&=0\end{aligned}$$

This polynomial is not factorable. Let's use the Quadratic Formula to find the solutions.

$x=\frac{23\pm\sqrt{(-23)^2-4\cdot 4\cdot(-30)}}{2\cdot 4}=\frac{23\pm\sqrt{1009}}{8}$

Approximately, the solutions are $\frac{23+\sqrt{1009}}{8} \approx 6.85$ and $\frac{23-\sqrt{1009}}{8} \approx -1.096$. It is harder to check these solutions. The easiest thing to do is to graph $\frac{3}{x}+\frac{4}{5}$ in $Y1$ and $\frac{6}{x-2}$ in $Y2$ (using your graphing calculator).

The x-values of the points of intersection (purple points in the graph) are approximately the same as the solutions we found.



Guided Practice

Solve the following equations.

1. $\frac{2x}{x-3} = 2+\frac{3x}{x^2-9}$
2. $\frac{4}{x-3}+5 = \frac{9}{x+2}$
3. $\frac{3}{x^2+4x+4}+\frac{1}{x+2} = \frac{2}{x^2-4}$

Answers

1. The LCD is x^2-9. Multiply each term by its factored form to cross-cancel.

$$\begin{aligned}
\frac{2x}{x-3} &= 2+\frac{3x}{x^2-9}\\
\frac{\cancel{(x-3)}(x+3)}{1}\cdot\frac{2x}{\cancel{x-3}} &= (x-3)(x+3)\cdot 2+\frac{\cancel{(x-3)(x+3)}}{1}\cdot\frac{3x}{\cancel{x^2-9}}\\
2x(x+3) &= 2(x^2-9)+3x\\
2x^2+6x &= 2x^2-18+3x\\
3x &= -18\\
x &= -6
\end{aligned}$$

Checking our answer, we have: $\frac{2(-6)}{-6-3} = 2+\frac{3(-6)}{(-6)^2-9} \rightarrow \frac{-12}{-9} = 2+\frac{-18}{27} \rightarrow \frac{4}{3} = 2-\frac{2}{3}$ $\boxed{\checkmark}$

2. The LCD is $(x-3)(x+2)$. Multiply each term by the LCD.

$$\begin{aligned}
\frac{4}{x-3}+5&=\frac{9}{x+2}\\
\cancel{(x-3)}(x+2)\cdot\frac{4}{\cancel{x-3}}+(x-3)(x+2)\cdot 5&=(x-3)\cancel{(x+2)}\cdot\frac{9}{\cancel{x+2}}\\
4(x+2)+5(x-3)(x+2)&=9(x-3)\\
4x+8+5x^2-5x-30&=9x-27\\
5x^2-10x+5&=0\\
5(x^2-2x+1)&=0
\end{aligned}$$

This polynomial factors to be $5(x-1)(x-1)=0$, so $x=1$ is a repeated solution. Checking our answer, we have $\frac{4}{1-3}+5=\frac{9}{1+2}\rightarrow -2+5=3\ \boxed{\checkmark}$

3. The LCD is $(x+2)(x+2)(x-2)$.

$$\begin{aligned}
\frac{3}{x^2+4x+4}+\frac{1}{x+2}&=\frac{2}{x^2-4}\\
\cancel{(x+2)(x+2)}(x-2)\cdot\frac{3}{\cancel{(x+2)(x+2)}}+\cancel{(x+2)}(x+2)(x-2)\cdot\frac{1}{\cancel{x+2}}&=(x+2)\cancel{(x+2)(x-2)}\cdot\frac{2}{\cancel{(x-2)(x+2)}}\\
3(x-2)+(x-2)(x+2)&=2(x+2)\\
3x-6+x^2-4&=2x+4\\
x^2+x-14&=0
\end{aligned}$$

This quadratic is not factorable, so we need to use the Quadratic Formula to solve for x.

$x=\frac{-1\pm\sqrt{1-4(-14)}}{2}=\frac{-1\pm\sqrt{57}}{2}\approx 3.27$ and -4.27

Using your graphing calculator, you can check the answer. The x-values of points of intersection of $y=\frac{3}{x^2+4x+4}+\frac{1}{x+2}$ and $y=\frac{2}{x^2-4}$ are the same as the values above.

Problem Set

Solve the following equations.

1. $\frac{6}{x+2}+1=\frac{5}{x}$
2. $\frac{5}{3x}-\frac{2}{x+1}=\frac{4}{x}$
3. $\frac{12}{x^2-9}=\frac{8x}{x-3}-\frac{2}{x+3}$
4. $\frac{6x}{x^2-1}+\frac{2}{x+1}=\frac{3x}{x-1}$
5. $\frac{5x-3}{4x}-\frac{x+1}{x+2}=\frac{1}{x^2+2x}$
6. $\frac{4x}{x^2+6x+9}-\frac{2}{x+3}=\frac{3}{x^2-9}$
7. $\frac{x^2}{x^2-8x+16}=\frac{x}{x-4}+\frac{3x}{x^2-16}$
8. $\frac{5x}{2x-3}+\frac{x+1}{x}=\frac{6x^2+x+12}{2x^2-3x}$
9. $\frac{3x}{x^2+2x-8}=\frac{x+1}{x^2+4x}+\frac{2x+1}{x^2-2x}$
10. $\frac{x+1}{x^2+7x}+\frac{x+2}{x^2-3x}=\frac{x}{x^2+4x-21}$

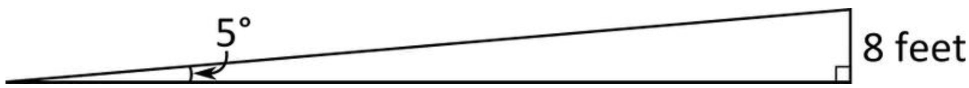
3.27 Basic Trigonometric Functions

Learning Objectives

- Find the values of the six trigonometric functions for angles in right triangles.

Introduction

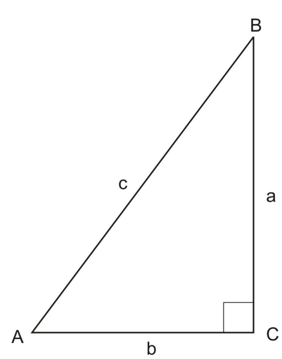
Consider a situation in which you are building a ramp for wheelchair access to a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?



Solving this kind of problem requires trigonometry. The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define six trigonometric functions. For each of these functions, the elements of the domain are angles. We will define these functions in two ways: first, using right triangles, and second, using angles of rotation. Once we have defined these functions, we will be able to solve problems like the one above.

The Sine, Cosine, and Tangent Functions

The first three trigonometric functions we will work with are the sine, cosine, and tangent functions. As noted above, the elements of the domains of these functions are angles. We can define these functions in terms of a right triangle: The elements of the range of the functions are particular ratios of sides of triangles.



We define the sine function as follows: For an acute angle x in a right triangle, the $\sin x$ is equal to the ratio of the side opposite of the angle over the hypotenuse of the triangle. For example, using this triangle, we have: $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Since all right triangles with the same acute angles are similar, this function will produce the same ratio, no matter which triangle is used. Thus, it is a well-defined function.

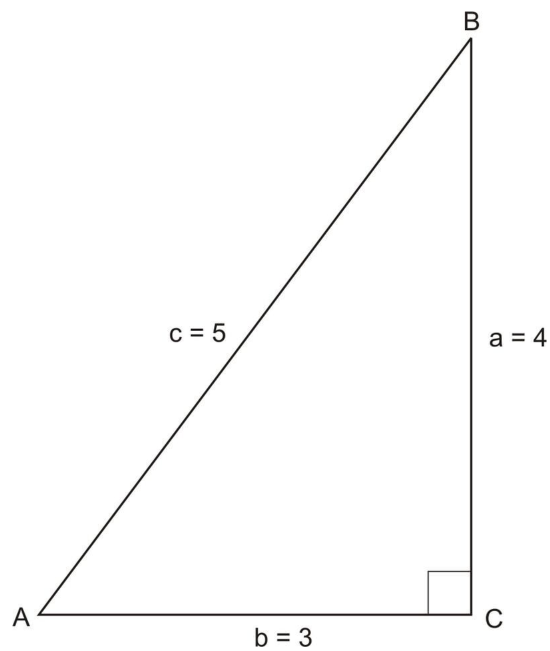
Similarly, the cosine of an angle is defined as the ratio of the side adjacent (next to) the angle over the hypotenuse of the triangle. Using this triangle, we have: $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Finally, the tangent of an angle is defined as the ratio of the side opposite the angle to the side adjacent to the angle. In the triangle above, we have: $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these functions. First, keep in mind that the abbreviations *sinx*, *cosx*, and *tanx* are just like $f(x)$. They simply stand for specific kinds of functions. Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write *sinx* it is still pronounced *sine*, with a long "*i*." When we write *cosx*, we still say co-sine. And when we write *tanx*, we still say tangent.

We can use these definitions to find the sine, cosine, and tangent values for angles in a right triangle.

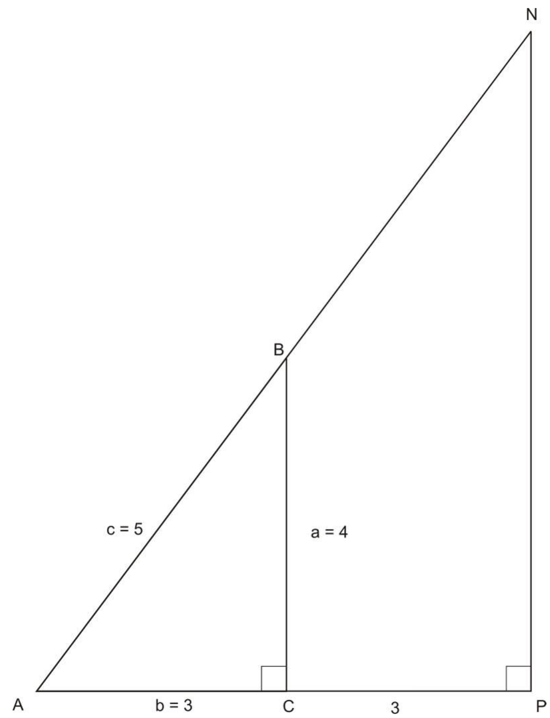
Example 1: Find the sine, cosine, and tangent of $\angle A$:



Solution:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{3}$$

One of the reasons that these functions will help us solve problems is that these ratios will always be the same, as long as the angles are the same. Consider for example, a triangle similar to triangle ABC.



If CP has length 3, then side AP of triangle NAP is 6. Because NAP is similar to ABC, side NP has length 8. This means the hypotenuse AN has length 10. (This can be shown either by using Pythagorean Triples or the Pythagorean Theorem.)

If we use triangle NAP to find the sine, cosine, and tangent of angle A, we get:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{8}{6} = \frac{4}{3}$$

Also notice that the tangent function is the same as the slope of the hypotenuse. $\tan A = \frac{4}{3}$, which is the same as $\frac{rise}{run}$ or $\frac{\text{change in } y}{\text{change in } x}$. The $\tan B$ does not equal the slope because it is the reciprocal of $tanA$.

Example 2: Find $\sin B$ using triangle ABC and triangle NAP.

Solution:

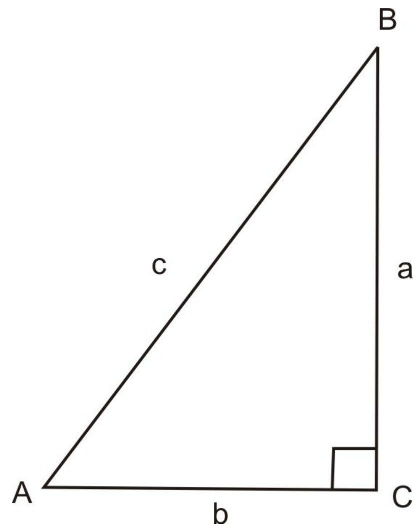
Using triangle $ABC : \sin B = \frac{3}{5}$

Using triangle $NAP : \sin B = \frac{6}{10} = \frac{3}{5}$

An easy way to remember the ratios of the sine, cosine, and tangent functions is SOH-CAH-TOA. $\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}}, \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$.

Secant, Cosecant, and Cotangent Functions

We can define three more functions also based on a right triangle. They are the reciprocals of sine, cosine and tangent.

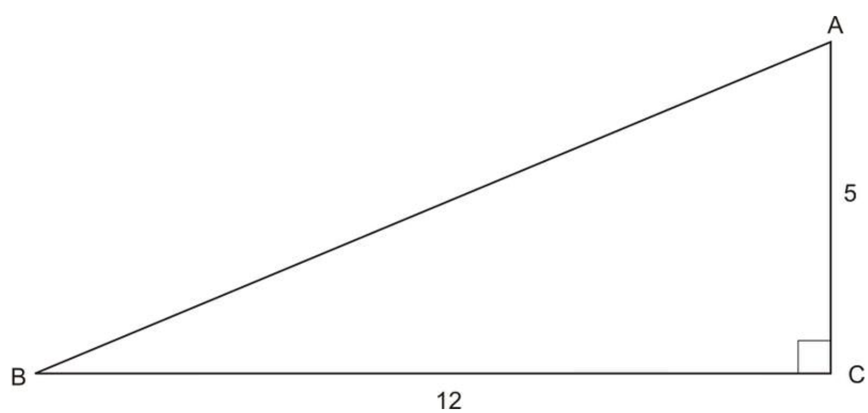


If $\sin A = \frac{a}{c}$, then the definition of cosecant, or csc, is $\csc A = \frac{c}{a}$.

If $\cos A = \frac{b}{c}$, then the definition of secant, or sec, is $\sec A = \frac{c}{b}$.

If $\tan A = \frac{a}{b}$, then the definition of cotangent, or cot, is $\cot A = \frac{b}{a}$.

Example 3: Find the secant, cosecant, and cotangent of angle B.



Solution:

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$\begin{aligned}
5^2 + 12^2 &= H^2 \\
25 + 144 &= H^2 \\
169 &= H^2 \\
H &= 13
\end{aligned}$$

Now we can find the secant, cosecant, and cotangent of angle B:

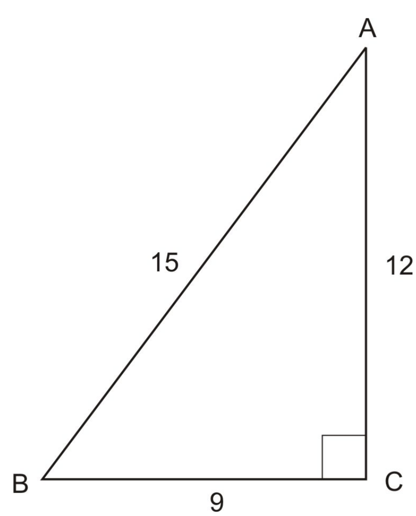
$$\sec B = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$
$$\csc B = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$
$$\cot B = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

Points to Consider

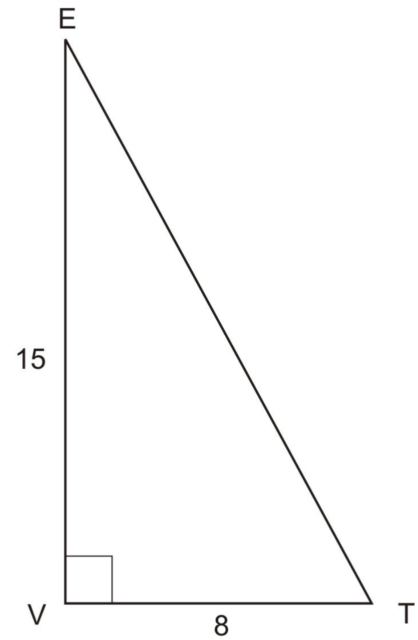
- Do you notice any similarities between the sine of one angle and the cosine of the other, in the same triangle?

Review Questions

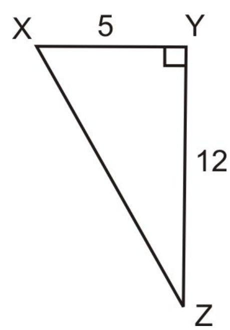
1. Find the values of the six trig functions of angle A.



2. Consider triangle VET below. Find the length of the hypotenuse and values of the six trig functions of angle T.



3. Consider the right triangle below.



   a. Find the hypotenuse.
   b. Find the six trigonometric functions of $\angle X$.
   c. Find the six trigonometric functions of $\angle Z$.

4. Looking back at #3, are any functions of $\angle X$ equal to any of the functions of $\angle Z$? If so, which ones? Do you think this could be generalized for ANY pair of acute angles in the same right triangle (also called complements)?
5. Consider an isosceles right triangle with legs of length 2. Find the sine, cosine and tangent of both acute angles.
6. Consider an isosceles right triangle with legs of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #5.
7. Consider a $30-60-90$ triangle with hypotenuse of length 10. Find the sine, cosine and tangent of both acute angles.
8. Consider a $30-60-90$ triangle with short leg of length x. Find the sine, cosine and tangent of both acute angles. Write down any similarities or patterns you notice with #7.
9. Consider a right triangle, ABC. If $\sin A = \frac{9}{41}$, find the length of the third side.

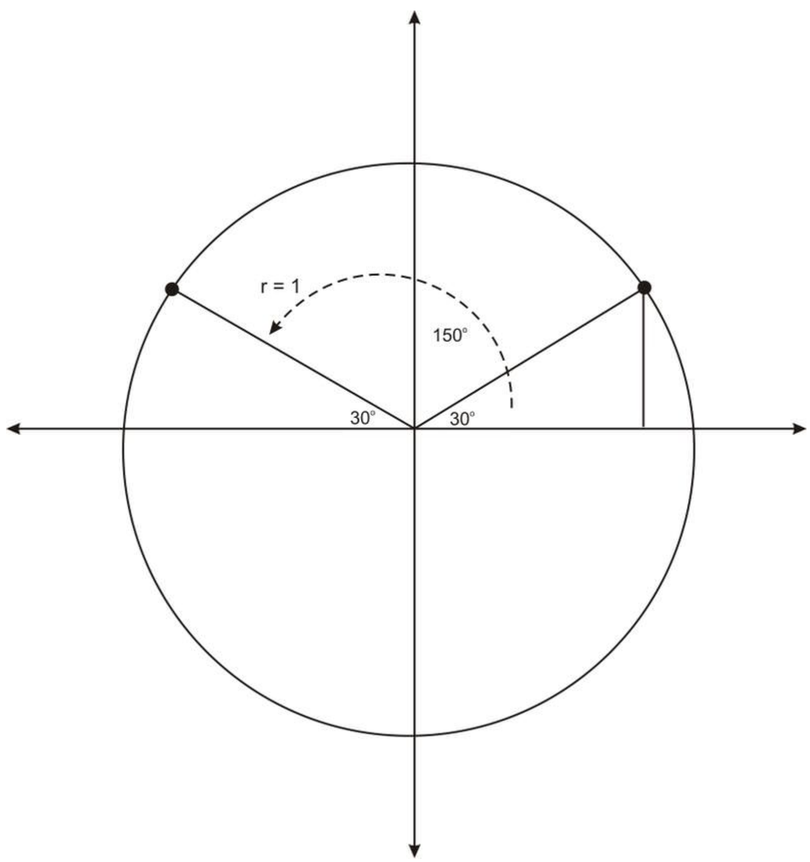
3.28 Trigonometric Functions of Any Angle

Learning Objectives

- Identify the reference angles for angles in the unit circle.
- Identify the ordered pair on the unit circle for angles whose reference angle is 30°, 45°, and 60°, or a quadrantal angle, including negative angles, and angles whose measure is greater than 360°.
- Use these ordered pairs to determine values of trig functions of these angles.
- Use calculators to find values of trig functions of any angle.

Reference Angles and Angles in the Unit Circle

In the previous lesson, one of the review questions asked you to consider the angle 150°. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of 30°, across the $y-$axis.



Notice that 150° makes a 30° angle with the negative $x-$axis. Therefore we say that 30° is the **reference angle** for 150°. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the $x-$axis. Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.

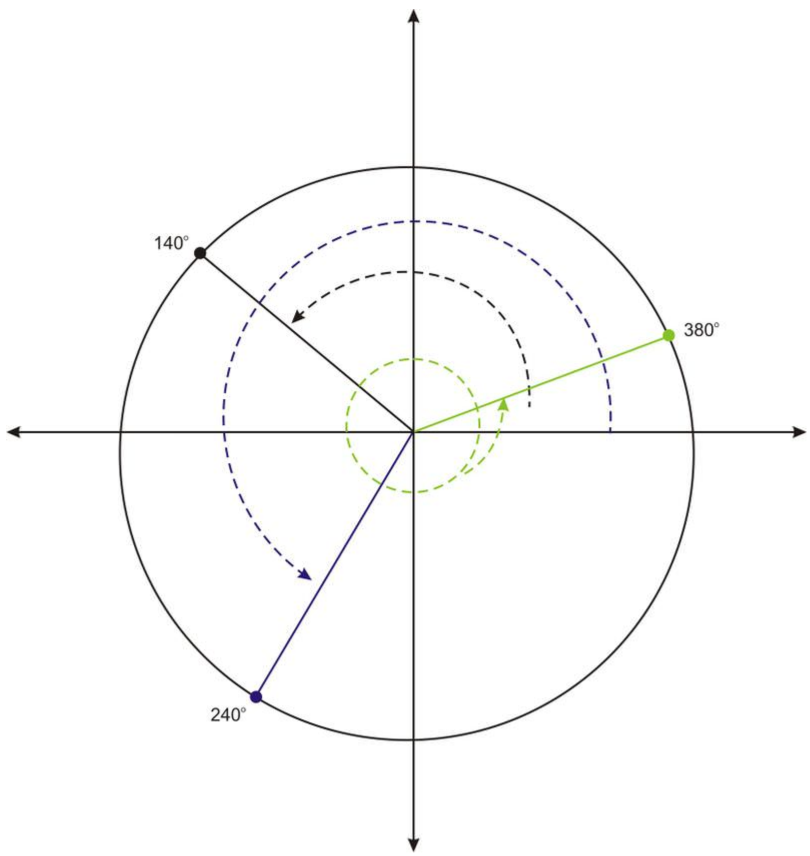
Example 1: Graph each angle and identify its reference angle.

a. 140°

b. 240°

c. 380°

Solution:



a. 140° makes a 40° angle with the $x-$axis. Therefore the reference angle is 40°.

b. 240° makes a 60° with the $x-$axis. Therefore the reference angle is 60°.

c. 380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

If an angle has a reference angle of 30°, 45°, or 60°, we can identify its ordered pair on the unit circle, and so we can find the values of the six trig functions of that angle. For example, above we stated that 150° has a reference angle of 30°. Because of its relationship to 30°, the ordered pair for 150° is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Now we can find the values of the six trig functions of 150°:

$$\cos 150 = x = \frac{-\sqrt{3}}{2} \qquad \sec 150 = \frac{1}{x} = \frac{1}{\frac{-\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}}$$

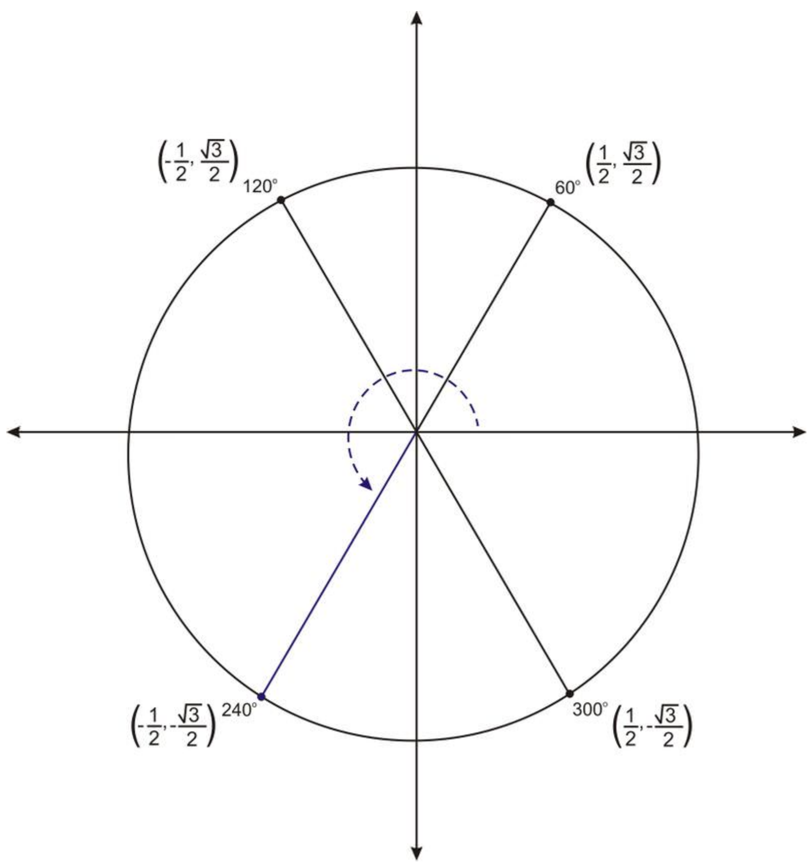
$$\sin 150 = y = \frac{1}{2} \qquad \csc 150 = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\tan 150 = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{-\sqrt{3}} \qquad \cot 150 = \frac{x}{y} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 2: Find the ordered pair for 240° and use it to find the value of sin 240°.

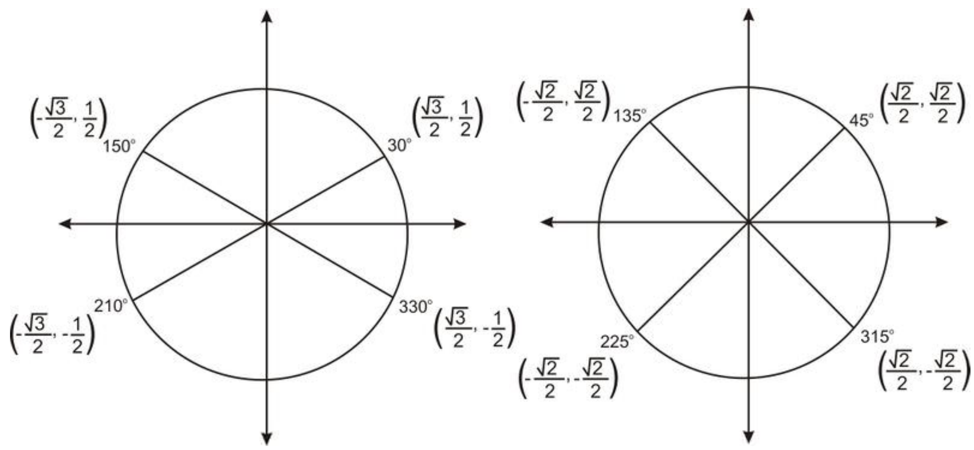
Solution: $\sin 240^\circ = \frac{-\sqrt{3}}{2}$

As we found in example 1, the reference angle for 240° is 60°. The figure below shows 60° and the three other angles in the unit circle that have 60° as a reference angle.



The terminal side of the angle 240° represents a reflection of the terminal side of 60° over both axes. So the coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The $y-$coordinate is the sine value, so $\sin 240^\circ = \frac{-\sqrt{3}}{2}$.

Just as the figure above shows 60° and three related angles, we can make similar graphs for 30° and 45°.



Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

Example 3: Find the value of $\cot 300^\circ$

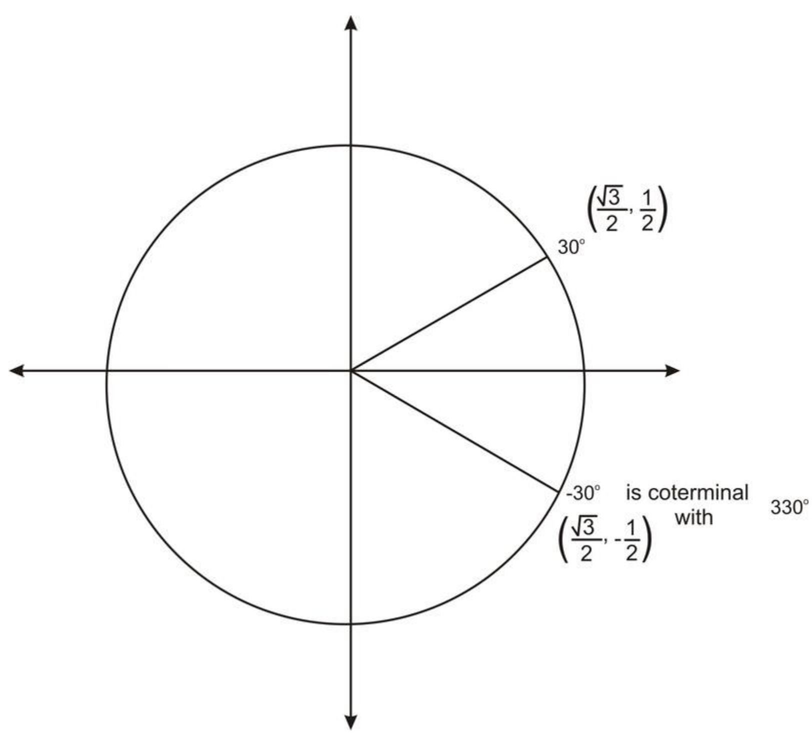
Solution: $\cot 300^\circ = -\frac{1}{\sqrt{3}}$

Using the graph above, you will find that the ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is $\cot 300 = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows -30°.



Notice that this angle is coterminal with 330°. So the ordered pair is $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$. We can use this ordered pair to find the values of any of the trig functions of -30°. For example, $\cos(-30^\circ)=x=\frac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.

Example 4: Find the value of each expression.

a. $\sin(-45^\circ)$

b. $\sec(-300^\circ)$

c. $\cos(-90^\circ)$

Solution:

a. $\sin(-45^\circ)=-\frac{\sqrt{2}}{2}$

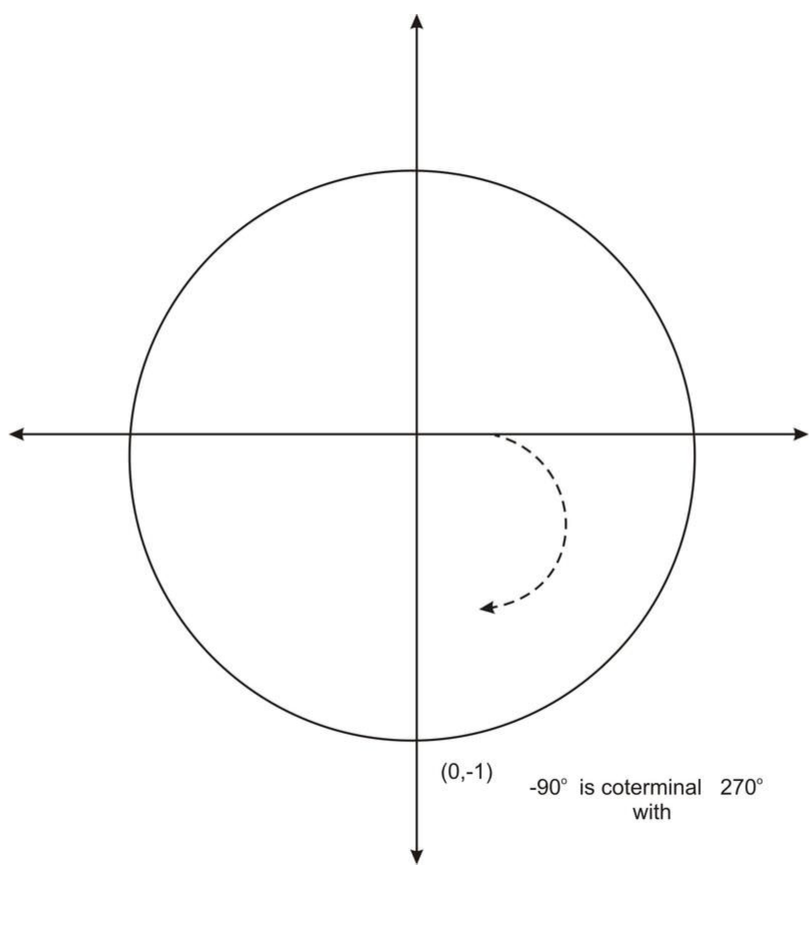
-45° is in the 4^{th} quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°. Therefore the ordered pair is $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ and the sine value is $-\frac{\sqrt{2}}{2}$.

b. $\sec(-300^\circ)=2$

The angle -300° is in the 1^{st} quadrant and has a reference angle of 60°. That is, this angle is coterminal with 60°. Therefore the ordered pair is $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and the secant value is $\frac{1}{x}=\frac{1}{\frac{1}{2}}=2$.

c. $\cos(-90^\circ)=0$

The angle -90° is coterminal with 270°. Therefore the ordered pair is (0, -1) and the cosine value is 0.

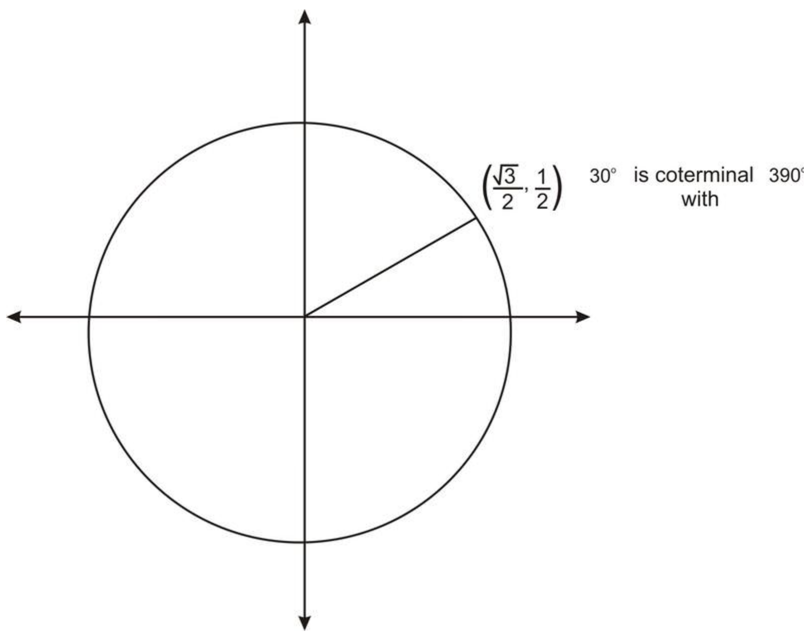


We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Trigonometric Functions of Angles Greater than 360 Degrees

Consider the angle 390°. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore 390° is coterminal with 30°. As you saw above with negative angles, this means that 390° has the same ordered pair as 30°, and so it has the same trig values. For example,

$$\cos 390^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$



In general, if an angle whose measure is greater than 360 has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. Again, determine the reference angle first.

Example 5: Find the value of each expression.

a. $\sin 420^\circ$

b. $\tan 840^\circ$

c. $\cos 540^\circ$

Solution:

a. $\sin 420^\circ = \frac{\sqrt{3}}{2}$

420° is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with 60°, and so it shares the same ordered pair, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The sine value is the $y-$coordinate.

b. $\tan 840^\circ = -\sqrt{3}$

840° is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore 840° is coterminal with 120°, so the ordered pair is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan 840^\circ = \tan 120^\circ = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

c. $\cos 540^\circ = -1$

540° is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with 180°, and the ordered pair is (-1, 0). So the cosine value is -1.

So far all of the angles we have worked with are multiples of 30, 45, 60, and 90. Next we will find approximate values of the trig functions of other angles.

Using a Calculator to Find Values

If you have a scientific calculator, you can determine the value of any trig function for any angle. Here we will focus on using a TI graphing calculator to find values.

First, your calculator needs to be in the correct "mode." In chapter 2 you will learn about a different system for measuring angles, known as radian measure. In this chapter, we are measuring angles in degrees. We need to make sure that the calculator is in degrees. To do this, press $\boxed{\text{MODE}}$. In the third row, make sure that Degree is highlighted. If Radian is highlighted, scroll down to this row, scroll over to Degree, and press $\boxed{\text{ENTER}}$. This will highlight Degree. Then press $\boxed{2^{\text{nd}}}$ $\boxed{\text{MODE}}$ to return to the main screen.

Now you can calculate any value. For example, we can verify the values from the table above. To find $\sin 130^\circ$, press $\boxed{\text{Sin}}$ $\boxed{130}$ $\boxed{\text{ENTER}}$. The calculator should return the value .7660444431.

Example 6: Find the approximate value of each expression. Round your answer to 4 decimal places.

a. $\sin 130^\circ$

b. $\cos 15^\circ$

c. $\tan 50^\circ$

Solution:

a. $\sin 130^\circ \approx 0.7660$

b. $\cos 15^\circ \approx 0.9659$

c. $\tan 50^\circ \approx 1.1918$

You may have noticed that the calculator provides a "(" after the SIN. In the previous calculations, you can actually leave off the ")". However, in more complicated calculations, leaving off the closing ")" can create problems. It is a good idea to get in the habit of closing parentheses.

You can also use a calculator to find values of more complicated expressions.

Example 7: Use a calculator to find an approximate value of $\sin 25^\circ + \cos 25^\circ$. Round your answer to 4 decimal places.

Solution: $\sin 25^\circ + \cos 25^\circ \approx 1.3289$

*This is an example where you need to close the parentheses.

Points to Consider

- What is the difference between the measure of an angle, and its reference angle? In what cases are these measures the same value?
- Which angles have the same cosine value, or the same sine value? Which angles have opposite cosine and sine values?

Review Questions

1. State the reference angle for each angle.
   a. 190°
   b. -60°
   c. 1470°
   d. -135°
2. State the ordered pair for each angle.
   a. 300°
   b. -150°
   c. 405°
3. Find the value of each expression.
   a. $\sin 210^\circ$
   b. $\tan 270^\circ$
   c. $\csc 120^\circ$
4. Find the value of each expression.
   a. $\sin 510^\circ$
   b. $\cos 930^\circ$
   c. $\csc 405^\circ$
5. Find the value of each expression.
   a. $\cos -150^\circ$
   b. $\tan -45^\circ$
   c. $\sin -240^\circ$
6. Use a calculator to find each value. Round to 4 decimal places.
   a. $\sin 118^\circ$
   b. $\tan 55^\circ$
   c. $\cos 100^\circ$
7. Recall, in lesson 1.4, we introduced inverse trig functions. Use your calculator to find the measure of an angle whose sine value is 0.2.
8. In example 6c, we found that $\tan 50^\circ \approx 1.1918$. Use your knowledge of a special angle to explain why this value is reasonable. *HINT: You will need to change the tangent of this angle into a decimal.*

9. Use the table below or a calculator to explore sum and product relationships among trig functions. Consider the following functions:

$$f(x) = \sin(x+x) \text{ and } g(x) = \sin(x) + \sin(x)$$
$$h(x) = \sin(x) \,^*\, \sin(x) \text{ and } j(x) = \sin(x^2)$$

Do you observe any patterns in these functions? Are there any equalities among the functions? Can you make a general conjecture about $\sin(a)+\sin(b)$ and $\sin(a+b)$ for all values of a,b? What about $\sin(a)\sin(a)$ and $\sin(a^2)$?

TABLE 3.19:

a°	b°	$\sin a + \sin b$	$\sin(a+b)$
10	30	.6736	.6428
20	60	1.2080	.9848
55	78	1.7973	.7314
122	25	1.2707	.5446
200	75	.6239	-.9962

10. Use a calculator or your knowledge of special angles to fill in the values in the table, then use the values to make a conjecture about the relationship between $(\sin a)^2$ and $(\cos a)^2$. If you use a calculator, round all values to 4 decimal places.

TABLE 3.20:

a	$(\sin a)^2$	$(\cos a)^2$
0		
25		
45		
80		
90		
120		
250		

3.29 Solving Trigonometric Equations

Objective

Here you'll learn how to solve equations with trigonometric functions.

Review Queue

Solve the following equations.

1. $x^2 - 7x - 18 = 0$
2. $9x^4 - 16 = 0$
3. $\sin x = \frac{\sqrt{2}}{2}, 0 < x < \frac{\pi}{2}$

Using Algebra

Objective

Here you'll solve trig equations using algebra.

Guidance

In the previous concept, we verified trigonometric identities, which are true for every real value of x. In this concept, we will solve trigonometric equations. An equation is only true for some values of x.

Example A

Verify that $\csc x - 2 = 0$ when $x = \frac{5\pi}{6}$.

Solution: Substitute in $x = \frac{5\pi}{6}$ to see if the equations holds true.

$$\begin{aligned}
\csc\left(\frac{5\pi}{6}\right) - 2 &= 0 \\
\frac{1}{\sin\left(\frac{5\pi}{6}\right)} - 2 &= 0 \\
\frac{1}{\frac{1}{2}} - 2 &= 0 \\
2 - 2 &= 0
\end{aligned}$$

This is a true statement, so $x = \frac{5\pi}{6}$ is a solution to the equation.

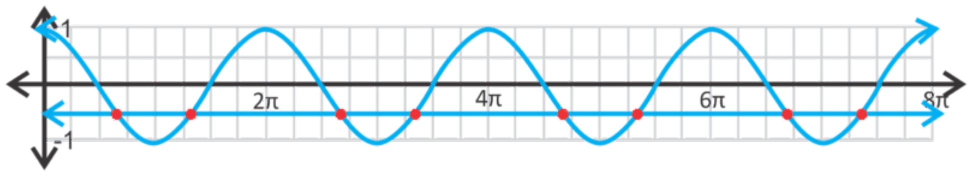
Example B

Solve $2\cos x + 1 = 0$.

Solution: To solve this equation, we need to isolate $\cos x$ and then use inverse to find the values of x when the equation is valid. You already did this to find the zeros in the graphing concepts earlier in this chapter.

$$\begin{aligned} 2\cos x + 1 &= 0 \\ 2\cos x &= -1 \\ \cos x &= -\frac{1}{2} \end{aligned}$$

So, when is the $\cos x = -\frac{1}{2}$? Between $0 \le x < 2\pi, x = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. But, the trig functions are periodic, so there are more solutions than just these two. You can write the general solutions as $x = \frac{2\pi}{3} \pm 2\pi n$ and $x = \frac{4\pi}{3} \pm 2\pi n$, where n is any integer. You can check your answer graphically by graphing $y = \cos x$ and $y = -\frac{1}{2}$ on the same set of axes. Where the two lines intersect are the solutions.



Example C

Solve $5\tan(x+2) - 1 = 0$, where $0 \le x < 2\pi$.

Solution: In this example, we have an interval where we want to find x. Therefore, at the end of the problem, we will need to add or subtract π, the period of tangent, to find the correct solutions within our interval.

$$\begin{aligned} 5\tan(x+2) - 1 &= 0 \\ 5\tan(x+2) &= 1 \\ \tan(x+2) &= \frac{1}{5} \end{aligned}$$

Using the $\tan^{-1}$ button on your calculator, we get that $\tan^{-1}\left(\frac{1}{5}\right) = 0.1974$. Therefore, we have:

$$\begin{aligned} x + 2 &= 0.1974 \\ x &= -1.8026 \end{aligned}$$

This answer is not within our interval. To find the solutions in the interval, add π a couple of times until we have found all of the solutions in $[0, 2\pi]$.

$$\begin{aligned} x &= -1.8026 + \pi = 1.3390 \\ &= 1.3390 + \pi = 4.4806 \end{aligned}$$

The two solutions are $x = 1.3390$ and 4.4806.

Guided Practice

1. Determine if $x = \frac{\pi}{3}$ is a solution for $2\sin x = \sqrt{3}$.

Solve the following trig equations in the interval $0 \le x < 2\pi$.

2. $3\cos^2 x - 5 = 0$

3. $3\sec(x-1)+2=0$

Answers

1. $2\sin\frac{\pi}{3}=\sqrt{3}\rightarrow 2\cdot\frac{\sqrt{3}}{2}=\sqrt{3}$ Yes, $x=\frac{\pi}{3}$ is a solution.

2. Isolate the $\cos^2 x$ and then take the square root of both sides. Don't forget about the $\pm$!

$$\begin{aligned}9\cos^2 x-5&=0\\ 9\cos^2 x&=5\\ \cos^2 x&=\frac{5}{9}\\ \cos x&=\pm\frac{\sqrt{5}}{3}\end{aligned}$$

The $\cos x=\frac{\sqrt{5}}{3}$ at $x=0.243$ rad (use your graphing calculator). To find the other value where cosine is positive, subtract 0.243 from 2π, $x=2\pi-0.243=6.037$ rad.

The $\cos x=-\frac{\sqrt{5}}{3}$ at $x=2.412$ rad, which is in the 2^{nd} quadrant. To find the other value where cosine is negative (the 3^{rd} quadrant), use the reference angle, 0.243, and add it to π. $x=\pi+0.243=3.383$ rad.

3. Here, we will find the solution within the given range, $0\le x<2\pi$.

$$\begin{aligned}3\sec(x-1)+2&=0\\ 3\sec(x-1)&=-2\\ \sec(x-1)&=-\frac{2}{3}\\ \cos(x-1)&=-\frac{3}{2}\end{aligned}$$

At this point, we can stop. The range of the cosine function is from 1 to -1. $-\frac{3}{2}$ is outside of this range, so there is no solution to this equation.

Problem Set

Determine if the following values for x. are solutions to the equation $5+6\csc x=17$.

1. $x=-\frac{7\pi}{6}$
2. $x=\frac{11\pi}{6}$
3. $x=\frac{5\pi}{6}$

Solve the following trigonometric equations. If no solutions exist, write *no solution*.

4. $1-\cos x=0$
5. $3\tan x-\sqrt{3}=0$
6. $4\cos x=2\cos x+1$
7. $5\sin x-2=2\sin x+4$
8. $\sec x-4=-\sec x$
9. $\tan^2(x-2)=3$

Sole the following trigonometric equations within the interval $0\le x<2\pi$. If no solutions exist, write *no solution*.

10. $\cos x = \sin x$
11. $-\sqrt{3}\csc x = 2$
12. $6\sin(x-2) = 14$
13. $7\cos x - 4 = 1$
14. $5 + 4\cot^2 x = 17$
15. $2\sin^2 x - 7 = -6$

By Using Quadratic Techniques

Objective

Here you'll solve trig equations by factoring and the Quadratic Formula.

Guidance

Another way to solve a trig equation is to use factoring or the quadratic formula. Let's look at a couple of examples.

Example A

Solve $\sin^2 x - 3\sin x + 2 = 0$.

Solution: This sine equation looks a lot like the quadratic $x^2 - 3x + 2 = 0$ which factors to be $(x-2)(x-1) = 0$ and the solutions are $x = 2$ and 1. We can factor the trig equation in the exact same manner. Instead of just x, we will have $\sin x$ in the factors.

$$\begin{aligned} \sin^2 x - 3\sin x + 2 &= 0 \\ (\sin x - 2)(\sin x - 1) &= 0 \\ \sin x = 2 \ \textit{and} \ \sin x &= 1 \end{aligned}$$

There is no solution for $\sin x = 2$ and $\sin x = 1$ when $x = \frac{\pi}{2} \pm 2\pi n$.

Example B

Solve $1 - \sin x = \sqrt{3}\cos x$ in the interval $0 \le x < 2\pi$.

Solution: To solve this equation, use the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$. Solve for either cosine and substitute into the equation. $\cos x = \sqrt{1 - \sin^2 x}$

$$\begin{aligned} 1 - \sin x &= \sqrt{3} \cdot \sqrt{1 - \sin^2 x} \\ (1 - \sin x)^2 &= \sqrt{3 - 3\sin^2 x}^2 \\ 1 - 2\sin x + \sin^2 x &= 3 - 3\sin^2 x \\ 4\sin^2 x - 2\sin x - 2 &= 0 \\ 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \end{aligned}$$

Solving each factor for x, we get $\sin x = -\frac{1}{2} \rightarrow x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ and $\sin x = 1 \rightarrow x = \frac{\pi}{2}$.

Example C

Solve $\tan^2 x - 5\tan x - 9 = 0$ in the interval $0 \le x < \pi$.

Solution: This equation is not factorable so you have to use the Quadratic Formula.

$$\begin{aligned}\tan x &= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-9)}}{2} \\ &= \frac{5 \pm \sqrt{61}}{2} \\ &\approx 6.41 \textit{ and } -1.41\end{aligned}$$

$x \approx \tan^{-1} 6.41 \approx 1.416$ rad and $x \approx \tan^{-1} -1.41 \approx -0.954$ rad

The first answer is within the range, but the second is not. To adjust -0.954 to be within the range, we need to find the answer in the second quadrant, $\pi - 0.954 = 2.186$ rad.

Guided Practice

Solve the following trig equations using any method in the interval $0 \leq x < 2\pi$.

1. $\sin^2 x \cos x = \cos x$

2. $\sin^2 x = 2\sin(-x) + 1$

3. $4\cos^2 x - 2\cos x - 1 = 0$

Answers

1. Put everything onto one side of the equation and factor out a cosine.

$$\begin{aligned}\sin^2 x \cos x - \cos x &= 0 \\ \cos x(\sin^2 x - 1) &= 0 \\ \cos x(\sin x - 1)(\sin x + 1) &= 0\end{aligned}$$

$$\begin{array}{lll}\cos x = 0 & \sin x = 1 & \sin x = -1 \\ x = \frac{\pi}{2} \textit{ and } \frac{3\pi}{2} & x = \frac{\pi}{2} & x = \frac{3\pi}{2}\end{array}$$

2. Recall that $\sin(-x) = -\sin x$ from the Negative Angle Identities.

$$\begin{aligned}\sin^2 x &= 2\sin(-x) + 1 \\ \sin^2 x &= -2\sin x + 1 \\ \sin^2 x + 2\sin x + 1 &= 0 \\ (\sin x + 1)^2 &= 0 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2}\end{aligned}$$

3. This quadratic is not factorable, so use the quadratic formula.

$$\begin{aligned}\cos x &= \frac{2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)} \\ &= \frac{2 \pm \sqrt{20}}{8} \\ &= \frac{1 \pm 2\sqrt{5}}{4}\end{aligned}$$

$$\begin{aligned}x &\approx \cos^{-1}\left(\frac{1+\sqrt{5}}{4}\right) & x &\approx \cos^{-1}\left(\frac{1-\sqrt{5}}{4}\right) \\ &\approx \cos^{-1} 0.8090 \qquad \textit{and} & &\approx \cos^{-1} -0.3090 \\ &\approx 0.6283 & &\approx 1.8850 \text{ (reference angle is } \pi - 1.8850 \approx 1.2570)\end{aligned}$$

The other solutions in the range are $x \approx 2\pi - 0.6283 \approx 5.6549$ and $x \approx \pi + 1.2570 \approx 4.3982$.

Problem Set

Solve the following trig equations using any method. Find all solutions in the interval $0 \leq x < 2\pi$. Round any decimal answers to 4 decimal places.

1. $2\cos^2 x - \sin x - 1 = 0$
2. $4\sin^2 x + 5\sin x + 1 = 0$
3. $3\tan^2 x - \tan x = 0$
4. $2\cos^2 x + \cos(-x) - 1 = 0$
5. $1 - \sin x = \sqrt{2}\cos x$
6. $\sqrt{\sin x} = 2\sin x - 1$
7. $\sin^3 x - \sin x = 0$
8. $\tan^2 x - 8\tan x - 7 = 0$
9. $5\cos^2 x + 3\cos x - 2 = 0$
10. $\sin x - \sin x \cos^2 x = 1$
11. $\cos^2 x - 3\cos x - 2 = 0$
12. $\sin^2 x \cos x = 4\cos x$
13. $\cos x \csc^2 x + 2\cos x = 6\cos x$

Using your graphing calculator, graph the following equations and determine the points of intersection in the interval $0 \leq x < 2\pi$.

14.

$$\begin{aligned}y &= \sin^2 x \\ y &= 2\sin x - 1\end{aligned}$$

15.

$$\begin{aligned}y &= 4\cos x - 3 \\ y &= -2\tan x\end{aligned}$$

Chapter 4 Advanced Trigonometry

Chapter Outline

4.1 Angles in Radians and Degrees

Learning Objectives

Here you will learn how to translate between different ways of measuring angles.

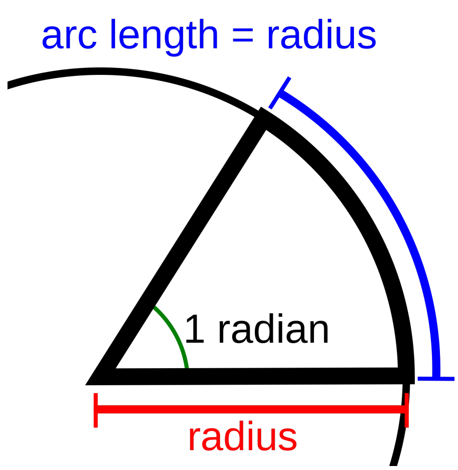
Most people are familiar with measuring angles in degrees. It is easy to picture angles like 30°, 45° or 90° and the fact that 360° makes up an entire circle. Over 2000 years ago the Babylonians used a base 60 number system and divided up a circle into 360 equal parts. This became the standard and it is how most people think of angles today.

However, there are many units with which to measure angles. For example, the gradian was invented along with the metric system and it divides a circle into 400 equal parts. The sizes of these different units are very arbitrary.

A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

Radians and Degrees

A **radian** is defined to be the central angle where the subtended **arc length** is the same length as the radius.



Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius r is $2\pi r$. Just over six radii (exactly 2π radii) would stretch around any circle.

To define a radian in terms of degrees, equate a circle measured in degrees to a circle measured in radians.

$360\ degrees = 2\pi\ radians$, so $\frac{180}{\pi}\ degrees = 1\ radian$

Alternatively; $360\ degrees = 2\pi\ radians$, so $1\ degree = \frac{\pi}{180}\ radians$

The conversion factor to convert degrees to radians is: $\frac{\pi}{180^\circ}$

The conversion factor to convert radians to degrees is: $\frac{180^\circ}{\pi}$

If an angle has no units, it is assumed to be in radians.

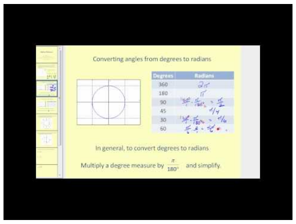
If you were to convert 150° into radians, you would multiply 150° by the correct conversion factor. You would get:

$150^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{18} = \frac{5\pi}{6}\ radians$

You can check your work by making sure the degree units cancel.

If you were to convert $\frac{\pi}{6}$ radians into degrees, you would multiply $\frac{\pi}{6}$ by the correct conversion factor. You would get $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$

Often the π's will cancel.



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Examples

Example 1

Earlier, you were asked how many radians make up a circle. Exactly 2π radians describe a circular arc. This is because 2π radii wrap around the circumference of any circle.

Example 2

Convert $(6\pi)^\circ$ into radians.

Don't be fooled just because this has π. This number is about 19°

$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{3}$

It is very unusual to ever have a π^2 term, but it can happen.

Example 3

Convert $\frac{5\pi}{6}$ into degrees.

$\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{5 \cdot 30^\circ}{1} = 150^\circ$

Example 4

Convert 210° into radians.

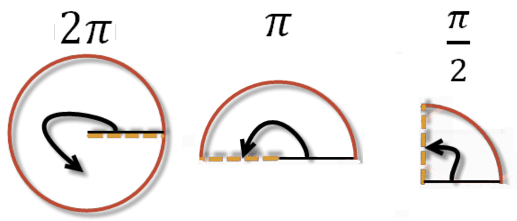
INSERT 4

$210^\circ \cdot \frac{\pi}{180^\circ} = \frac{7 \cdot 30 \cdot \pi}{6 \cdot 30} = \frac{7\pi}{6}$

Example 5

Draw a $\frac{\pi}{2}$ angle by first drawing a 2π angle, halving it and halving the result.

$\frac{\pi}{2} = 90^\circ$



Review

Find the radian measure of each angle.

1. 120°
2. 300°
3. 90°
4. 330°
5. 270°
6. 45°
7. $(5\pi)^\circ$

Find the degree measure of each angle.

8. $\frac{7\pi}{6}$
9. $\frac{5\pi}{4}$
10. $\frac{3\pi}{2}$
11. $\frac{5\pi}{3}$
12. π
13. $\frac{\pi}{6}$
14. 3
15. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.1.

4.2 Phase Shift of Sinusoidal Functions

Learning Objectives

Here you will apply all the different transformations, including horizontal shifting, to sinusoidal functions.

A periodic function that does not start at the sinusoidal axis or at a maximum or a minimum has been shifted horizontally. This horizontal movement invites different people to see different starting points since a sine wave does not have a beginning or an end.

What are five other ways of writing the function $f(x) = 2 \cdot \sin x$?

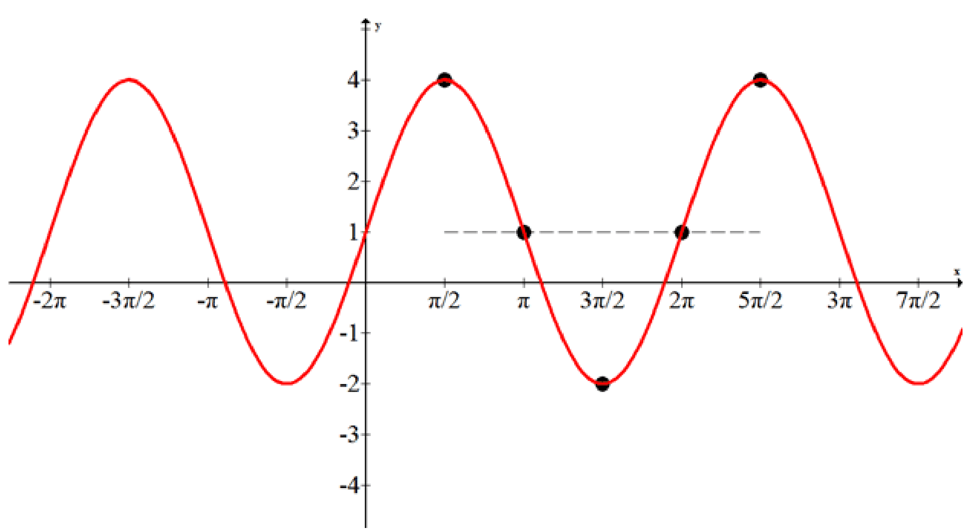
Phase Shift of Sinusoidal Functions

The general sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

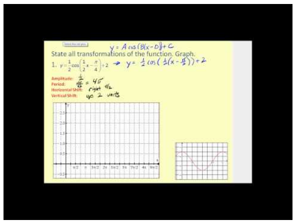
The constant c controls the phase shift. **Phase shift** is the horizontal shift left or right for periodic functions. If $c = \frac{\pi}{2}$ then the sine wave is shifted left by $\frac{\pi}{2}$. If $c = -3$ then the sine wave is shifted right by 3. This is the opposite direction than you might expect, but it is consistent with the rules of transformations for all functions.

To graph a function such as $f(x) = 3 \cdot \cos\left(x - \frac{\pi}{2}\right) + 1$, first find the start and end of one period. Then sketch only that portion of the sinusoidal axis. Finally, plot the 5 important points for a cosine graph while keeping the amplitude in mind. The graph is shown below:



Generally b is always written to be positive. If you run into a situation where b is negative, use your knowledge of even and odd functions to rewrite the function.

$$\cos(-x) = \cos(x)$$
$$\sin(-x) = -\sin(x)$$



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Examples

Example 1

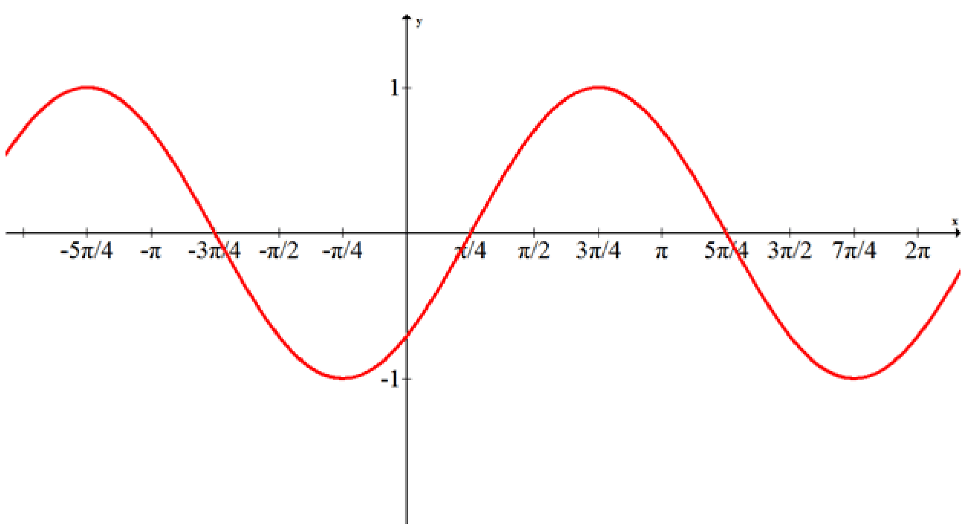
Earlier, you were asked to write $f(x) = 2 \cdot \sin x$ in five different ways. The function $f(x) = 2 \cdot \sin x$ can be rewritten an infinite number of ways.

$$2 \cdot \sin x = -2 \cdot \cos\left(x + \frac{\pi}{2}\right) = 2 \cdot \cos\left(x - \frac{\pi}{2}\right) = -2 \cdot \sin(x - \pi) = 2 \cdot \sin(x - 8\pi)$$

It all depends on where you choose start and whether you see a positive or negative sine or cosine graph.

Example 2

Given the following graph, identify equivalent sine and cosine algebraic models.



Either this is a sine function shifted right by $\frac{\pi}{4}$ or a cosine graph shifted left $\frac{5\pi}{4}$.

$f(x) = \sin\left(x - \frac{\pi}{4}\right) = \cos\left(x + \frac{5\pi}{4}\right)$

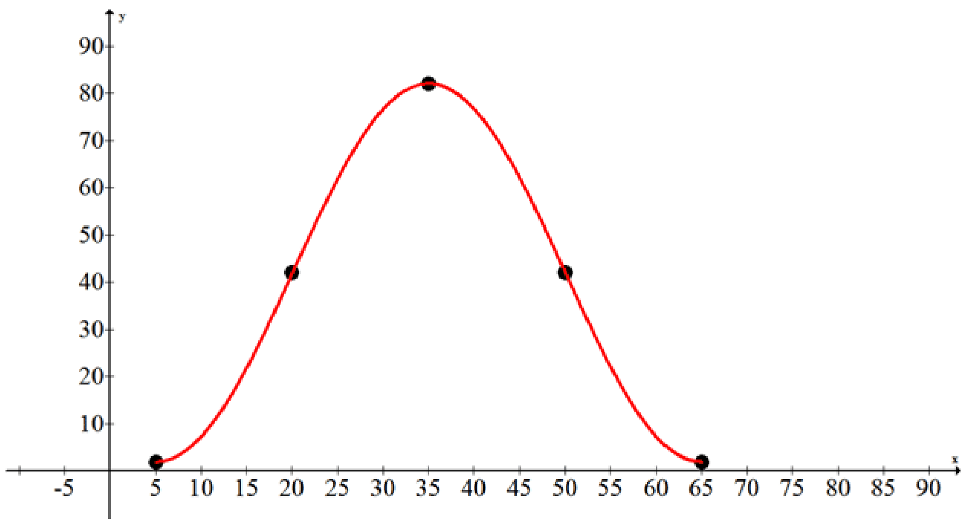
Example 3

At $t = 5$ minutes William steps up 2 feet to sit at the lowest point of the Ferris wheel that has a diameter of 80 feet. A full hour later he finally is let off the wheel after making only a single revolution. During that hour he wondered how to model his height over time in a graph and equation.

Since the period is 60 which works extremely well with the 360° in a circle, this problem will be shown in degrees.

TABLE 4.1:

Time (minutes)	Height (feet)
5	2
20	42
35	82
50	42
65	2



William chooses to see a negative cosine in the graph. He identifies the amplitude to be 40 feet. The vertical shift of the sinusoidal axis is 42 feet. The horizontal shift is 5 minutes to the right.

The period is 60 (not 65) minutes which implies $b = 6$ when graphed in degrees.

$60 = \frac{360}{b}$

Thus one equation would be:

$f(x) = -40 \cdot \cos(6(x-5)) + 42$

Example 4

Tide tables report the times and depths of low and high tides. Here is part of tide report from Salem, Massachusetts dated September 19, 2006.

TABLE 4.2:

10:15 AM	9 ft.	High Tide
4:15 PM	1 ft.	Low Tide
10:15 PM	9 ft.	High Tide

Find an equation that predicts the height based on the time. Choose when $t = 0$ carefully.

There are two logical places to set $t = 0$. The first is at midnight the night before and the second is at 10:15 AM. The first option illustrates a phase shift that is the focus of this concept, but the second option produces a simpler equation. Set $t = 0$ to be at midnight and choose units to be in minutes.

TABLE 4.3:

Time (hours : minutes)	Time (minutes)	Tide (feet)
10:15	615	9
16:15	975	1
22:15	1335	9
	$\frac{615+975}{2} = 795$	5
	$\frac{1335+975}{2} = 1155$	5

These numbers seem to indicate a positive cosine curve. The amplitude is four and the vertical shift is 5. The horizontal shift is 615 and the period is 720.

$720 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{360}$

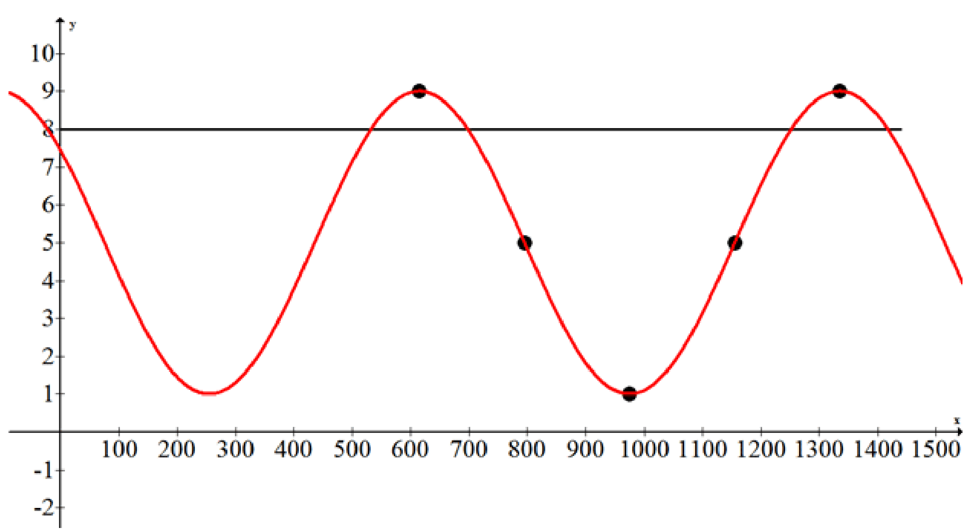
Thus one equation is:

$f(x) = 4 \cdot \cos\left(\frac{\pi}{360}(x - 615)\right) + 5$

Example 5

Use the equation from Example 4 to find out when the tide will be at exactly 8 ft on September 19^{th}.

This problem gives you the y and asks you to find the x. Later you will learn how to solve this algebraically, but for now use the power of the intersect button on your calculator to intersect the function with the line $y = 8$. Remember to find all the x values between 0 and 1440 to account for the entire 24 hours.



There are four times within the 24 hours when the height is exactly 8 feet. You can convert these times to hours and minutes if you prefer.

$t \approx 532.18$ (8:52), 697.82 (11:34), 1252.18 (20 : 52), 1417.82 (23:38)

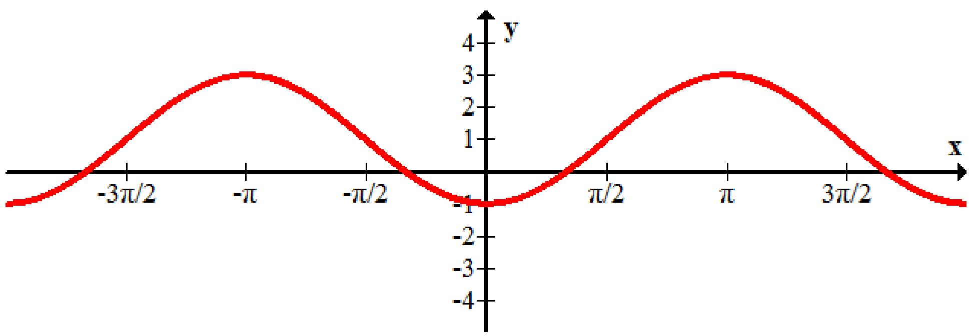
Review

Graph each of the following functions.

1. $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) - 1$
2. $g(x) = -\sin(x - \pi) + 3$
3. $h(x) = 3\cos(2(x - \pi))$
4. $k(x) = -2\sin(2x - \pi) + 1$
5. $j(x) = -\cos\left(x + \frac{\pi}{2}\right)$

Give one possible sine equation for each of the graphs below.

6.



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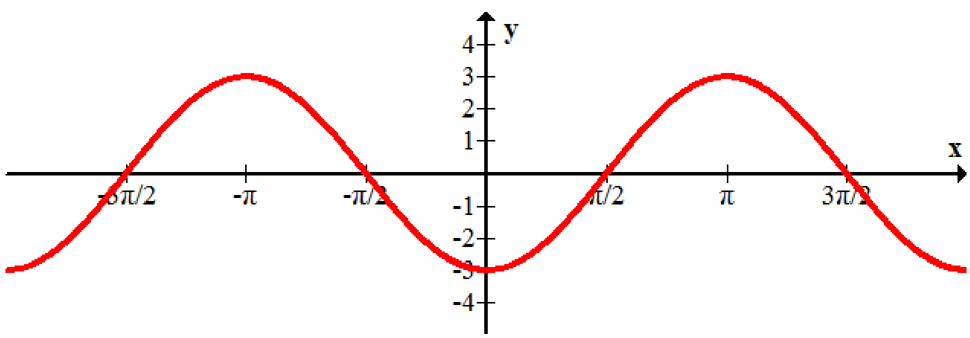

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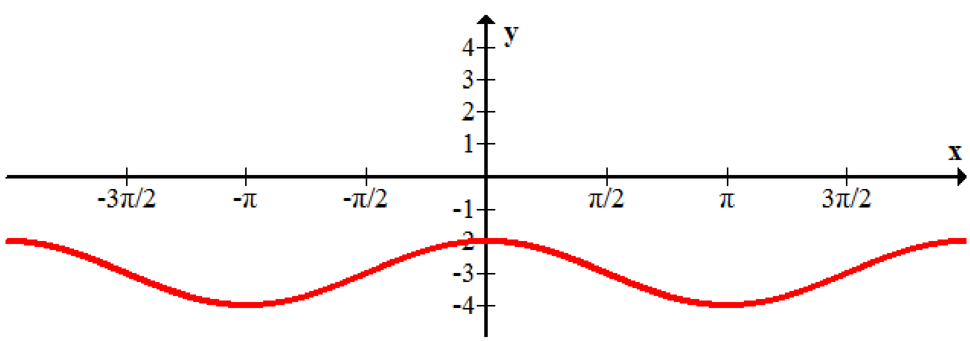
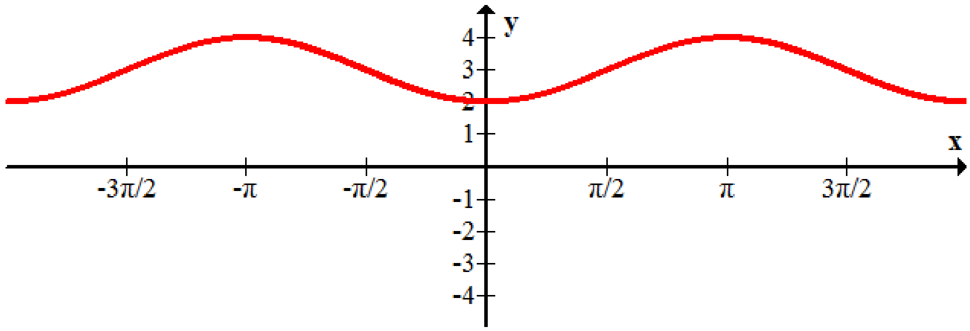


Give one possible cosine function for each of the graphs below.

9.



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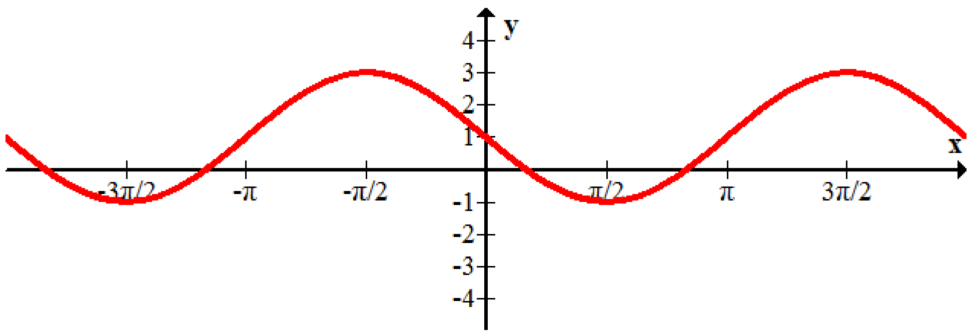

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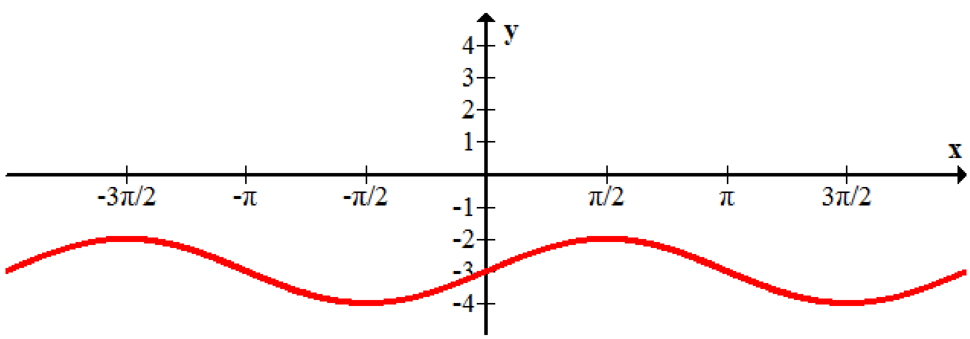


The temperature over a certain 24 hour period can be modeled with a sinusoidal function. At 3:00, the temperature for the period reaches a low of $22^\circ F$. At 15:00, the temperature for the period reaches a high of $40^\circ F$.

12. Find an equation that predicts the temperature based on the time in minutes. Choose $t = 0$ to be midnight.
13. Use the equation from #12 to predict the temperature at 4:00 PM.
14. Use the equation from #12 to predict the temperature at 8:00 AM.
15. Use the equation from #12 to predict the time(s) it will be $32^\circ F$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.6.

4.3 Graphs of Other Trigonometric Functions

Learning Objectives

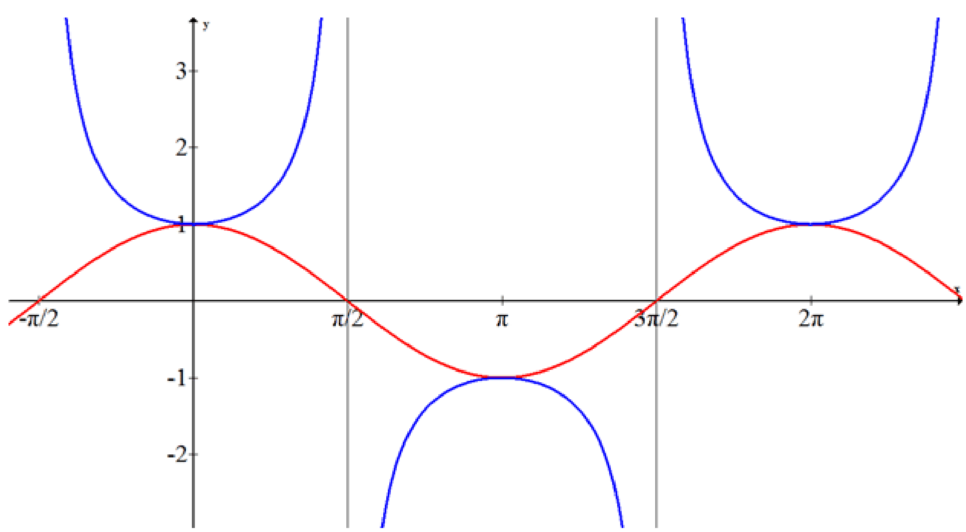
Here you will extend what you know about rational functions and sine and cosine functions to produce graphs of the four other trigonometric functions: tangent, secant, cosecant, and cotangent. You will also see how the four coefficients of a general sinusoidal function affect the new functions in similar ways.

If you already know the relationship between the equation and graph of sine and cosine functions then the other four functions can be found by identifying zeroes, asymptotes and key points. Are the four new functions transformations of the sine and cosine functions?

Graphing Other Trigonometric Functions

Secant and Cosecant

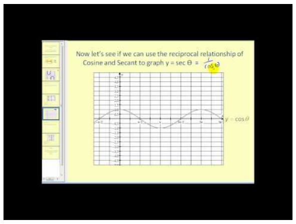
Since secant is the inverse of cosine the graphs are very closely related.



Notice wherever cosine is zero, secant has a vertical asymptote and where $\cos x = 1$ then $\sec x = 1$ as well. These two logical pieces allow you to graph any secant function of the form:

$f(x) = \pm a \cdot \sec(b(x+c)) + d$

The method is to graph it as you would a cosine and then insert asymptotes and the secant curves so they touch the cosine curve at its maximum and minimum values. This technique is identical to graphing cosecant graphs. Simply use the sine graph to find the location and asymptotes.



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Tangent and Cotangent

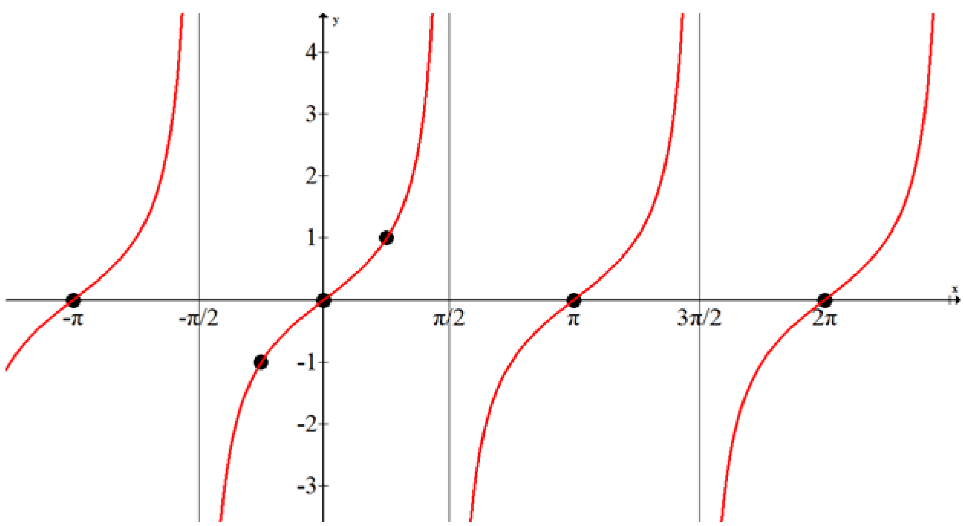
The tangent and cotangent graphs are more difficult because they are a ratio of the sine and cosine functions.

- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$

The way to think through the graph of $f(x) = \tan x$ is to first determine its asymptotes. The asymptotes occur when the denominator, cosine, is zero. This happens at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\ldots$ The next thing to plot is the zeros which occur when the numerator, sine, is zero. This happens at $0, \pm\pi, \pm, 2\pi\ldots$ From the unit circle and basic right triangle trigonometry, you already know some values of $\tan x$:

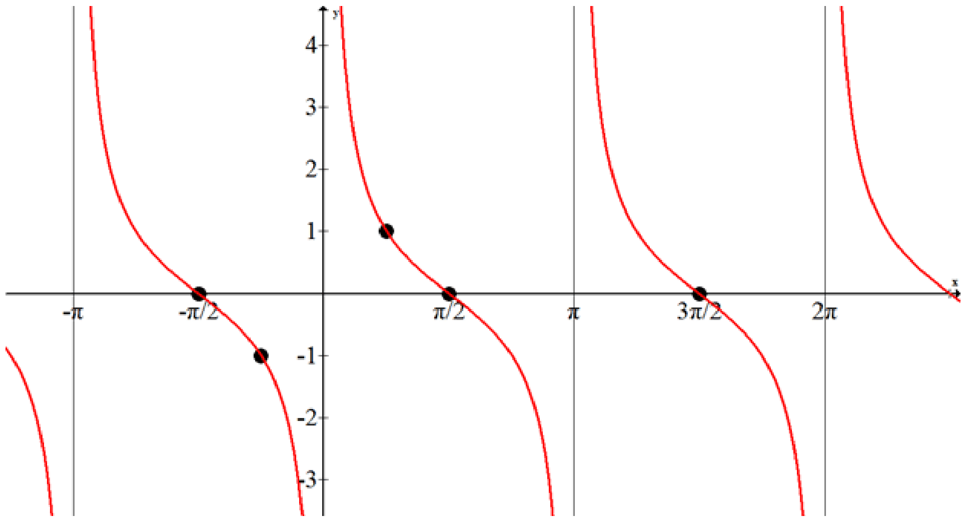
- $\tan\frac{\pi}{4} = 1$
- $\tan\left(-\frac{\pi}{4}\right) = -1$

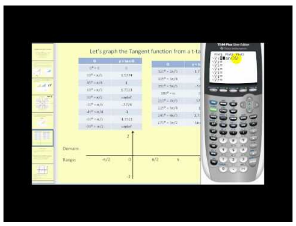
By plotting all this information, you get a very good sense as to what the graph of tangent looks like and you can fill in the rest.



Notice that the period of tangent is π not 2π, because it has a shorter cycle.

The graph of cotangent can be found using identical logic as tangent. You know $\cot x = \frac{1}{\tan x}$. This means that the graph of cotangent will have zeros wherever tangent has asymptotes and asymptotes wherever tangent has zeroes. You also know that where tangent is 1, cotangent is also 1. Thus the graph of cotangent is:





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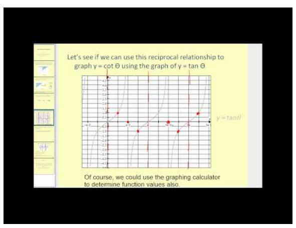

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Examples

Example 1

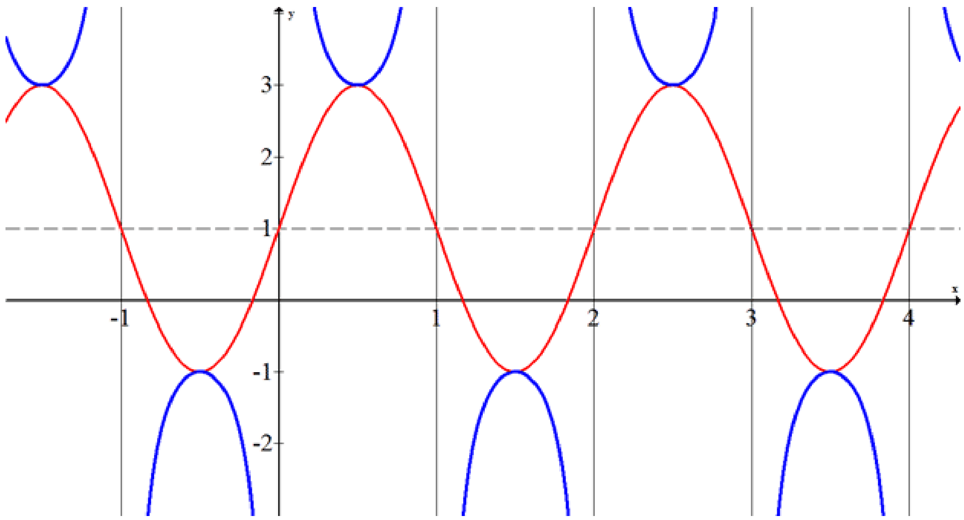
Earlier, you were asked if the four new functions are transformations of sine and cosine. The four new functions are not purely transformations of the sine and cosine functions. However, secant and cosecant are transformations of each other as are tangent and cotangent.

Example 2

Graph the function $f(x) = -2 \cdot \csc(\pi(x-1)) + 1$.

Graph the function as if it were a sine function. Then insert asymptotes wherever the sine function crosses the sinusoidal axis. Lastly add in the cosecant curves.

The amplitude is 2. The shape is negative sine. The function is shifted up one unit and to the right one unit.



Note that only the blue portion of the graph represents the given function.

Example 3

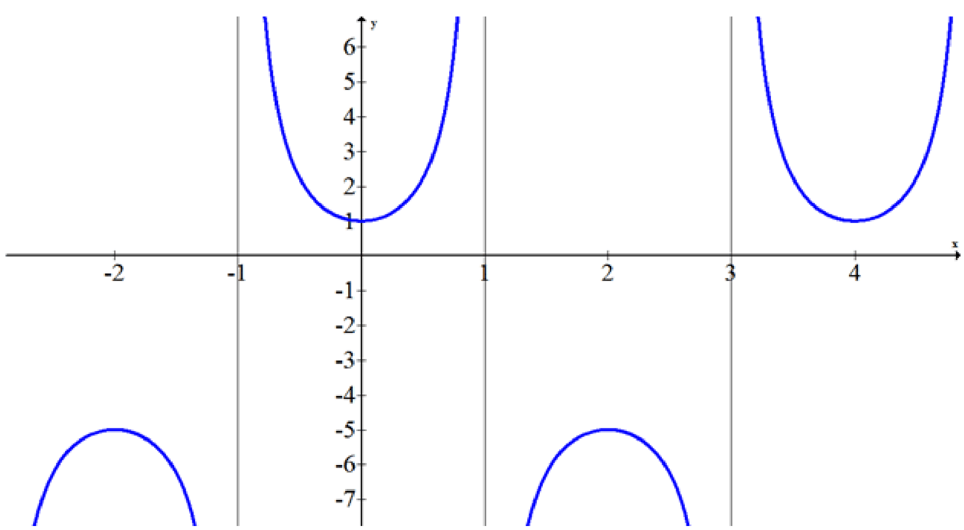
How do you write a tangent function as a cotangent function?

There are two main ways to go between a tangent function and a cotangent function. The first method was discussed in Example A: $f(x) = \tan x = \frac{1}{\cot x}$.

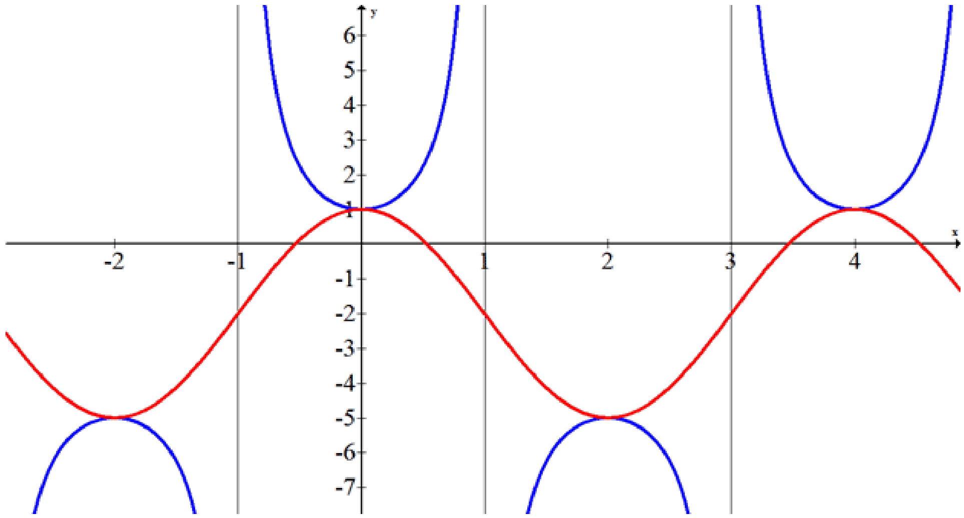
The second approach involves two transformations. Start by reflecting across the x or the y axis. Notice that this produces an identical result. Next shift the function to the right or left by $\frac{\pi}{2}$. Again this produces an identical result. $f(x) = \tan x = -\cot\left(x - \frac{\pi}{2}\right)$.

Example 4

Find the equation of the function in the following graph.



If you connect the relative maximums and minimums of the function, it produces a shifted cosine curve that is easier to work with.



The amplitude is 3. The vertical shift is 2 down. The period is 4 which implies that $b = \frac{\pi}{2}$. The shape is positive cosine and if you choose to start at $x = 0$ there is no phase shift.

$$f(x) = 3 \cdot \csc\left(\frac{\pi}{2}x\right) - 2$$

Example 5

Where are the asymptotes for tangent and why do they occur?

Since $\tan x = \frac{\sin x}{\cos x}$ the asymptotes occur whenever $\cos x = 0$ which is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Review

1. What function can you use to help you make a sketch of $f(x) = \sec x$? Why?

2. What function can you use to help you make a sketch of $g(x) = \csc x$? Why?

Make a sketch of each of the following from memory.

3. $f(x) = \sec x$

4. $g(x) = \csc x$

5. $h(x) = \tan x$

6. $k(x) = \cot x$

Graph each of the following.

7. $f(x) = 2\csc(x) + 1$

8. $g(x) = 2\csc\left(\frac{\pi}{2}x\right) + 1$

9. $h(x) = 2\csc\left(\frac{\pi}{2}(x-3)\right) + 1$

10. $j(x) = \cot\left(\frac{\pi}{2}x\right) + 3$

11. $k(x) = -\sec\left(\frac{\pi}{3}(x+1)\right) - 4$

12. $m(x) = -\tan(x) + 1$

13. $p(x) = -2\tan\left(x - \frac{\pi}{2}\right) + 1$
14. Find two ways to write $\sec x$ in terms of other trigonometric functions.
15. Find two ways to write $\csc x$ in terms of other trigonometric functions.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.7.

4.4 Graphs of Inverse Trigonometric Functions

Learning Objectives

Here you will graph the final form of trigonometric functions, the inverse trigonometric functions. You will learn why the entire inverses are not always included and you will apply basic transformation techniques.

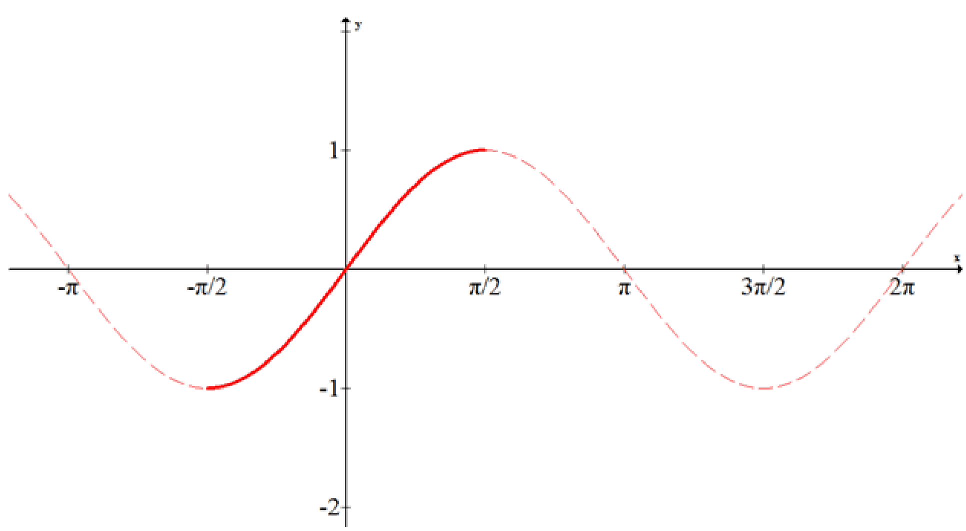
In order for inverses of functions to be functions, the original function must pass the **horizontal line test**. Though none of the trigonometric functions pass the horizontal line test, you can restrict their domains so that they can pass. Then the inverses are produced just like with normal functions. Once you have the basic inverse functions, the normal transformation rules apply.

Why is $\sin^{-1}(\sin 370^\circ) \neq 370^\circ$? Don't the arcsin and sin just cancel out?

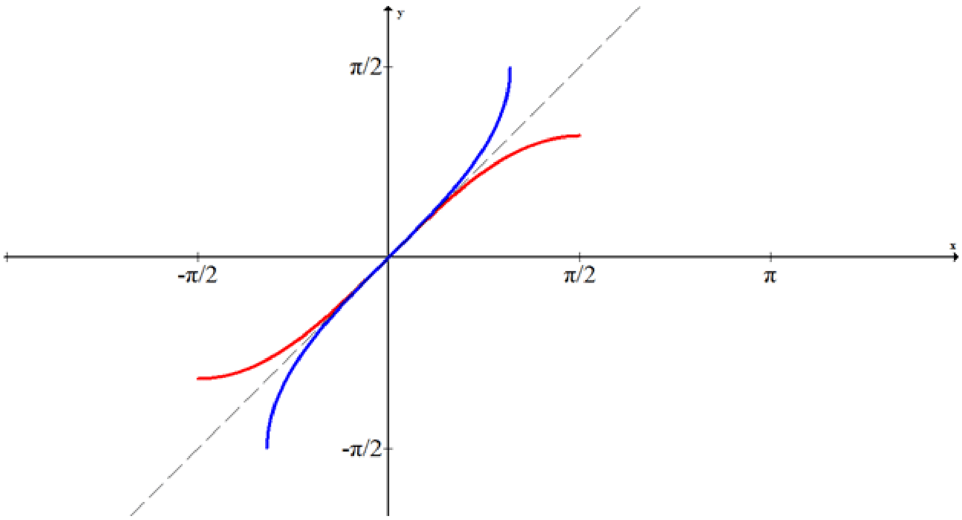
Graphs of Inverse Trigonometric Functions

Since none of the six trigonometric functions pass the horizontal line test, you must **restrict their domains** before finding inverses of these functions. This is just like the way $y = \sqrt{x}$ is the inverse of $y = x^2$ when you restrict the domain to $x \geq 0$.

Consider the sine graph:



As a general rule, the restrictions to the domain are either the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[0, \pi]$ to keep things simple. In this case sine is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as shown above. To find the inverse, reflect the bold portion across the line $y = x$. The blue curve below shows $f(x) = \sin^{-1} x$.



The result of this inversion is that arcsine will only ever produce angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. You must use logic and common sense to interpret these numbers in context.

The blue curve below shows $f(x) = \cos^{-1} x$?

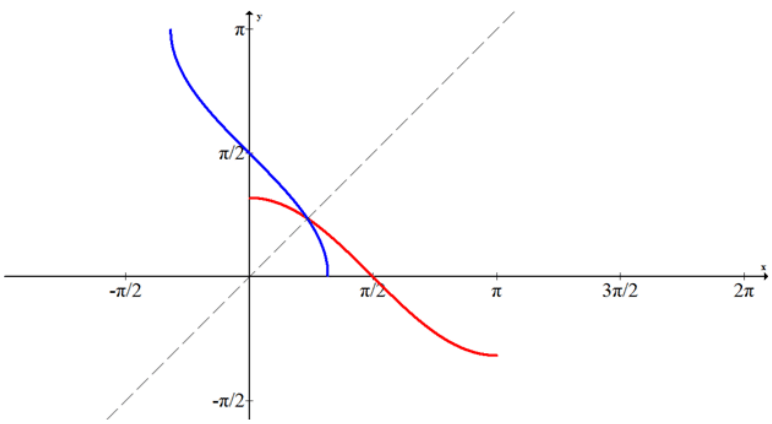
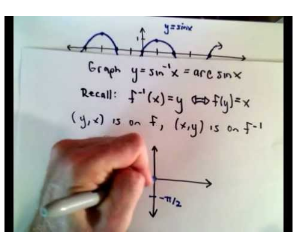


FIGURE 4.1

The portion of cosine that fits the horizontal line test is the interval $[0, \pi]$. To find the inverse, that portion is reflected across the line $y = x$.



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Examples

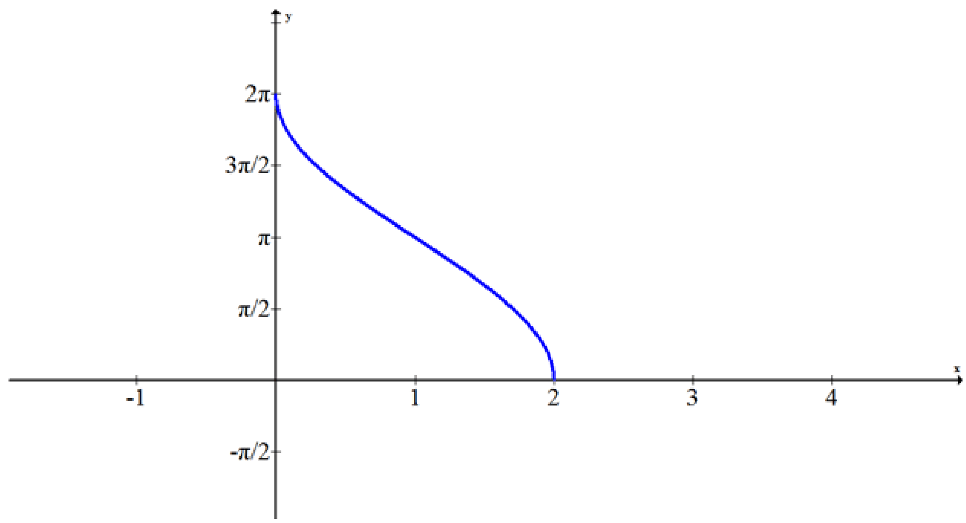
Example 1

Earlier, you were asked why sine and arcsine don't always just cancel out. Since arcsine only produces angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ or -90° to $+90^\circ$ the result of $\sin^{-1}(\sin 370^\circ)$ is 10° which is coterminal to 370°.

Example 2

Graph the function $f(x) = 2\cos^{-1}(x-1)$.

Since the graph of $f(x) = \cos^{-1} x$ was done in Example A, now you just need to shift it right one unit and stretch it vertically by a factor of 2. It intersected the x axis at 1 before and now it will intersect at 2. It reached a height of π before and now it will reach a height of 2π.



Example 3

Evaluate the following expression with and without a calculator using right triangles and your knowledge of inverse trigonometric functions.

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)$

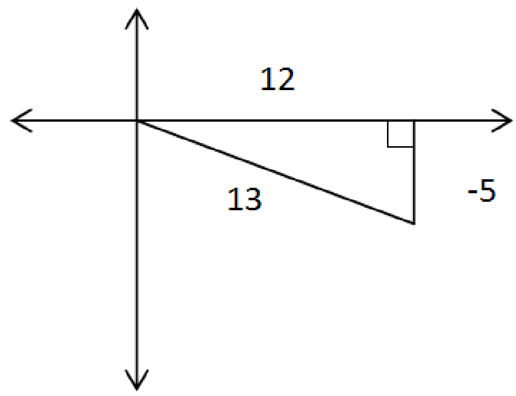
When using a calculator it can be extremely confusing trying to tell the difference between $\sin^{-1} x$ and $(\sin x)^{-1}$. In order to be able to effectively calculate this out it is best to write the expression explicitly only in terms of functions that your calculator does have.

The hardest part of this question is seeing the csc as a function (which produces an angle) on a ratio of a hypotenuse of 13 and an opposite side of -5. The sine of the inverse ratio must produce the same angle, so you can substitute it.

- $\csc^{-1}\left(-\frac{13}{5}\right) = \sin^{-1}\left(-\frac{5}{13}\right)$
- $\cot(\theta) = \frac{1}{\tan\theta}$

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = \frac{1}{\tan\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)} = -\frac{12}{5}$

Not using a calculator is usually significantly easier. Start with your knowledge that $\csc^{-1}\left(-\frac{13}{5}\right)$ describes an angle in the fourth or the second quadrant because those are the two quadrants where cosecant is negative. Since $\csc^{-1}\theta$ has range $-\frac{\pi}{2}, \frac{\pi}{2}$, it only produces angles in quadrant I or quadrant IV (see Guided Practice 2). This triangle must then be in the fourth quadrant. All you need to do is draw the triangle and identify the cotangent ratio.



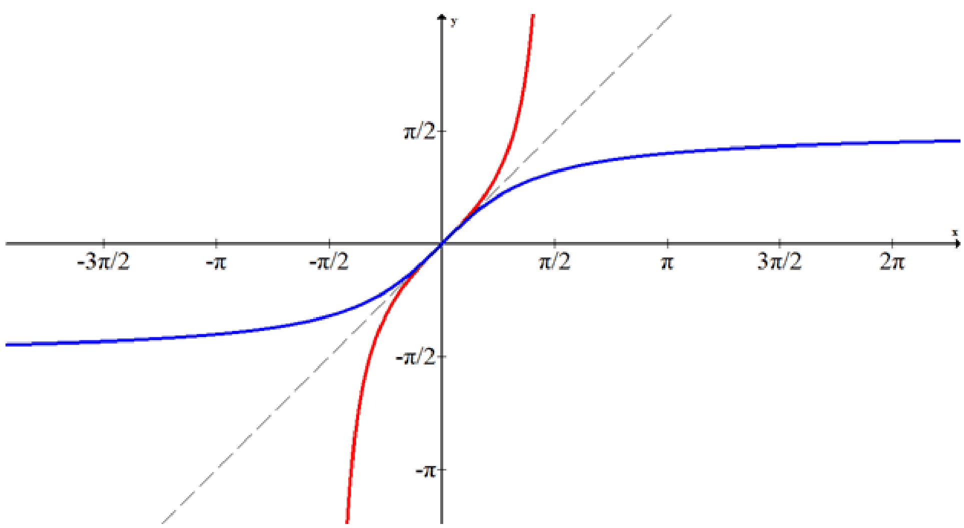
Cotangent is adjacent over opposite.

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)=-\frac{12}{5}$

Example 4

What is the graph of $y=\tan^{-1}x$?

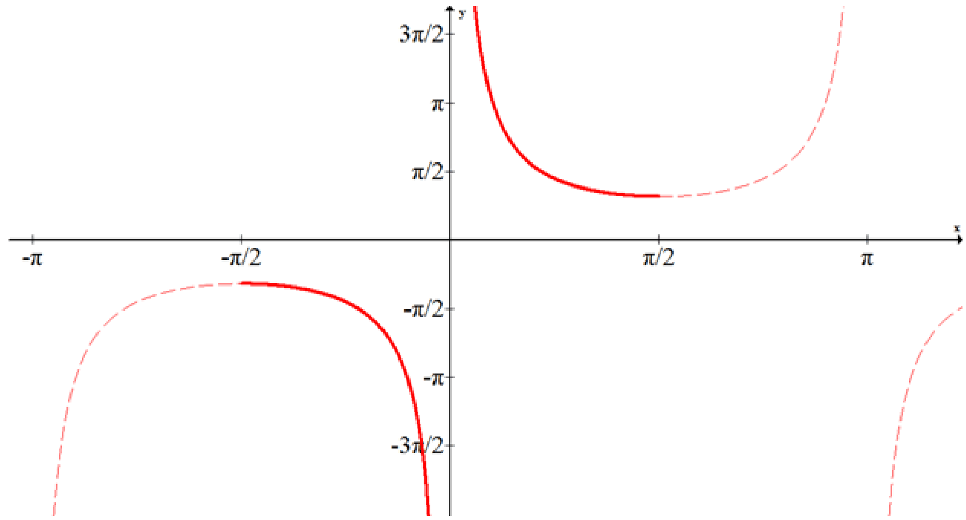
Graph the portion of tangent that fits the horizontal line test and reflect across the line $y=x$. Note that the graph of arctan is in blue.

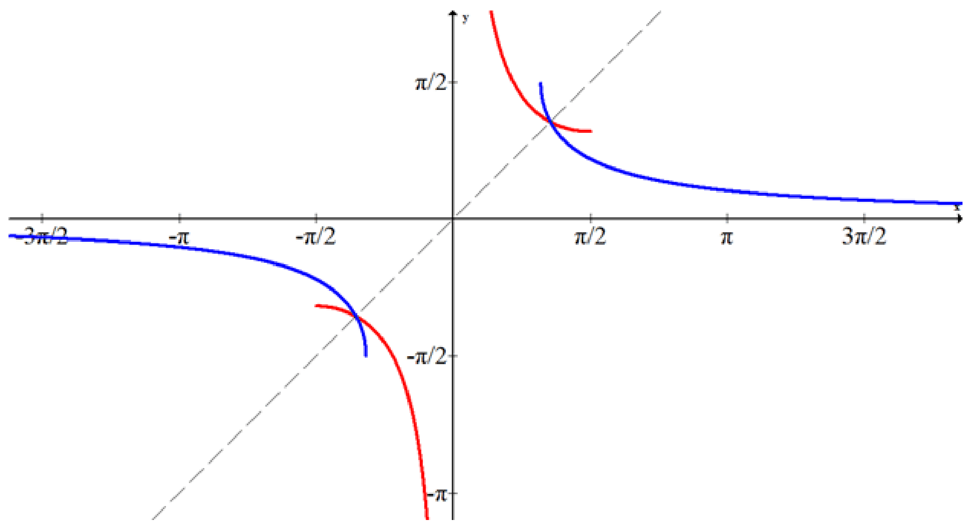


Example 5

What is the graph of $y=\csc^{-1}x$

Graph the portion of cosecant that fits the horizontal line test and reflect across the line $y=x$.





Note that $f(x) = \csc^{-1} x$ is in blue. Also note that the word "arc-co-secant" is too cumbersome to use because of the train of prefixes.

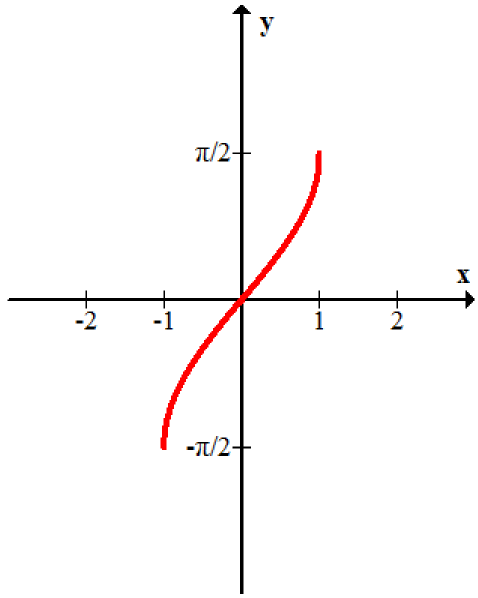
Review

1. Graph $f(x) = \cot^{-1} x$.

2. Graph $g(x) = \sec^{-1} x$.

Name each of the following graphs.

3.



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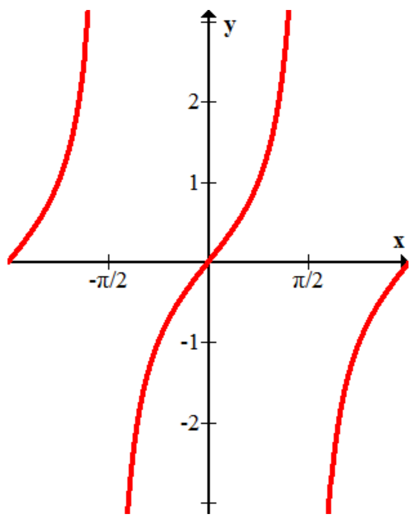

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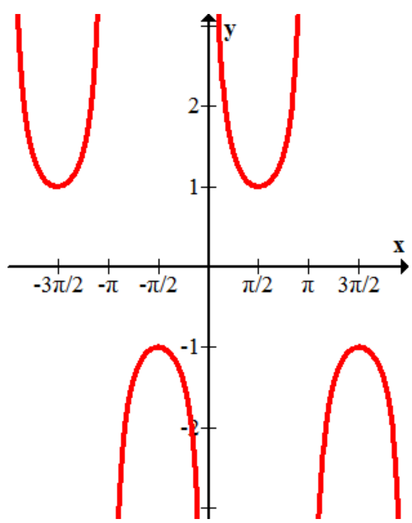

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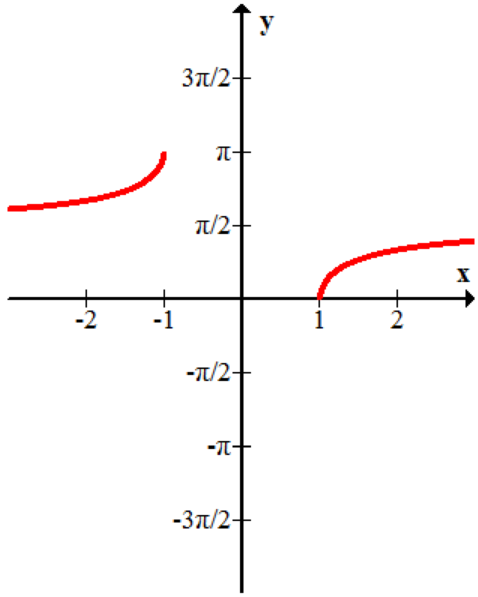

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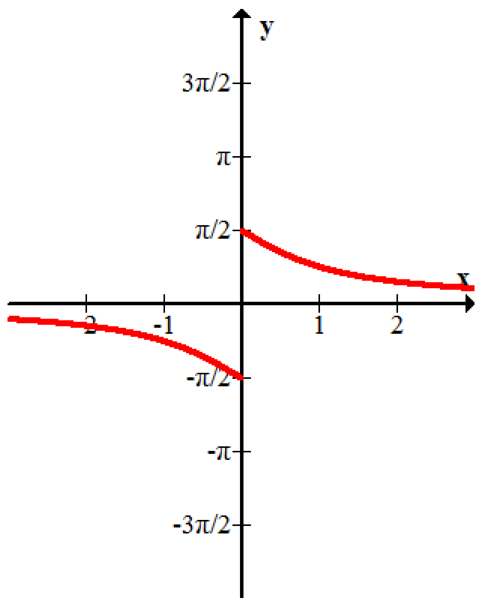


Graph each of the following functions using your knowledge of function transformations.

8. $h(x) = 3\sin^{-1}(x+1)$

9. $k(x) = 2\sin^{-1}(x) + \frac{\pi}{2}$

10. $m(x) = -\cos^{-1}(x-2)$

11. $j(x) = \cot^{-1}(x) + \pi$

12. $p(x) = -2\tan^{-1}(x-1)$

13. $q(x) = \csc^{-1}(x-2)$

14. $r(x) = -\sec^{-1}(x) + 4$

15. $t(x) = \csc^{-1}(x+1) - \frac{3\pi}{2}$

16. $v(x) = 2\sec^{-1}(x+2) + \frac{\pi}{2}$

17. $w(x) = -\cot^{-1}(x) - \frac{\pi}{2}$

Evaluate each expression.

18. $\sec\left(\tan^{-1}\left[\frac{3}{4}\right]\right)$

19. $\cot\left(\csc^{-1}\left[\frac{13}{12}\right]\right)$

20. $\csc\left(\tan^{-1}\left[\frac{4}{3}\right]\right)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.8.

4.5 Basic Trigonometric Identities

Learning Objectives

Here you will simplify trigonometric expressions using the reciprocal, quotient, odd-even and cofunction identities. You will also apply these simplification techniques in trigonometric proofs.

The basic trigonometric identities are ones that can be logically deduced from the definitions and graphs of the six trigonometric functions. Previously, some of these identities have been used in a casual way, but now they will be formalized and added to the toolbox of trigonometric identities.

How can you use the trigonometric identities to simplify the following expression?

$$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$$

Trigonometric Identities

An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.

Reciprocal Identities

The **reciprocal identities** refer to the connections between the trigonometric functions like sine and cosecant. Sine is opposite over hypotenuse and cosecant is hypotenuse over opposite. This logic produces the following six identities.

- $\sin\theta = \frac{1}{\csc\theta}$
- $\cos\theta = \frac{1}{\sec\theta}$
- $\tan\theta = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

Quotient Identities

The **quotient identities** follow from the definition of sine, cosine and tangent.

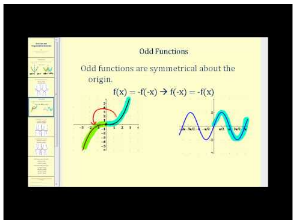
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Odd/Even Identities

The **odd-even identities** follow from the fact that only cosine and its reciprocal secant are even and the rest of the trigonometric functions are odd.

- $\sin(-\theta) = -\sin\theta$

- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\csc(-\theta) = -\csc\theta$



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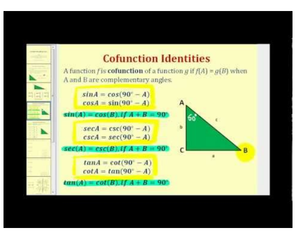
Cofunction Identities

The **cofunction identities** make the connection between trigonometric functions and their "co" counterparts like sine and cosine. Graphically, all of the cofunctions are reflections and horizontal shifts of each other.

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$



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Examples

Example 1

Earlier, you were asked how you could simplify the trigonometric expression:

$$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$$

It can be simplified to be equivalent to negative tangent as shown below:

$$\begin{aligned}\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1} &= \frac{\sin(-\theta)}{\sin\left(\frac{\pi}{2}-\theta\right)}\\ &= \frac{-\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

Example 2

If $\sin\theta = 0.87$, find $\cos\left(\theta - \frac{\pi}{2}\right)$.

While it is possible to use a calculator to find θ, using identities works very well too.

First you should factor out the negative from the argument. Next you should note that cosine is even and apply the odd-even identity to discard the negative in the argument. Lastly recognize the cofunction identity.

$\cos\left(\theta-\frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2}-\theta\right)\right) = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = 0.87$

Example 3

If $\cos\left(\theta-\frac{\pi}{2}\right) = 0.68$ then determine $\csc(-\theta)$.

You need to show that $\cos\left(\theta-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}-\theta\right)$.

$$\begin{aligned}0.68 &= \cos\left(\theta-\frac{\pi}{2}\right)\\ &= \cos\left(\frac{\pi}{2}-\theta\right)\\ &= \sin(\theta)\end{aligned}$$

Then, $\csc(-\theta) = -\csc\theta$

$$\begin{aligned}&= -\frac{1}{\sin\theta}\\ &= -(0.68)^{-1}\end{aligned}$$

Example 4

Use identities to prove the following: $\cot(-\beta)\cot\left(\frac{\pi}{2}-\beta\right)\sin(-\beta) = \cos\left(\beta-\frac{\pi}{2}\right)$.

When doing trigonometric proofs, it is vital that you start on one side and only work with that side until you derive what is on the other side. Sometimes it may be helpful to work from both sides and find where the two sides meet, but this work is not considered a proof. You will have to rewrite your steps so they follow from only one side. In this case, work with the left side and keep rewriting it until you have $\cos\left(\beta-\frac{\pi}{2}\right)$.

$$\begin{aligned}
\cot(-\beta)\cot\left(\frac{\pi}{2}-\beta\right)\sin(-\beta) &= -\cot\beta\tan\beta\cdot -\sin\beta \\
&= -1\cdot -\sin\beta \\
&= \sin\beta \\
&= \cos\left(\frac{\pi}{2}-\beta\right) \\
&= \cos\left(-\left(\beta-\frac{\pi}{2}\right)\right) \\
&= \cos\left(\beta-\frac{\pi}{2}\right)
\end{aligned}$$

Example 5

Prove the following trigonometric identity by working with only one side.

$\cos x \sin x \tan x \cot x \sec x \csc x = 1$

$$\begin{aligned}
\cos x \sin x \tan x \cot x \sec x \csc x &= \cos x \sin x \tan x \cdot \frac{1}{\tan x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\sin x} \\
&= 1
\end{aligned}$$

Review

1. Prove the quotient identity for cotangent using sine and cosine.
2. Explain why $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ using graphs and transformations.
3. Explain why $\sec\theta=\frac{1}{\cos\theta}$.
4. Prove that $\tan\theta\cdot\cot\theta=1$.
5. Prove that $\sin\theta\cdot\csc\theta=1$.
6. Prove that $\sin\theta\cdot\sec\theta=\tan\theta$.
7. Prove that $\cos\theta\cdot\csc\theta=\cot\theta$.
8. If $\sin\theta=0.81$, what is $\sin(-\theta)$?
9. If $\cos\theta=0.5$, what is $\cos(-\theta)$?
10. If $\cos\theta=0.25$, what is $\sec(-\theta)$?
11. If $\csc\theta=0.7$, what is $\sin(-\theta)$?
12. How can you tell from a graph if a function is even or odd?
13. Prove $\frac{\tan x\cdot\sec x}{\csc x}\cdot\cot x=\tan x$.
14. Prove $\frac{\sin^2 x\cdot\sec x}{\tan x}\cdot\csc x=1$.
15. Prove $\cos x\cdot\tan x=\sin x$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

4.6 Pythagorean Identities

Learning Objectives

Here you will prove and use the Pythagorean identities for the six trigonometric functions to simplify expressions and write proofs.

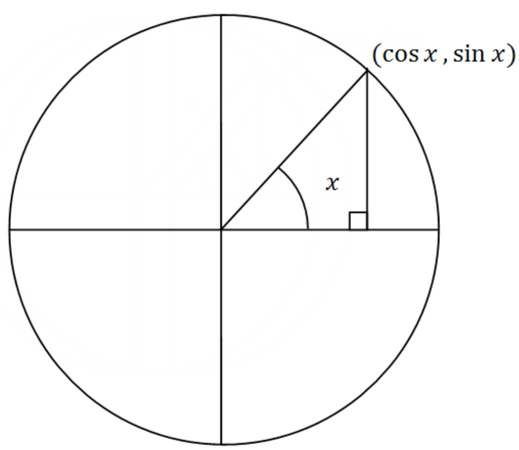
The Pythagorean Theorem works on right triangles. If you consider the x coordinate of a point along the unit circle to be the cosine and the y coordinate of the point to be the sine and the distance to the origin to be 1 then the Pythagorean Theorem immediately yields the identity:

$$y^2 + x^2 = 1$$
$$\sin^2 x + \cos^2 x = 1$$

An observant student may guess that other Pythagorean identities exist with the rest of the trigonometric functions. Is $\tan^2 x + \cot^2 x = 1$ a legitimate identity?

Pythagorean Identities

The proof of the **Pythagorean identity** for sine and cosine is essentially just drawing a right triangle in a unit circle, identifying the cosine as the x coordinate, the sine as the y coordinate and 1 as the hypotenuse.



$\cos^2 x + \sin^2 x = 1$

Most people rewrite the order of the sine and cosine so that the sine comes first.

$\sin^2 x + \cos^2 x = 1$

The two other Pythagorean identities are:

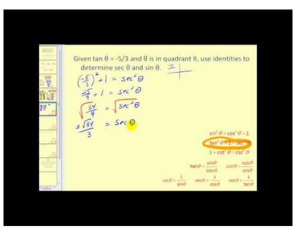
- $1 + \cot^2 x = \csc^2 x$
- $\tan^2 x + 1 = \sec^2 x$

To derive these two Pythagorean identities, divide the original Pythagorean identity by $\sin^2 x$ and $\cos^2 x$ respectively. To derive the Pythagorean identity $1+\cot^2 x=\csc^2 x$ divide through by $\sin^2 x$ and simplify.

$$\begin{aligned}\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}&=\frac{1}{\sin^2 x}\\ 1+\cot^2 x&=\csc^2 x\end{aligned}$$

Similarly, to derive the Pythagorean identity $\tan^2 x+1=\sec^2 x$, divide through by $\cos^2 x$ and simplify.

$$\begin{aligned}\frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}&=\frac{1}{\cos^2 x}\\ \tan^2 x+1&=\sec^2 x\end{aligned}$$



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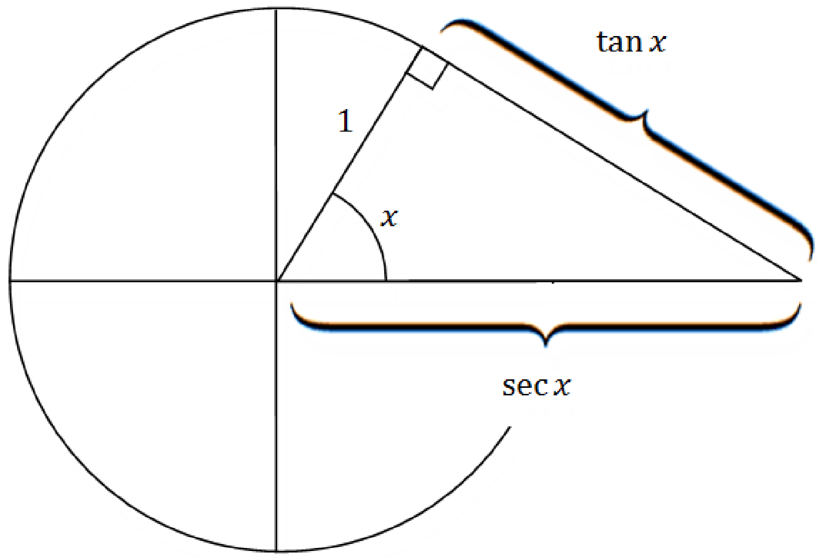
Examples

Example 1

Earlier, you were asked if $\tan^2 x+\cot^2 x=1$ is a legitimate identity. Cofunctions are not always connected directly through a Pythagorean identity.

$\tan^2 x+\cot^2 x\neq 1$

Visually, the right triangle connecting tangent and secant can also be observed in the unit circle. Most people do not know that tangent is named "tangent" because it refers to the distance of the line tangent from the point on the unit circle to the x axis. Look at the picture below and think about why it makes sense that $\tan x$ and $\sec x$ are as marked. $\tan x=\frac{opp}{adj}$. Since the adjacent side is equal to 1 (the radius of the circle), $\tan x$ simply equals the opposite side. Similar logic can explain the placement of $\sec x$.



Example 2

Simplify the following expression: $\frac{\sin x(\csc x - \sin x)}{1-\sin x}$.

$$\begin{aligned}
\frac{\sin x(\csc x - \sin x)}{1-\sin x} &= \frac{\sin x \cdot \csc x - \sin^2 x}{1-\sin x} \\
&= \frac{1-\sin^2 x}{1-\sin x} \\
&= \frac{(1-\sin x)(1+\sin x)}{1-\sin x} \\
&= 1+\sin x
\end{aligned}$$

Note that factoring the Pythagorean identity is one of the most powerful applications. This is very common and is a technique that you should feel comfortable using.

Example 3

Prove the following trigonometric identity: $(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) = 2$

Group the terms and apply a different form of the second two Pythagorean identities which are $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$.

$$\begin{aligned}
(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) &= \sec^2 x - \tan^2 x + \csc^2 x - \cot^2 x \\
&= 1+1 \\
&= 2
\end{aligned}$$

Example 4

Simplify the following expression.

$(\sec^2 x)(1-\sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right)$

$$\begin{aligned}
&(\sec^2 x)(1-\sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right) \\
&= \sec^2 x \cdot \cos^2 x - (\sin^2 x + \cos^2 x) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

Example 5

Simplify the following expression.

$(\cos t - \sin t)^2 + (\cos t + \sin t)^2$

Note that initially, the expression is not the same as the Pythagorean identity.

$$\begin{aligned}
&(\cos t - \sin t)^2 + (\cos t + \sin t)^2 \\
&= \cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t \\
&= 1 - 2\cos t \sin t + 1 + 2\cos t \sin t \\
&= 2
\end{aligned}$$

Review

Prove each of the following:

1. $(1-\cos^2 x)(1+\cot^2 x) = 1$
2. $\cos x(1-\sin^2 x) = \cos^3 x$
3. $\sin^2 x = (1-\cos x)(1+\cos x)$
4. $\sin x = \frac{\sin^2 x + \cos^2 x}{\csc x}$
5. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$
6. $\sin^2 x \cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$

Simplify each expression as much as possible.

7. $\tan^3 x \csc^3 x$
8. $\frac{\csc^2 x - 1}{\sec^2 x}$
9. $\frac{1-\sin^2 x}{1+\sin x}$
10. $\sqrt{1-\cos^2 x}$
11. $\frac{\sin^2 x - \sin^4 x}{\cos^2 x}$
12. $(1+\tan^2 x)(\sec^2 x)$
13. $\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$
14. $\frac{1+\tan^2 x}{\csc^2 x}$
15. $\frac{1-\sin^2 x}{\cos x}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

4.7 Sum and Difference Identities

Learning Objectives

Here you will add six identities to your toolbox: the sum and difference identities for sine, cosine and tangent. You will use these identities along with previous identities for proofs and simplifying expressions.

With your knowledge of special angles like the sine and cosine of 30° and 45°, you can find the sine and cosine of 15°, the difference of 45° and 30°, and 75°, the sum of 45° and 30°. Using what you know about the unit circle and the sum and difference identities, how do you determine $\sin 15^\circ$ and $\sin 75^\circ$?

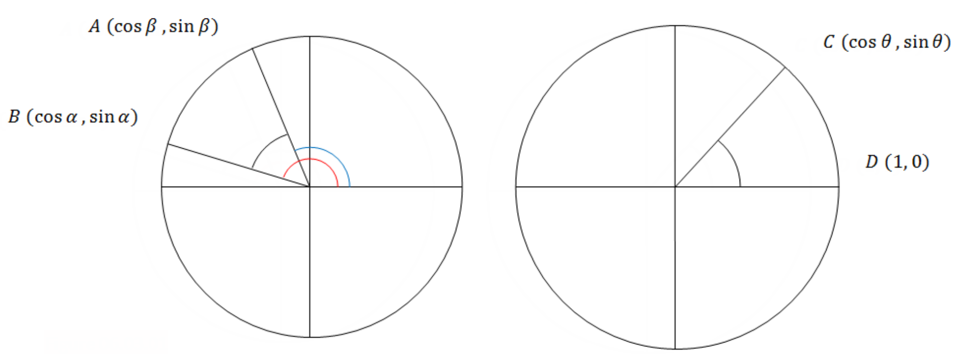
Sum and Difference Identities

There are some intuitive but incorrect formulas for sums and differences with respect to trigonometric functions. The form below does not work for any trigonometric function and is one of the most common incorrect guesses as to the sum and difference identity.

$\sin(\theta+\beta) \neq \sin\theta + \sin\beta$

First look at the derivation of the **cosine difference identity**:

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



Start by drawing two arbitrary angles α and β. In the image above α is the angle in red and β is the angle in blue. The difference $\alpha-\beta$ is noted in black as θ. The reason why there are two pictures is because the image on the right has the same angle θ in a rotated position. This will be useful to work with because the length of $\overline{AB}$ will be the same as the length of $\overline{CD}$.

$$\overline{AB} = \overline{CD}$$
$$\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} = \sqrt{(\cos\theta-1)^2+(\sin\theta-0)^2}$$
$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 = (\cos\theta-1)^2+(\sin\theta-0)^2$$

$$(\cos\alpha)^2-2\cos\alpha\cos\beta+(\cos\beta)^2+(\sin\alpha)^2-2\sin\alpha\sin\beta+(\sin\beta)^2 = (\cos\theta-1)^2+(\sin\theta)^2$$

$$\begin{aligned}
2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= (\cos\theta)^2-2\cos\theta+1+(\sin\theta)^2\\
2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= 1-2\cos\theta+1\\
-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= -2\cos\theta\\
\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos\theta\\
&= \cos(\alpha-\beta)
\end{aligned}$$

You can use this identity to prove the **cosine of a sum identity**. First, start with the cosine of a difference and make a substitution. Then use the odd-even identity.

$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$

Let $\gamma = -\beta$

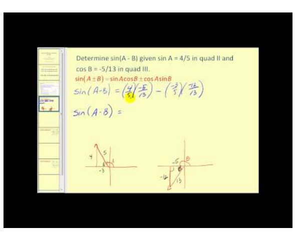
$$\begin{aligned}
\cos\alpha\cos(-\gamma)+\sin\alpha\sin(-\gamma) &= \cos(\alpha+\gamma)\\
\cos\alpha\cos\gamma-\sin\alpha\sin\gamma &= \cos(\alpha+\gamma)
\end{aligned}$$

The proofs for sine and tangent are left to the videos and examples. They are listed here for your reference. Cotangent, secant and cosecant are excluded because you can use reciprocal identities to get those once you have sine, cosine and tangent.

Summary

- $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta\mp\sin\alpha\sin\beta$
- $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta\pm\cos\alpha\sin\beta$
- $\tan(\alpha\pm\beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

Note that the symbol $\pm$ is short hand for "**plus or minus**" and the symbol $\mp$ is shorthand for "**minus or plus**." The order is important because for cosine of a sum, the negative sign is used on the other side of the identity. This is the opposite of sine of a sum, where a positive sign is used on the other side of the identity.



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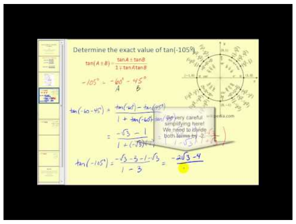

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Examples

Example 1

Earlier, you were asked to evaluate $\sin 15^\circ$ and $\sin 75^\circ$ exactly without a calculator. To do this you need to use the sine of a difference and sine of a sum.

$$\sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$
$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 2

Find the exact value of $\tan 15^\circ$ without using a calculator.

$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$

A final solution will not have a radical in the denominator. In this case multiplying through by the conjugate of the denominator will eliminate the radical. This technique is very common in PreCalculus and Calculus.

$$\begin{aligned}
&= \frac{(3 - \sqrt{3}) \cdot (3 - \sqrt{3})}{(3 + \sqrt{3}) \cdot (3 - \sqrt{3})} \\
&= \frac{(3 - \sqrt{3})^2}{9 - 3} \\
&= \frac{(3 - \sqrt{3})^2}{6}
\end{aligned}$$

Example 3

Evaluate the expression exactly without using a calculator.

$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ$

Once you know the general form of the sum and difference identities then you will recognize this as cosine of a difference.

$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ = \cos(50^\circ - 5^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$

Example 4

Use a sum or difference identity to find an exact value of $\cot\left(\frac{5\pi}{12}\right)$.

Start with the definition of cotangent as the inverse of tangent.

$$\begin{aligned}
\cot\left(\frac{5\pi}{12}\right) &= \frac{1}{\tan\left(\frac{5\pi}{12}\right)} \\
&= \frac{1}{\tan\left(\frac{9\pi}{12}-\frac{4\pi}{12}\right)} \\
&= \frac{1}{\tan(135^\circ-60^\circ)} \\
&= \frac{1+\tan 135^\circ \tan 60^\circ}{\tan 135^\circ - \tan 60^\circ} \\
&= \frac{1+(-1)\cdot\sqrt{3}}{(-1)-\sqrt{3}} \\
&= \frac{(1-\sqrt{3})}{(-1-\sqrt{3})} \\
&= \frac{(1-\sqrt{3})^2}{(-1+\sqrt{3})\cdot(1-\sqrt{3})} \\
&= \frac{(1-\sqrt{3})^2}{-(1-3)} \\
&= \frac{(1-\sqrt{3})^2}{2}
\end{aligned}$$

Example 5

Prove the following identity:

$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

Here are the steps:

$$\begin{aligned}
\frac{\sin(x-y)}{\sin(x+y)} &= \frac{\tan x - \tan y}{\tan x + \tan y} \\
\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} &= \\
\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} \cdot \frac{\left(\frac{1}{\cos x \cdot \cos y}\right)}{\left(\frac{1}{\cos x \cdot \cos y}\right)} &= \\
\frac{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) - \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)}{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) + \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)} &= \\
\frac{\tan x - \tan y}{\tan x + \tan y} &=
\end{aligned}$$

Review

Find the exact value for each expression by using a sum or difference identity.

1. $\cos 75^\circ$
2. $\cos 105^\circ$
3. $\cos 165^\circ$
4. $\sin 105^\circ$
5. $\sec 105^\circ$
6. $\tan 75^\circ$
7. Prove the sine of a sum identity.
8. Prove the tangent of a sum identity.
9. Prove the tangent of a difference identity.
10. Evaluate without a calculator: $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$.
11. Evaluate without a calculator: $\sin 35^\circ \cos 5^\circ - \cos 35^\circ \sin 5^\circ$.
12. Evaluate without a calculator: $\sin 55^\circ \cos 5^\circ + \cos 55^\circ \sin 5^\circ$.
13. If $\cos\alpha\cos\beta = \sin\alpha\sin\beta$, then what does $\cos(\alpha+\beta)$ equal?
14. Prove that $\tan\left(x+\frac{\pi}{4}\right) = \frac{1+\tan x}{1-\tan x}$.
15. Prove that $\sin(x+\pi) = -\sin x$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.3.

4.8 Double, Half, and Power Reducing Identities

Learning Objectives

Here you will prove and use the double, half, and power reducing identities.

These identities are significantly more involved and less intuitive than previous identities. By practicing and working with these advanced identities, your toolbox and fluency substituting and proving on your own will increase. Each identity in this concept is named aptly. Double angles work on finding $\sin 80^\circ$ if you already know $\sin 40^\circ$. Half angles allow you to find $\sin 15^\circ$ if you already know $\sin 30^\circ$. Power reducing identities allow you to find $\sin^2 15^\circ$ if you know the sine and cosine of 30°.

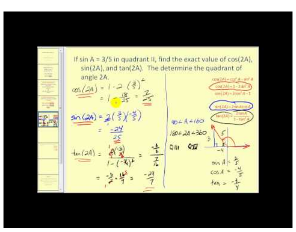
What is $\sin^2 15^\circ$?

Double Angle, Half Angle, and Power Reducing Identities

Double Angle Identities

The **double angle identities** are proved by applying the sum and difference identities. They are left as review problems. These are the double angle identities.

- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$



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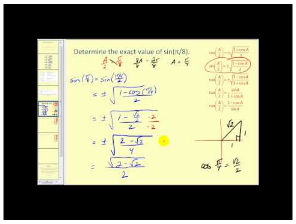
Half Angle Identities

The **half angle identities** are a rewritten version of the power reducing identities. The proofs are left as review problems.

- $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

- $\tan \frac{x}{2} = \pm \sqrt{\frac{1-\cos x}{1+\cos x}}$



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Power Reducing Identities

The **power reducing identities** allow you to write a trigonometric function that is squared in terms of smaller powers. The proofs are left as examples and review problems.

- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

Power reducing identities are most useful when you are asked to rewrite expressions such as $\sin^4 x$ as an expression without powers greater than one. While $\sin x \cdot \sin x \cdot \sin x \cdot \sin x$ does technically simplify this expression as necessary, you should try to get the terms to sum together not multiply together.

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 \\
&= \left(\frac{1-\cos 2x}{2}\right)^2 \\
&= \frac{1-2\cos 2x+\cos^2 2x}{4} \\
&= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)
\end{aligned}$$

Examples

Example 1

Earlier, you were asked to find $\sin^2 15^\circ$. In order to fully identify $\sin^2 15^\circ$ you need to use the power reducing formula.

$$\begin{aligned}
\sin^2 x &= \frac{1-\cos 2x}{2} \\
\sin^2 15^\circ &= \frac{1-\cos 30^\circ}{2} = \frac{1}{2} - \frac{\sqrt{3}}{4}
\end{aligned}$$

Example 2

Write the following expression with only $\sin x$ and $\cos x$: $\sin 2x + \cos 3x$.

$$\begin{aligned}
\sin 2x + \cos 3x &= 2\sin x \cos x + \cos(2x + x) \\
&= 2\sin x \cos x + \cos 2x \cos x - \sin 2x \sin x \\
&= 2\sin x \cos x + (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x \\
&= 2\sin x \cos x + \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= 2\sin x \cos x + \cos^3 x - 3\sin^2 x \cos x
\end{aligned}$$

Example 3

Use half angles to find an exact value of $\tan 22.5^\circ$ without using a calculator.

$\tan \frac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$\tan 22.5^\circ = \tan \frac{45^\circ}{2} = \pm\sqrt{\dfrac{1-\cos 45^\circ}{1+\cos 45^\circ}} = \pm\sqrt{\dfrac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\dfrac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}}$

Sometimes you may be requested to get all the radicals out of the denominator.

Example 4

Prove the power reducing identity for sine.

$\sin^2 x = \frac{1-\cos 2x}{2}$

Start with the double angle identity for cosine.

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\cos 2x &= (1-\sin^2 x) - \sin^2 x \\
\cos 2x &= 1 - 2\sin^2 x
\end{aligned}$$

This expression is an equivalent expression to the double angle identity and is often considered an alternate form.

$$\begin{aligned}
2\sin^2 x &= 1 - \cos 2x \\
\sin^2 x &= \frac{1-\cos 2x}{2}
\end{aligned}$$

Example 5

Simplify the following identity: $\sin^4 x - \cos^4 x$.

Here are the steps:

$$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)\\ &= -(\cos^2 x - \sin^2 x)\\ &= -\cos 2x\end{aligned}$$

Review

Prove the following identities.

1. $\sin 2x = 2\sin x\cos x$

2. $\cos 2x = \cos^2 x - \sin^2 x$

3. $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

4. $\cos^2 x = \frac{1+\cos 2x}{2}$

5. $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

6. $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

7. $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

8. $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

9. $\csc 2x = \frac{1}{2}\csc x\sec x$

10. $\cot 2x = \frac{\cot^2 x-1}{2\cot x}$

Find the value of each expression using half angle identities.

11. $\tan 15^\circ$

12. $\tan 22.5^\circ$

13. $\sec 22.5^\circ$

14. Show that $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x}$.

15. Show that $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.4.

4.9 Trigonometric Equations

Learning Objectives

Here you will solve equations that contain trigonometric functions. You will also learn to identify when an equation is an identity and when it is has no solutions.

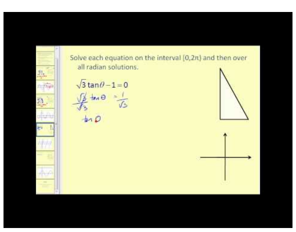
Solving a trigonometric equation is just like solving a regular equation. You will use factoring and other algebraic techniques to get the variable on one side. The biggest difference with trigonometric equations is the opportunity for there to be an infinite number of solutions that must be described with a pattern. The equation $\cos x = 1$ has many solutions including 0 and 2π. How would you describe all of them?

Solving Trigonometric Equations

The identities you have learned are helpful in solving **trigonometric equations**. The goal of solving an equation hasn't changed. Do whatever it takes to get the variable alone on one side of the equation. Factoring, especially with the Pythagorean identity, is critical.

When solving trigonometric equations, try to give exact (non-rounded) answers. If you are working with a calculator, keep in mind that while some newer calculators can provide exact answers like $\frac{\sqrt{3}}{2}$, most calculators will produce a decimal of 0.866... If you see a decimal like 0.866..., try squaring it. The result might be a nice fraction like $\frac{3}{4}$. Then you can logically conclude that the original decimal must be the square root of $\frac{3}{4}$ or $\frac{\sqrt{3}}{2}$.

When solving, if the two sides of the equation are always equal, then the equation is an identity. If the two sides of an equation are never equal, as with $\sin x = 3$, then the equation has no solution.



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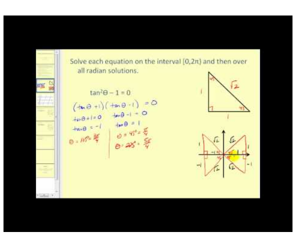

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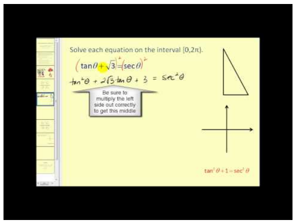

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Examples

Example 1

Earlier, you were asked how you could describe the many solutions of $\cos x = 1$. When you type $\cos^{-1} 1$ on your calculator, it will yield only one solution which is 0. In order to describe all the solutions you must use logic and the graph to figure out that cosine also has a height of 1 at $-2\pi, 2\pi, -4\pi, 4\pi \ldots$ Luckily all these values are sequences in a clear pattern so you can describe them all in general with the following notation:

$x = 0 \pm n \cdot 2\pi$ where n is an integer, or $x = \pm n \cdot 2\pi$ where n is an integer.

Example 2

Solve the following equation algebraically and confirm graphically on the interval $[-2\pi, 2\pi]$.

$\cos 2x = \sin x$

$$\begin{aligned}
\cos 2x &= \sin x \\
1 - 2\sin^2 x &= \sin x \\
0 &= 2\sin^2 x + \sin x - 1 \\
0 &= (2\sin x - 1)(\sin x + 1)
\end{aligned}$$

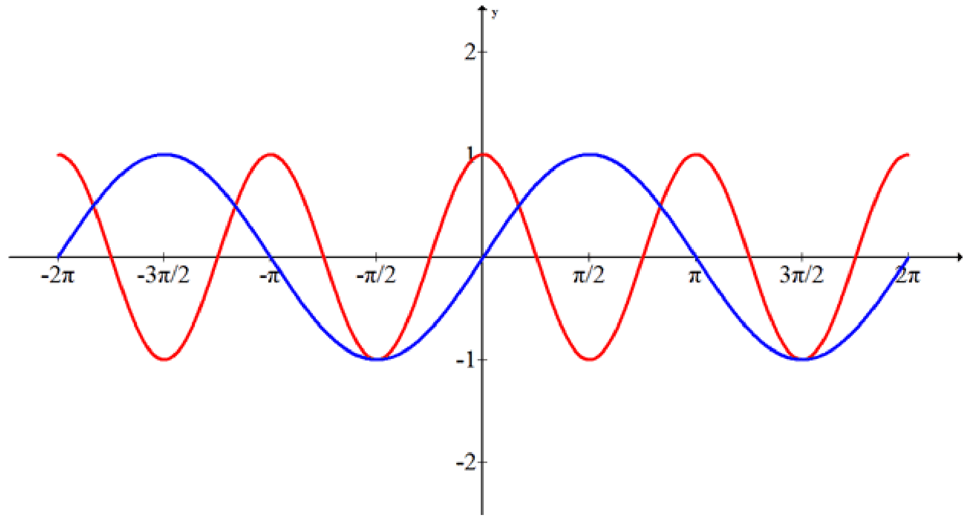
Solving the first part set equal to zero within the interval yields:

$$\begin{aligned}
&0 = 2\sin x - 1 \\
&\frac{1}{2} = \sin x \\
&x = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}
\end{aligned}$$

Solving the second part set equal to zero yields:

$$\begin{aligned}
0 &= \sin x + 1 \\
-1 &= \sin x \\
x &= -\frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

These are the six solutions that will appear as intersections of the two graphs $f(x) = \cos 2x$ and $g(x) = \sin x$.



Note that the terms "general solution," "completely solve", and"solve exactly"

Example 3

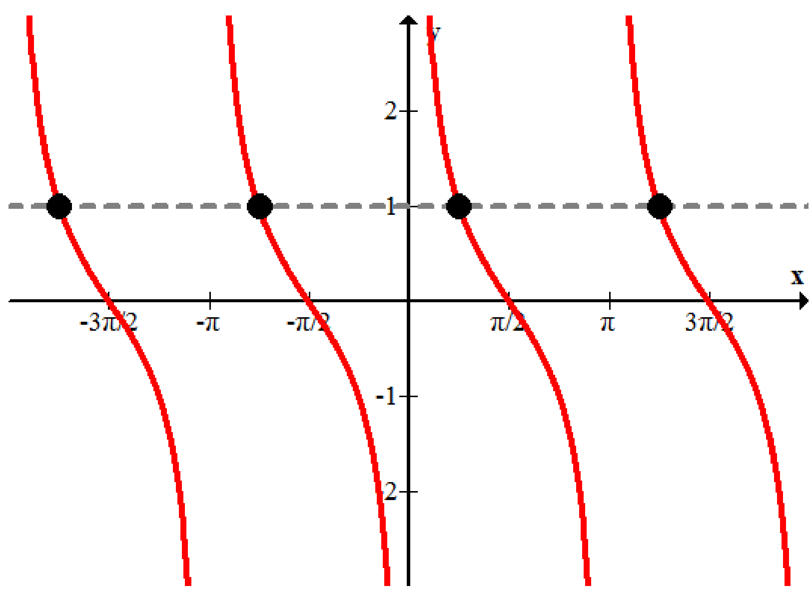
Determine the general solution to the following equation.

$\cot x - 1 = 0$

$$\begin{aligned}\cot x - 1 &= 0\\ \cot x &= 1\end{aligned}$$

One solution is $x = \frac{\pi}{4}$. However, since this question asks for the general solution, you need to find every possible solution. You have to know that cotangent has a period of π which means if you add or subtract π from $\frac{\pi}{4}$ then it will also yield a height of 1. To capture all these other possible x values you should use this notation.

$x = \frac{\pi}{4} \pm n \cdot \pi$ where n is a integer



Notice that trigonometric equations may have an infinite number of solutions that repeat in a certain pattern because they are periodic functions. When you see these directions remember to find all the solutions by using notation like in this example.

Example 4

Solve the following equation.

$4\cos^2 x - 1 = 3 - 4\sin^2 x$

$$\begin{aligned}4\cos^2 x - 1 &= 3 - 4\sin^2 x\\ 4\cos^2 x + 4\sin^2 x &= 3 + 1\\ 4(\cos^2 x + \sin^2 x) &= 4\\ 4 &= 4\end{aligned}$$

This equation is always true which means the right side is always equal to the left side. This is an identity.

Example 5

Solve the following equation exactly.

$2\cos^2 x + 3\cos x - 2 = 0$

Start by factoring:

$$\begin{aligned}2\cos^2 x + 3\cos x - 2 &= 0\\ (2\cos x - 1)(\cos x + 2) &= 0\end{aligned}$$

Note that $\cos x \neq -2$ which means only one equation needs to be solved for solutions.

$$\begin{aligned}2\cos x - 1 &= 0\\ \cos x &= \frac{1}{2}\\ x &= \frac{\pi}{3}, -\frac{\pi}{3}\end{aligned}$$

These are the solutions within the interval $-\pi$ to π. Since this represents one full period of cosine, the rest of the solutions are just multiples of 2π added and subtracted to these two values.

$x = \pm\frac{\pi}{3} \pm n \cdot 2\pi$ where n is an integer

Review

Solve each equation on the interval $[0, 2\pi)$.

1. $3\cos^2 \frac{x}{2} = 3$
2. $4\sin^2 x = 8\sin^2 \frac{x}{2}$

Find approximate solutions to each equation on the interval $[0, 2\pi)$.

3. $3\cos^2 x + 10\cos x + 2 = 0$
4. $\sin^2 x + 3\sin x = 5$
5. $\tan^2 x + \tan x = 3$
6. $\cot^2 x + 5\tan x + 14 = 0$
7. $\sin^2 x + \cos^2 x = 1$

Solve each equation on the interval $[0, 360^\circ)$.

8. $2\sin\left(x - \frac{\pi}{2}\right) = 1$
9. $4\cos(x - \pi) = 4$

Solve each equation on the interval $[2\pi, 4\pi)$.

10. $\cos^2 x + 2\cos x + 1 = 0$
11. $3\sin x = 2\cos^2 x$
12. $\tan x \sin^2 x = \tan x$
13. $\sin^2 x + 1 = 2\sin x$
14. $\sec^2 x = 4$
15. $\sin^2 x - 4 = \cos^2 x - \cos 2x - 4$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.5.

4.10 Graphing Inverse Trigonometric Functions

Learning Objectives

- Understand the meaning of restricted domain as it applies to the inverses of the six trigonometric functions.
- Apply the domain, range and quadrants of the six inverse trigonometric functions to evaluate expressions.

Finding the Inverse by Mapping

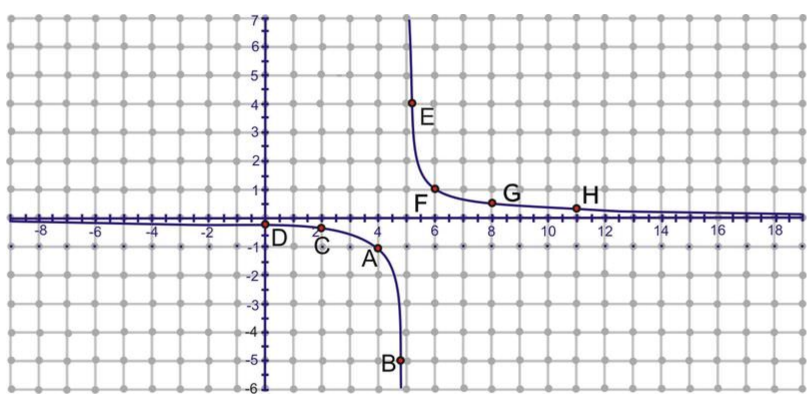
Determining an inverse function algebraically can be both involved and difficult, so it is useful to know how to map f to f^{-1}. The graph of f can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a,b) and (b,a) in the coordinate plane are symmetric with respect to the line $y=x$.

The points (a,b) and (b,a) are reflections of each other across the line $y=x$.

Example 1: Find the inverse of $f(x)=\frac{1}{x-5}$ mapping.

Solution: From the last section, we know that the inverse of this function is $y=\frac{5x+1}{x}$. To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot. Note: The coordinates of some of the points are rounded.



A: (4, -1)

B: (4.8, -5)

C: (2, -0.3)

D: (0, -0.2)

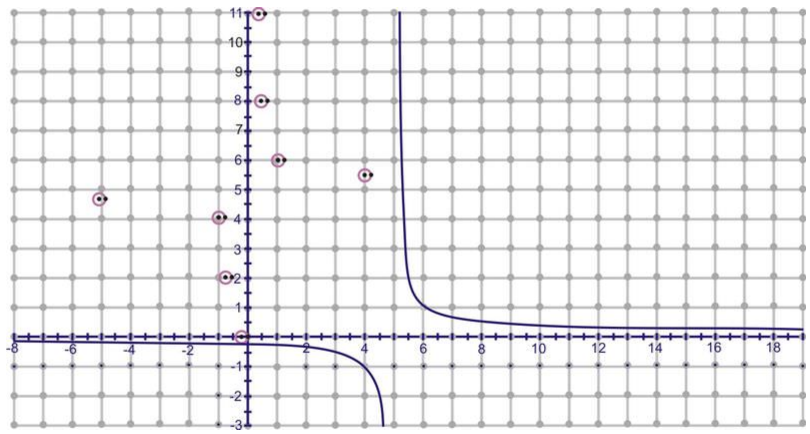
E: (5.3, 3.3)

F: (6, 1)

G: (8, 0.3)

H: (11, 0.2)

Now, take these eight points, switch the x and y and plot (y,x). Connect them to make the inverse function.



$A^{-1}:\ (-1,4)$

$B^{-1}:\ (-5,4.8)$

$C^{-1}:\ (-0.3,2)$

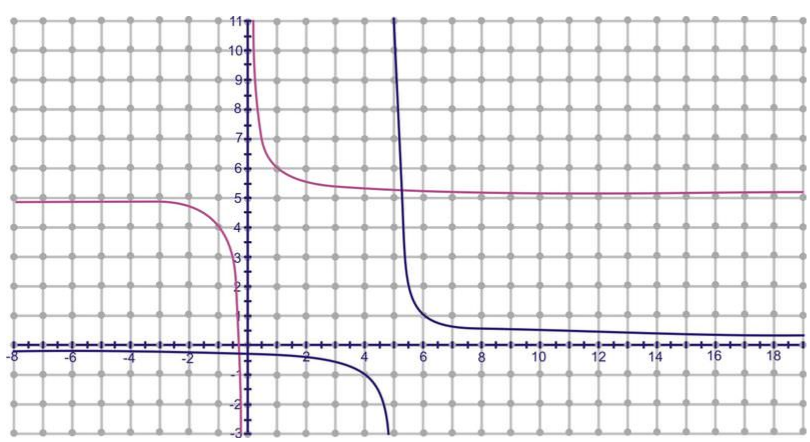
$D^{-1}:\ (-0.2,0)$

$E^{-1}:\ (3.3,5.3)$

$F^{-1}:\ (1,6)$

$G^{-1}:\ (0.3,8)$

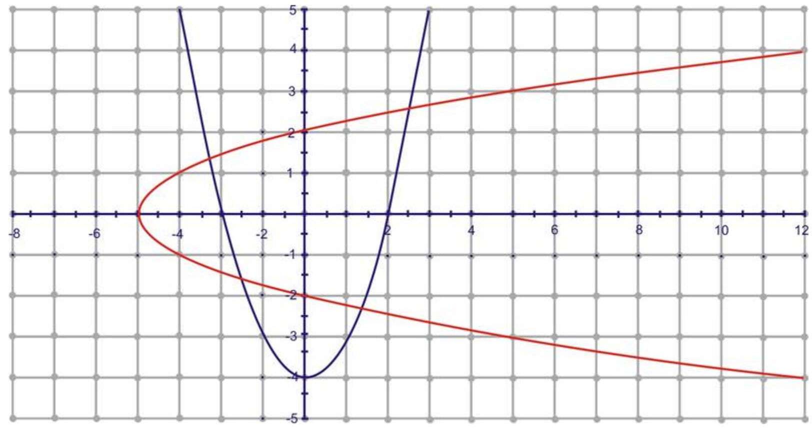
$H^{-1}:\ (0.2,11)$



Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Example 2: Find the inverse of $f(x) = x^2 - 4$.

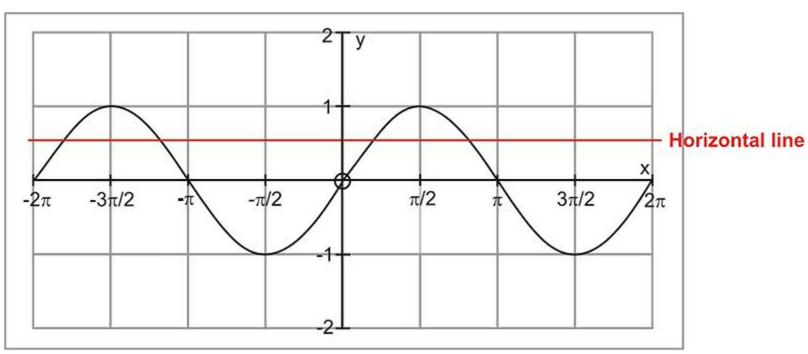
Solution: Let's use the graphic approach for this one. The function is graphed in blue and its inverse is red.



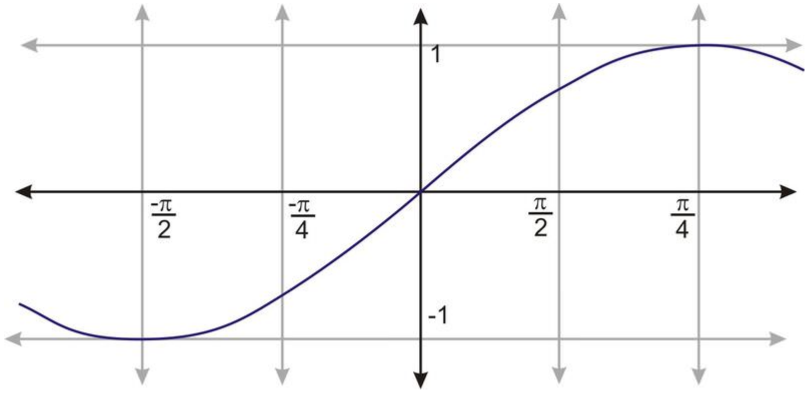
Clearly, the inverse relation is not a function because it does not pass the vertical line test. This is because all parabolas fail the horizontal line test. To "make" the inverse a function, we restrict the domain of the original function. For parabolas, this is fairly simple. To find the inverse of this function algebraically, we get $f^{-1}(x) = \sqrt{x+4}$. Technically, however, the inverse is $\pm\sqrt{x+4}$ because the square root of any number could be positive or negative. So, the inverse of $f(x) = x^2 - 4$ is both parts of the square root equation, $\sqrt{x+4}$ and $-\sqrt{x+4}$. $\sqrt{x+4}$ will yield the top portion of the horizontal parabola and $-\sqrt{x+4}$ will yield the bottom half. Be careful, because if you just graph $f^{-1}(x) = \sqrt{x+4}$ in your graphing calculator, it will only graph the top portion of the inverse.

This technique of sectioning the inverse is applied to finding the inverse of trigonometric functions because it is periodic.

Finding the Inverse of the Trigonometric Functions



In order to consider the inverse of this function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of f is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ a new function $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ takes on all values once and only once.

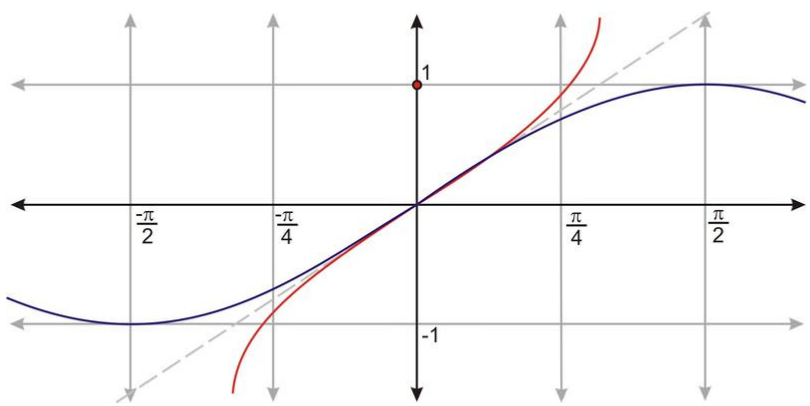


In the previous lesson the inverse of $f(x)$ was represented by the symbol $f^{-1}(x)$, and $y = f^{-1}(x) \Leftrightarrow f(y) = x$. The inverse of $\sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ will be written as $\sin^{-1} x$. or $\arcsin x$.

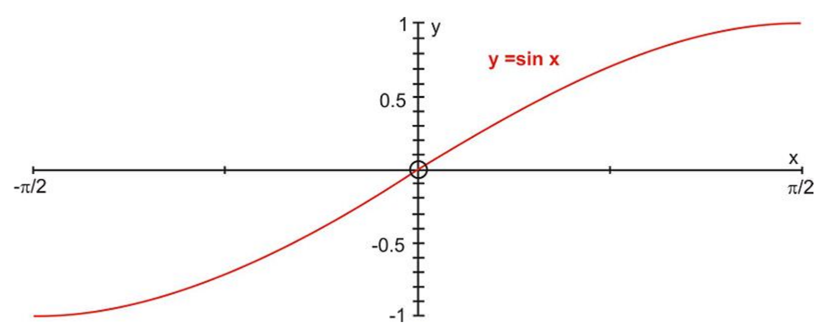
$$\left\{\begin{matrix} y = \sin^{-1} x \\ or \\ y = \arcsin x \end{matrix}\right\} \Leftrightarrow \sin y = x$$

In this lesson we will use both $\sin^{-1} x$ and $\arcsin x$ and both are read as "the inverse sine of x" or "the number between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ whose sine is x."

The graph of $y = \sin^{-1} x$ is obtained by applying the inverse reflection principle and reflecting the graph of $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ in the line $y = x$. The domain of $y = \sin x$ becomes the range of $y = \sin^{-1} x$, and hence the range of $y = \sin x$ becomes the domain of $y = \sin^{-1} x$.

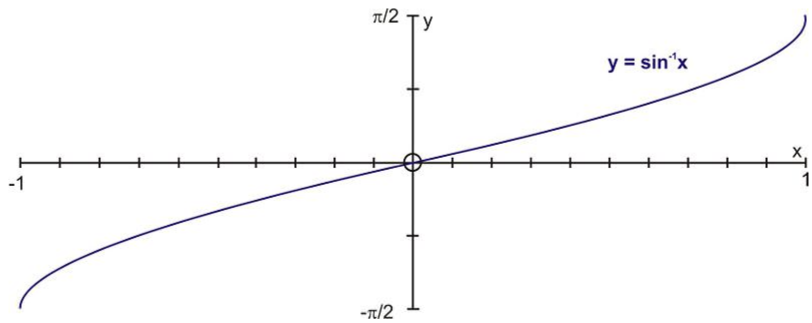


Another way to view these graphs is to construct them on separate grids. If the domain of $y = \sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse sine function $y = \sin^{-1} x$ is the inverse of the restricted section of the sine function.



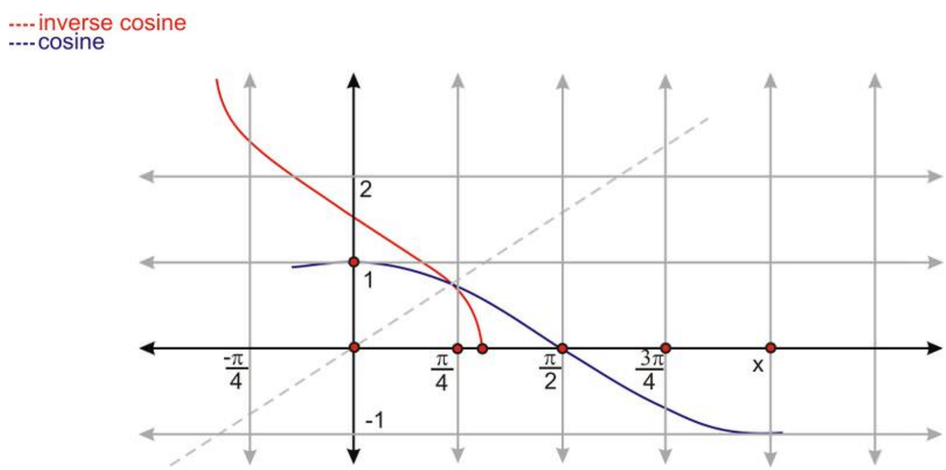
The domain of $y = \sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and the range is** [-1, 1].

The restriction of $y = \sin x$ is a one-to-one function and it has an inverse that is shown below.

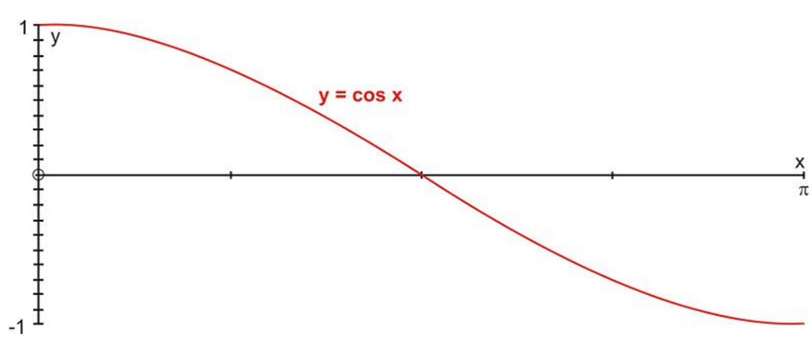


The domain of $y = \sin^{-1}$ **is** [-1, 1] **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse functions for cosine and tangent are defined by following the same process as was applied for the inverse sine function. However, in order to create one-to-one functions, different intervals are used. The cosine function is restricted to the interval $0 \leq x \leq \pi$ and the new function becomes $y = \cos x, 0 \leq x \leq \pi$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$ The result is the graph of $y = \cos^{-1} x$ (also expressed as $y = \arccos x$).

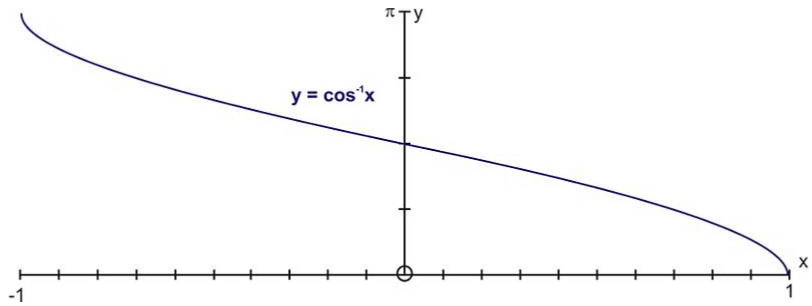


Again, construct these graphs on separate grids to determine the domain and range. If the domain of $y = \cos x$ is restricted to the interval $[0, \pi]$, the result is a restricted one-to one function. The inverse cosine function $y = \cos^{-1} x$ is the inverse of the restricted section of the cosine function.



The domain of $y = \cos x$ is $[0, \pi]$ **and the range is** [-1, 1].

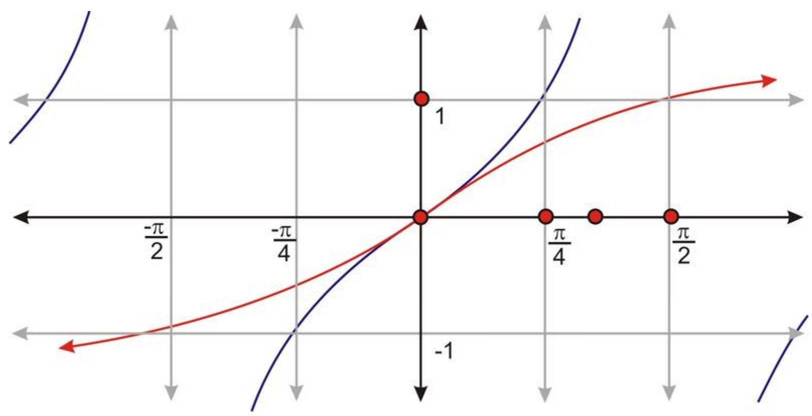
The restriction of $y = \cos x$ is a one-to-one function and it has an inverse that is shown below.



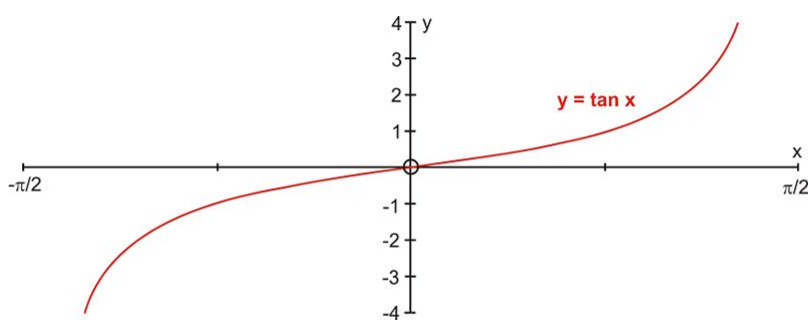
The statements $y = \cos x$ **and** $x = \cos y$ **are equivalent for** $y-$**values in the restricted domain** $[0, \pi]$ **and** $x-$**values between** -1 **and** 1.

The domain of $y = \cos^{-1} x$ is [-1, 1] **and the range is** $[0, \pi]$.

The tangent function is restricted to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the new function becomes $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$. The result is the graph of $y = \tan^{-1} x$ (also expressed as $y = \arctan x$).

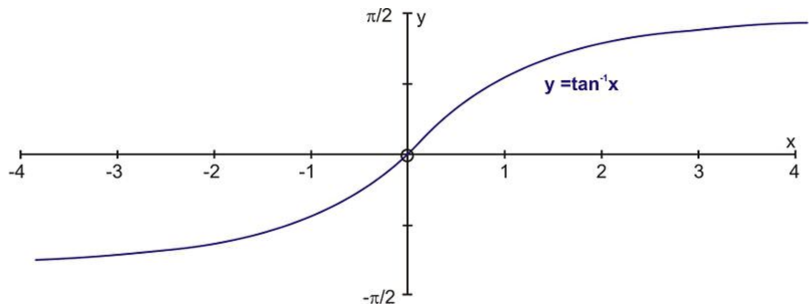


Graphing the two functions separately will help us to determine the domain and range. If the domain of $y = \tan x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse tangent function $y = \tan^{-1} x$ is the inverse of the restricted section of the tangent function.



The domain of $y = \tan x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and the range is** $[-\infty, \infty]$.

The restriction of $y = \tan x$ is a one-to-one function and it has an inverse that is shown below.



The statements $y = \tan x$ **and** $x = \tan y$ **are equivalent for** $y-$**values in the restricted domain** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and** $x-$**values between** -4 **and** +4.

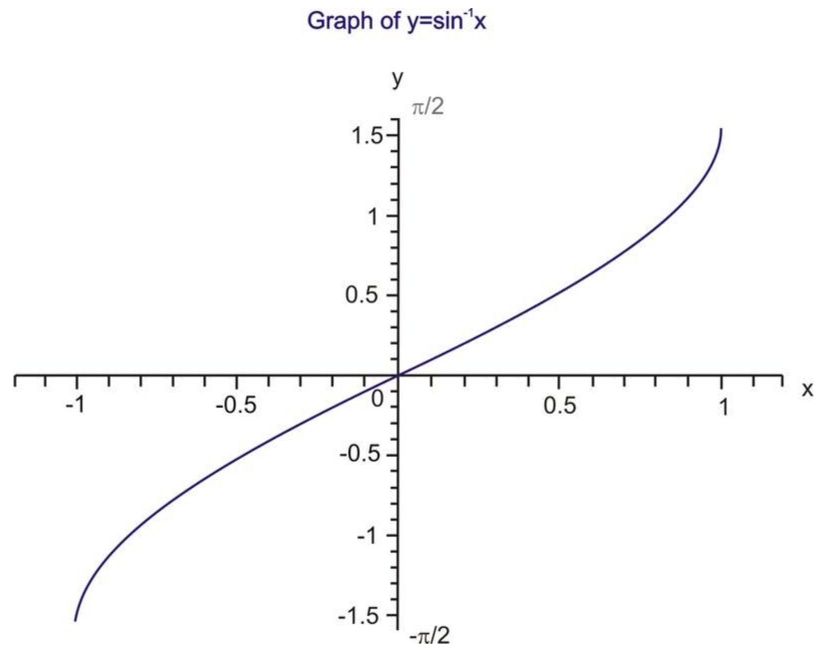
The domain of $y = \tan^{-1} x$ is $[-\infty, \infty]$ **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

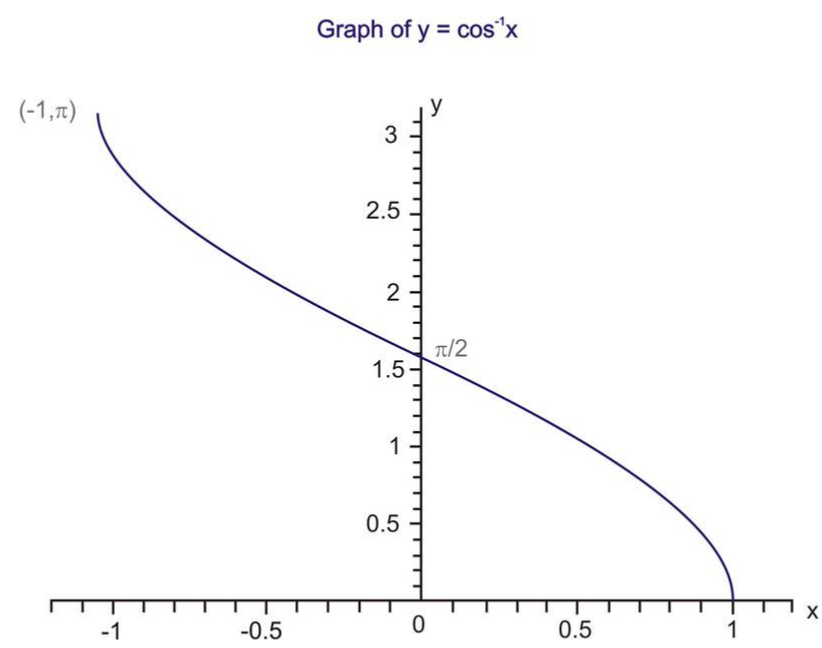
The above information can be readily used to evaluate inverse trigonometric functions without the use of a calculator. These calculations are done by applying the restricted domain functions to the unit circle. To summarize:

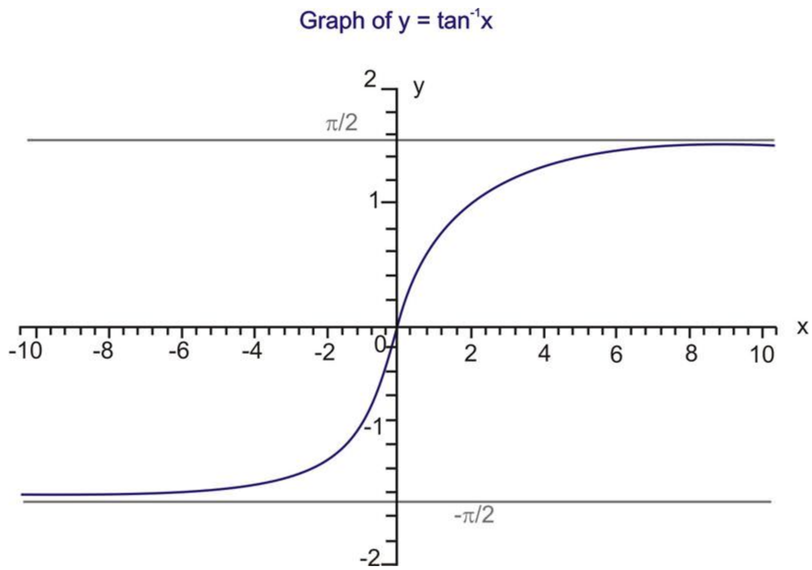
TABLE 4.4:

Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
$y = \sin x$		$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	[-1, 1]	1 AND 4
	$y = \arcsin x$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
	$y = \sin^{-1} x$			
$y = \cos x$		$[0, \pi]$	[-1, 1]	1 AND 2
	$y = \arccos x$	[-1, 1]	$[0, \pi]$	
	$y = \cos^{-1} x$			
$y = \tan x$		$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$	1 AND 4
	$y = \arctan x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
	$y = \tan^{-1} x$			

Now that the three trigonometric functions and their inverses have been summarized, let's take a look at the graphs of these inverse trigonometric functions.


Graph of y=sin⁻¹x
y
π/2
1.5
1
0.5
-1
-0.5
0
0.5
1
x
-0.5
-1
-1.5
-π/2


Graph of y = cos⁻¹x
(-1,π)
y
3
2.5
2
π/2
1.5
1
0.5
-1
-0.5
0
0.5
1
x



Points to Consider

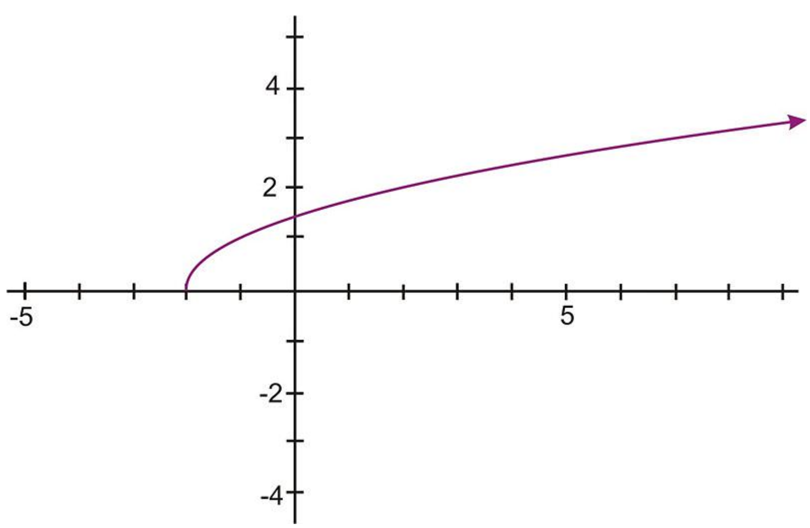
- What are the restricted domains for the inverse relations of the trigonometric functions?
- Can the values of the special angles of the unit circle be applied to the inverse trigonometric functions?

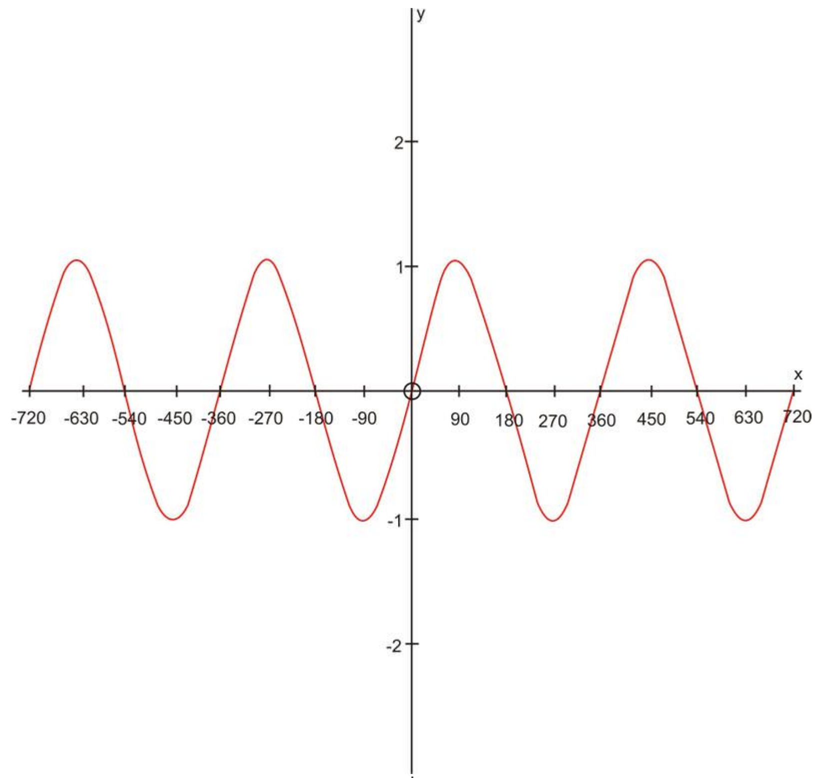
Review Questions

Study each of the following graphs and answer these questions:

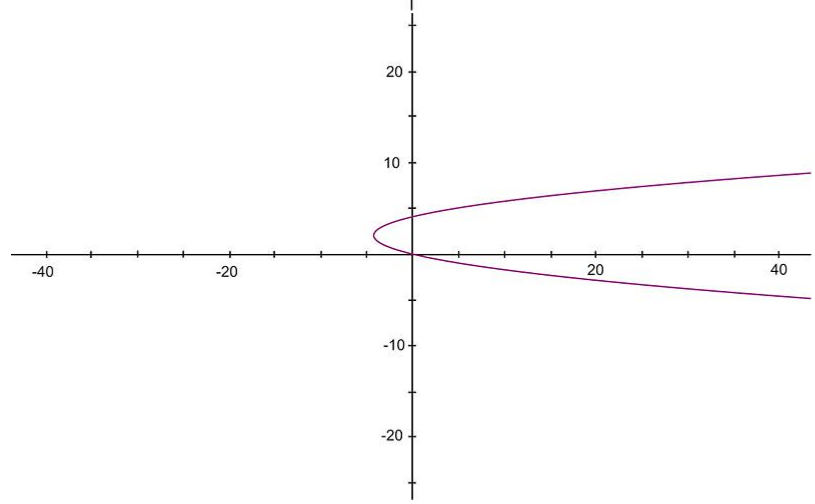
(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

1. 



2.



3.

Find the inverse of the following functions using the mapping principle.

4. $f(x) = x^2 + 2x - 15$
5. $y = 1 + 2\sin x$
6. Sketch a graph of $y = \frac{1}{2}\cos^{-1}(3x+1)$. Sketch $y = \cos^{-1} x$ on the same set of axes and compare how the two differ.
7. Sketch a graph of $y = 3 - \tan^{-1}(x-2)$. Sketch $y = \tan^{-1} x$ on the same set of axes and compare how the two differ.
8. Graph $y = 2\sin^{-1}(2x)$
9. Graph $y = 4 + \cos^{-1}\frac{1}{3}x$
10. Remember that sine and cosine are out of phase with each other, $\sin x = \cos\left(x - \frac{\pi}{2}\right)$. Find the inverse of $y = \cos\left(x - \frac{\pi}{2}\right)$. Is the inverse of $y = \cos\left(x - \frac{\pi}{2}\right)$ the same as $y = \sin^{-1} x$? Why or why not?

4.11 Relating Trigonometric Functions

Learning Objectives

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.
- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.
- State the domain and range of each trig function.
- State the sign of a trig function, given the quadrant in which an angle lies.
- State the Pythagorean identities and use these identities to find values of trig functions.

Reciprocal identities

The first set of identities we will establish are the reciprocal identities. A **reciprocal** of a fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin\theta = \frac{y}{r}$. Now consider the cosecant function: $\csc\theta = \frac{r}{y}$. In the unit circle, these values are $\sin\theta = \frac{y}{1} = y$ and $\csc\theta = \frac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin\theta = \frac{1}{2}$, then $\csc\theta = \frac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{or} \qquad \cos\theta = \frac{1}{\sec\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} \qquad \text{or} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Example 1: Find the value of each expression using a reciprocal identity.

a. $\cos\theta = .3, \sec\theta = ?$

b. $\cot\theta = \frac{4}{3}, \tan\theta = ?$

Solution:

a. $\sec\theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos\theta = .3$, then $\sec\theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos\theta = .3 = \frac{3}{10} \Rightarrow \sec\theta = \frac{10}{3}$.

b. $\tan\theta = \frac{3}{4}$

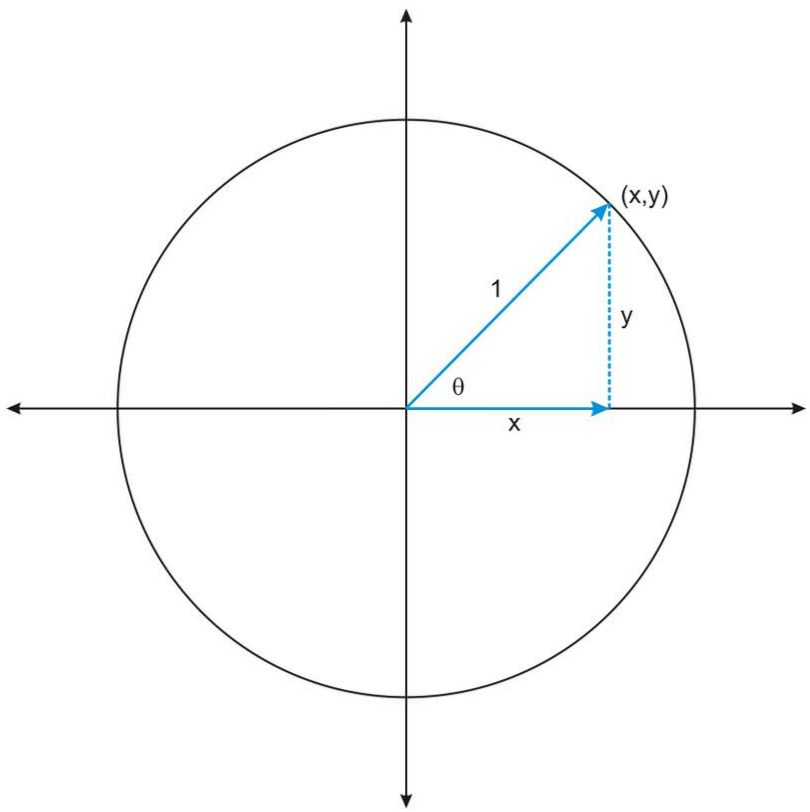
These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

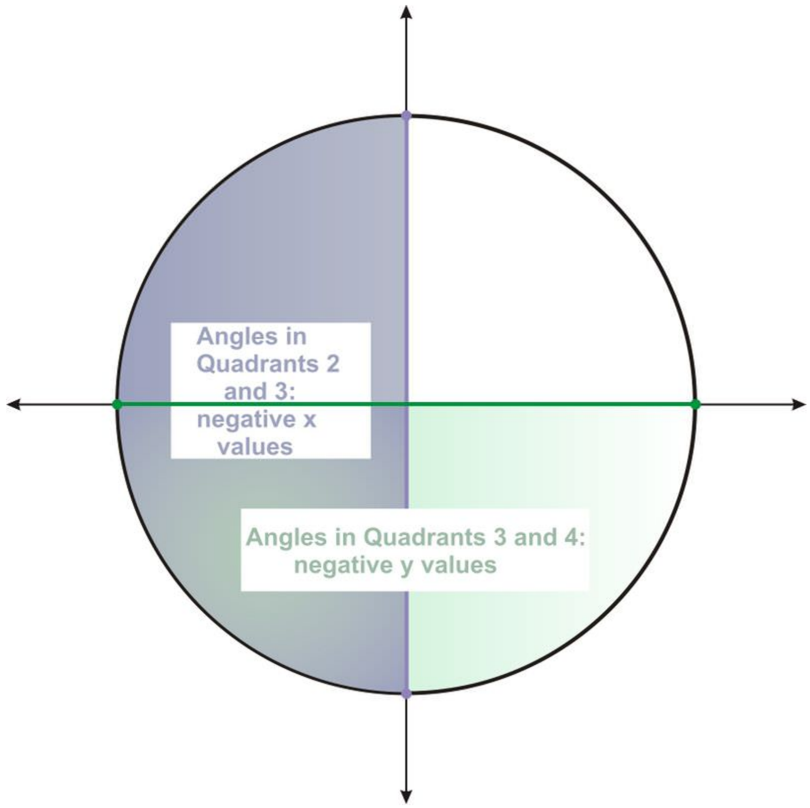
Domain, Range, and Signs of Trig Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous sections, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, R. We can determine the range of the functions if we think about the fact that the sine of an angle is the $y-$coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the $x-$coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.



In this right triangle, x and y are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative $y-$coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative $x-$coordinate.



In either case, the minimum value is -1. For example, $\cos 180^\circ = -1$ and $\sin 270^\circ = -1$. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

TABLE 4.5:

	Domain	Range
Sine	$\theta = R$	$-1 \leq y \leq 1$
Cosine	$\theta = R$	$-1 \leq y \leq 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function in order to begin to verify this range, as well as the domain and range of the secant function.

TABLE 4.6:

	Domain	Range
Cosecant	$\theta \varepsilon R, \theta \neq 0, 180, 360 \ldots$	$\csc \theta \leq -1$ or $\csc \theta \geq 1$
Secant	$\theta \varepsilon R, \theta \neq 90, 270, 450 \ldots$	$\sec \theta \leq -1$ or $\sec \theta \geq 1$

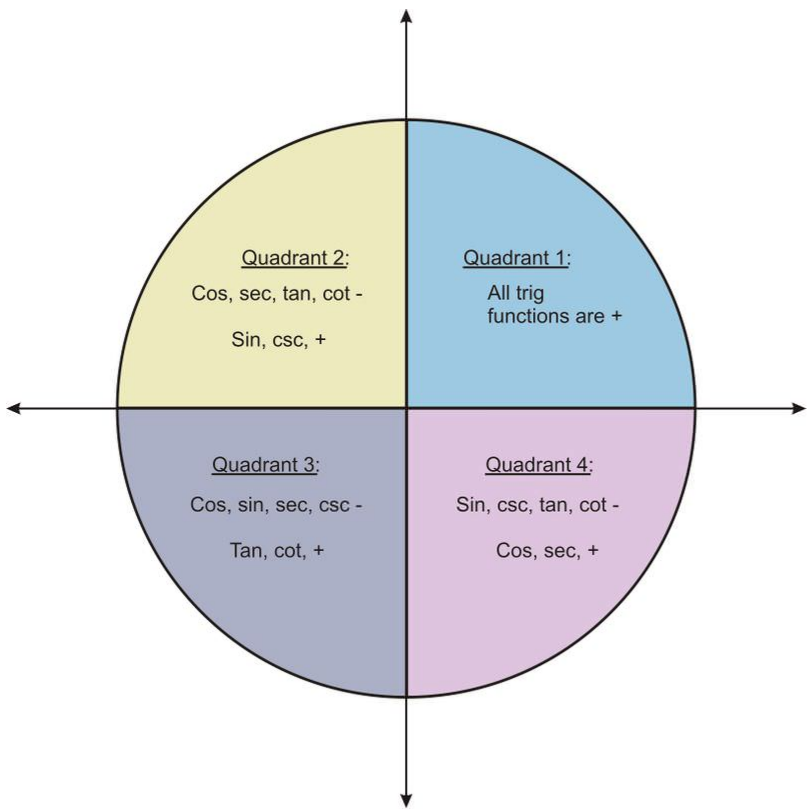
Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan \theta = \frac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an $x-$coordinate of $0 : 90^\circ, 270^\circ$, etc. The cotangent function is defined as $\cot\theta = \frac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a $y-$coordinate of $0 : 0°, 180°, 360°$, etc.

TABLE 4.7:

Function	Domain	Range
Tangent	$\theta \varepsilon R, \theta \neq 90, 270, 450 \ldots$	All reals
Cotangent	$\theta \varepsilon R, \theta \neq 0, 180, 360 \ldots$	All reals

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.



An easy way to remember this is "All Students Take Calculus." Quadrant I: All values are positive, Quadrant II: Sine is positive, Quadrant III: Tangent is positive, and Quadrant IV: Cosine is positive. This simple memory device will help you remember which trig functions are positive and where.

Example 2: State the sign of each expression.

a. $\cos 100°$

b. $\csc 220°$

c. $\tan 370°$

Solution:

a. The angle $100°$ is in the second quadrant. Therefore the $x-$coordinate is negative and so $\cos 100°$ is negative.

b. The angle 220° is in the third quadrant. Therefore the $y-$coordinate is negative. So the sine, and the cosecant are negative.

c. The angle 370° is in the first quadrant. Therefore the tangent value is positive.

So far we have considered relationships between pairs of functions: the six trig functions can be grouped in pairs as reciprocals. Now we will consider relationships among three trig functions.

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities above. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, as long as $\cos\theta \neq 0$:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta.$$

The equation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example 3: If $\cos\theta = \frac{5}{13}$ and $\sin\theta = \frac{12}{13}$, what is the value of $\tan\theta$?

Solution: $\tan\theta = \frac{12}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example 4: Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solution:

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot\theta$$

This is also an identity that you can use to find the value of the cotangent function, given values of sine and cosine. Both of the quotient identities will also be useful in chapter 3, in which you will prove other identities.

Cofunction Identities and Reflection

These identities relate to the problems you did in section 1.3. Recall, #3 and #4 from the review questions, where $\sin X = \cos Z$ and $\cos X = \sin Z$, where X and Z were complementary angles. These are called cofunction identities because the functions have common values. These identities are summarized below.

$$\sin\theta = \cos(90^\circ - \theta) \qquad \cos\theta = \sin(90^\circ - \theta)$$
$$\tan\theta = \cot(90^\circ - \theta) \qquad \cot\theta = \tan(90^\circ - \theta)$$

Example 5: Find the value of each trig function.

a. $\cos 120^\circ$

b. $\cos(-120^\circ)$

c. $\sin 135^\circ$

d. $\sin(-135^\circ)$

Solution: Because these angles have reference angles of 60° and 45°, the values are:

a. $\cos 120^\circ = -\frac{1}{2}$

b. $\cos(-120^\circ) = \cos 240^\circ = -\frac{1}{2}$

c. $\sin 135^\circ = \frac{\sqrt{2}}{2}$

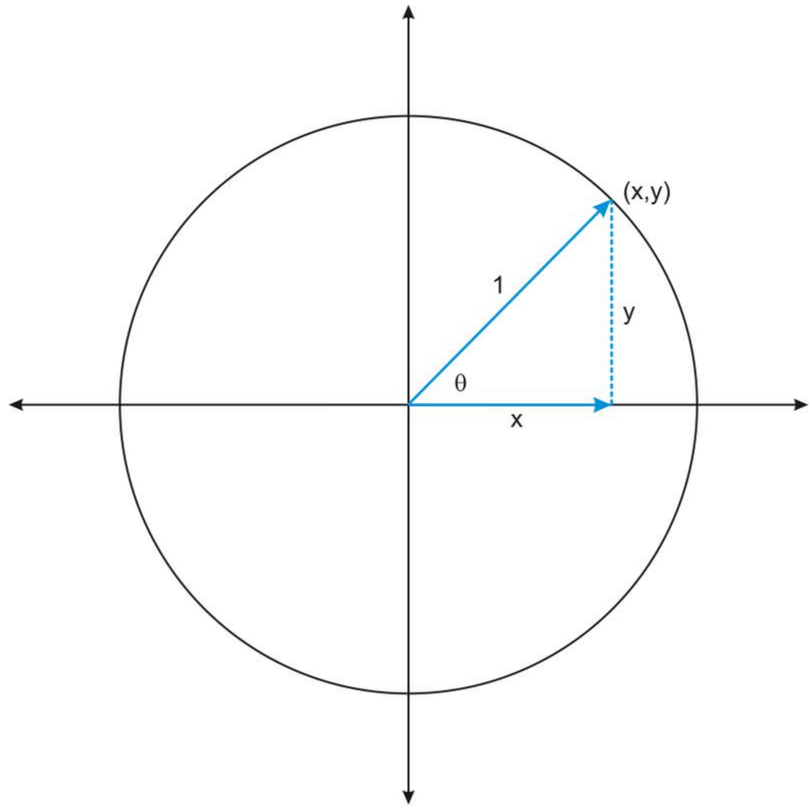
d. $\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2}$

These values show us that sine and cosine also reflect over the x axis. This allows us to generate three more identities.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Pythagorean Identities

The final set of identities are called the Pythagorean Identities because they rely on the Pythagorean Theorem. In previous lessons we used the Pythagorean Theorem to find the sides of right triangles. Consider once again the way that we defined the trig functions in 1.3. Let's look at the unit circle:



The legs of the right triangle are x, and y. The hypotenuse is 1. Therefore the following equation is true for all x and y on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ *means* $(\cos\theta)^2$ *and* $\sin^2\theta$ *means* $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example 6: If $\cos\theta = \frac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \frac{\sqrt{15}}{4}$

$$\begin{aligned}
\cos^2\theta+\sin^2\theta &= 1\\
\left(\frac{1}{4}\right)^2+\sin^2\theta &= 1\\
\frac{1}{16}+\sin^2\theta &= 1\\
\sin^2\theta &= 1-\frac{1}{16}\\
\sin^2\theta &= \frac{15}{16}\\
\sin\theta &= \pm\sqrt{\frac{15}{16}}\\
\sin\theta &= \pm\frac{\sqrt{15}}{4}
\end{aligned}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \frac{\sqrt{15}}{4}$.

Example 7: Use the identity $\cos^2\theta+\sin^2\theta=1$ to show that $\cot^2\theta+1=\csc^2\theta$

Solution:

$\cos^2\theta+\sin^2\theta=1$	Divide both sides by $\sin^2\theta$.
$\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$	
$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$	$\frac{\sin^2\theta}{\sin^2\theta}=1$
$\frac{\cos^2\theta}{\sin^2\theta}+1=\frac{1}{\sin^2\theta}$	
$\frac{\cos\theta}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}+1=\frac{1}{\sin\theta}\times\frac{1}{\sin\theta}$	Write the squared functions in terms of their factors.
$\cot\theta\times\cot\theta+1=\csc\theta\times\csc\theta$	Use the quotient and reciprocal identities.
$\cot^2\theta+1=\csc^2\theta$	Write the functions as squared functions.

Points to Consider

1. How do you know if an equation is an identity? *HINT: you could consider using a the calculator and graphing a related function, or you could try to prove it mathematically.*
2. How can you verify the domain or range of a function?

Review Questions

1. Use reciprocal identities to give the value of each expression.
   a. $\sec\theta = 4, \cos\theta = ?$
   b. $\sin\theta = \frac{1}{3}, \csc\theta = ?$
2. In the lesson, the range of the cosecant function was given as: $\csc\theta \leq -1$ or $\csc\theta \geq 1$.
   a. Use a calculator to fill in the table below. Round values to 4 decimal places.
   b. Use the values in the table to explain in your own words what happens to the values of the cosecant function as the measure of the angle approaches 0 degrees.
   c. Explain what this tells you about the range of the cosecant function.
   d. Discuss how you might further explore values of the sine and cosecant to better understand the range of the cosecant function.

TABLE 4.8:

Angle	Sin	Csc
10		
5		
1		
0.5		
0.1		
0		
-.1		
-.5		
-1		
-5		
-10		

3. In the lesson the domain of the secant function were given: Domain: $\theta\varepsilon^\circ, \theta \neq 90, 270, 450\ldots$ Explain why certain values are excluded from the domain.
4. State the quadrant in which each angle lies, and state the sign of each expression
   a. $\sin 80^\circ$
   b. $\cos 200^\circ$
   c. $\cot 325^\circ$
   d. $\tan 110^\circ$
5. If $\cos\theta = \frac{6}{10}$ and $\sin\theta = \frac{8}{10}$, what is the value of $\tan\theta$?
6. Use quotient identities to explain why the tangent and cotangent function have positive values for angles in the third quadrant.
7. If $\sin\theta = 0.4$, what is the value of $\cos\theta$? Assume that θ is an angle in the first quadrant.
8. If $\cot\theta = 2$, what is the value of $\csc\theta$? Assume that θ is an angle in the first quadrant.
9. Show that $1 + \tan^2\theta = \sec^2\theta$.
10. Explain why it is necessary to state the quadrant in which the angle lies for problems such as #7.

4.12 Fundamental Identities

Introduction

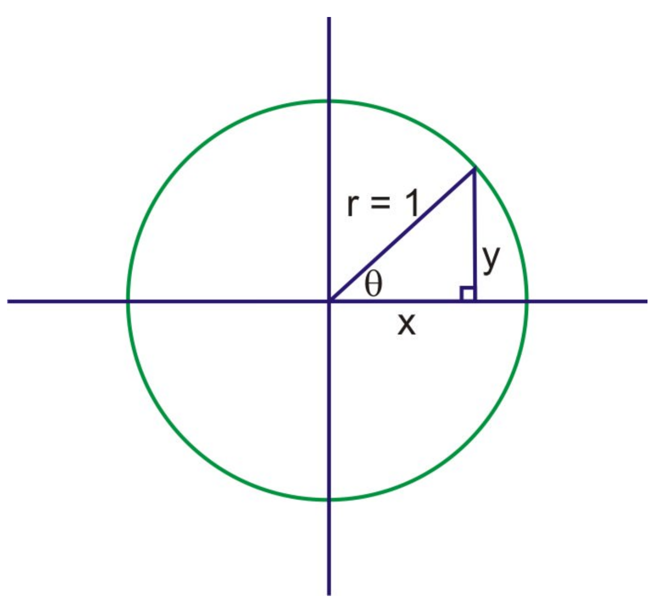
We now enter into the proof portion of trigonometry. Starting with the basic definitions of sine, cosine, and tangent, identities (or fundamental trigonometric equations) emerge. Students will learn how to prove certain identities, using other identities and definitions. Finally, students will be able solve trigonometric equations for theta, also using identities and definitions.

Learning Objectives

- use the fundamental identities to prove other identities.
- apply the fundamental identities to values of θ and show that they are true.

Quotient Identity

In Chapter 1, the three fundamental trigonometric functions sine, cosine and tangent were introduced. All three functions can be defined in terms of a right triangle or the unit circle.



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{y}{1} = y$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} = \frac{x}{1} = x$$
$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

The Quotient Identity is $\tan\theta = \frac{\sin\theta}{\cos\theta}$. We see that this is true because tangent is equal to $\frac{y}{x}$, in the unit circle. We know that y is equal to the sine values of θ and x is equal to the cosine values of θ. Substituting $\sin\theta$ for y and $\cos\theta$ for x and we have a new identity.

Example 1: Use $\theta = 45^\circ$ to show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ holds true.

Solution: Plugging in 45°, we have: $\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ}$. Then, substitute each function with its actual value and simplify both sides.

$\frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = 1$ and we know that $\tan 45^\circ = 1$, so this is true.

Example 2: Show that $\tan 90^\circ$ is undefined using the Quotient Identity.

Solution: $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$, because you cannot divide by zero, the tangent at 90° is undefined.

Reciprocal Identities

Chapter 1 also introduced us to the idea that the three fundamental reciprocal trigonometric functions are cosecant (csc), secant (sec) and cotangent (cot) and are defined as:

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

If we apply the Quotient Identity to the reciprocal of tangent, an additional quotient is created:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

Example 3: Prove $\tan\theta = \sin\theta\sec\theta$

Solution: First, you should change everything into sine and cosine. Feel free to work from either side, as long as the end result from both sides ends up being the same.

$$\begin{aligned}\tan\theta &= \sin\theta\ \sec\theta \\ &= \sin\theta \cdot \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Here, we end up with the Quotient Identity, which we know is true. Therefore, this identity is also true and we are finished.

Example 4: Given $\sin\theta = -\frac{\sqrt{6}}{5}$ and θ is in the fourth quadrant, find $\sec\theta$.

Solution: Secant is the reciprocal of cosine, so we need to find the adjacent side. We are given the opposite side, $\sqrt{6}$ and the hypotenuse, 5. Because θ is in the fourth quadrant, cosine will be positive. From the Pythagorean Theorem, the third side is:

$$\begin{aligned}\left(\sqrt{6}\right)^2 + b^2 &= 5^2\\ 6 + b^2 &= 25\\ b^2 &= 19\\ b &= \sqrt{19}\end{aligned}$$

From this we can now find $\cos\theta = \frac{\sqrt{19}}{5}$. Since secant is the reciprocal of cosine, $\sec\theta = \frac{5}{\sqrt{19}}$, or $\frac{5\sqrt{19}}{19}$.

Pythagorean Identity

Using the fundamental trig functions, sine and cosine and some basic algebra can reveal some interesting trigonometric relationships. Note when a trig function such as $\sin\theta$ is multiplied by itself, the mathematical convention is to write it as $\sin^2\theta$. ($\sin\theta^2$ can be interpreted as the sine of the square of the angle, and is therefore avoided.)

$\sin^2\theta = \frac{y^2}{r^2}$ and $\cos^2\theta = \frac{x^2}{r^2}$ or $\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2+y^2}{r^r}$

Using the Pythagorean Theorem for the triangle above: $x^2 + y^2 = r^2$

Then, divide both sides by r^2, $\frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1$. So, because $\frac{x^2+y^2}{r^2} = 1, \sin^2\theta + \cos^2\theta$ also equals 1. This is known as the Trigonometric Pythagorean Theorem or the Pythagorean Identity and is written $\sin^2\theta + \cos^2\theta = 1$. Alternative forms of the Theorem are: $1 + \cot^2\theta = \csc^2\theta$ and $\tan^2\theta + 1 = \sec^2\theta$. The second form is found by taking the original form and dividing each of the terms by $\sin^2\theta$, while the third form is found by dividing all the terms of the first by $\cos^2\theta$.

Example 5: Use 30° to show that $\sin^2\theta + \cos^2\theta = 1$ holds true.

Solution: Plug in 30° and find the values of $\sin 30^\circ$ and $\cos 30^\circ$.

$$\begin{gathered}\sin^2 30^\circ + \cos^2 30^\circ\\ \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\\ \frac{1}{4} + \frac{3}{4} = 1\end{gathered}$$

Even and Odd Identities

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the $y-$axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions and therefore the parabola is not symmetric about the $y-$axis. What about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

Even Function	Odd Function
$y=(-x)^2=x^2$	$y=(-x)^3=-x^3$

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30^\circ)=\sin 330^\circ=-\frac{1}{2}=-\sin 30^\circ$$
$$\sin(-135^\circ)=\sin 225^\circ=-\frac{\sqrt{2}}{2}=-\sin 135^\circ$$

From this we see that sine is **odd**. Therefore, $\sin(-x)=-\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30^\circ)=\cos 330^\circ=\frac{\sqrt{3}}{2}=\cos 30^\circ$$
$$\cos(-135^\circ)=\cos 225^\circ=-\frac{\sqrt{2}}{2}=\cos 135^\circ$$

This tells us that the cosine is **even**. Therefore, $\cos(-x)=\cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x)=-\tan x$$
$$\csc(-x)=-\csc x$$
$$\sec(-x)=\sec x$$
$$\cot(-x)=-\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.

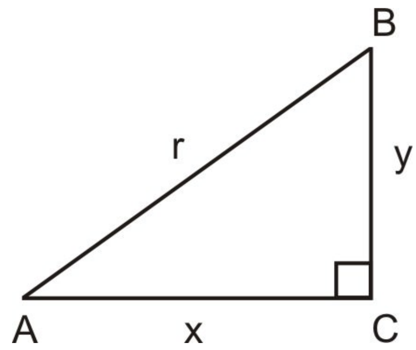
Example 6: If $\cos(-x)=\frac{3}{4}$ and $\tan(-x)=-\frac{\sqrt{7}}{3}$, find $\sin x$.

Solution: We know that sine is odd. Cosine is even, so $\cos x=\frac{3}{4}$. Tangent is odd, so $\tan x=\frac{\sqrt{7}}{3}$. Therefore, sine is positive and $\sin x=\frac{\sqrt{7}}{4}$.

Cofunction Identities

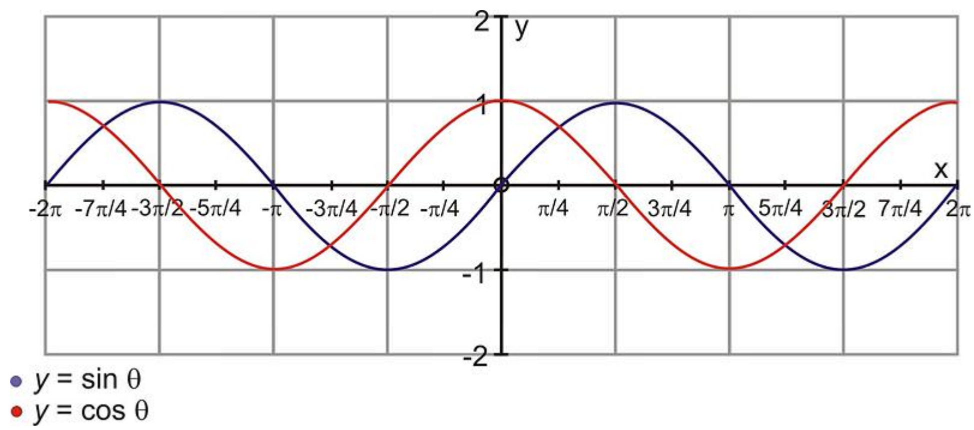
Recall that two angles are complementary if their sum is 90°. In every triangle, the sum of the interior angles is 180° and the right angle has a measure of 90°. Therefore, the two remaining acute angles of the triangle have a sum equal to 90°, and are complementary. Let's explore this concept to identify the relationship between a function of one angle and the function of its complement in any right triangle, or the cofunction identities. A cofunction is a pair of trigonometric functions that are equal when the variable in one function is the complement in the other.

In $\triangle ABC$, $\angle C$ is a right angle, $\angle A$ and $\angle B$ are complementary.



Chapter 1 introduced the cofunction identities (section 1.8) and because $\angle A$ and $\angle B$ are complementary, it was found that $\sin A = \cos B, \cos A = \sin B, \tan A = \cot B, \cot A = \tan B, \csc A = \sec B$ and $\sec A = \csc B$. For each of the above $\angle A = \frac{\pi}{2} - \angle B$. To generalize, $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ and $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta, \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ and $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$.

The following graph represents two complete cycles of $y = \sin x$ and $y = \cos\theta$.



Notice that a phase shift of $\frac{\pi}{2}$ on $y = \cos x$, would make these graphs exactly the same. These cofunction identities hold true for all real numbers for which both sides of the equation are defined.

Example 7: Use the cofunction identities to evaluate each of the following expressions:

a. If $\tan\left(\frac{\pi}{2} - \theta\right) = -4.26$ determine $\cot\theta$

b. If $\sin\theta = 0.91$ determine $\cos\left(\frac{\pi}{2} - \theta\right)$.

Solution:

a. $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ therefore $\cot\theta = -4.26$

b. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ therefore $\cos\left(\frac{\pi}{2} - \theta\right) = 0.91$

Example 8: Show $\sin\left(\frac{\pi}{2} - x\right) = \cos(-x)$ is true.

Solution: Using the identities we have derived in this section, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, and we know that cosine is an even function so $\cos(-x) = \cos x$. Therefore, each side is equal to $\cos x$ and thus equal to each other.

Points to Consider

- Why do you think secant is even like cosine?
- How could you show that tangent is odd?

Review Questions

1. Use the Quotient Identity to show that the $\tan 270^\circ$ is undefined.
2. If $\cos\left(\frac{\pi}{2}-x\right)=\frac{4}{5}$, find $\sin(-x)$.
3. If $\tan(-x)=-\frac{5}{12}$ and $\sin x=-\frac{5}{13}$, find $\cos x$.
4. Simplify $\sec x \cos\left(\frac{\pi}{2}-x\right)$.
5. Verify $\sin^2\theta+\cos^2\theta=1$ using:
    a. the sides $5, 12$, and 13 of a right triangle, in the first quadrant
    b. the ratios from a $30-60-90$ triangle
6. Prove $1+\tan^2\theta=\sec^2\theta$ using the Pythagorean Identity
7. If $\csc z=\frac{17}{8}$ and $\cos z=-\frac{15}{17}$, find $\cot z$.
8. Factor:
    a. $\sin^2\theta-\cos^2\theta$
    b. $\sin^2\theta+6\sin\theta+8$
9. Simplify $\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta}$ using the trig identities
10. Rewrite $\frac{\cos x}{\sec x-1}$ so that it is only in terms of cosine. Simplify completely.
11. Prove that tangent is an odd function.

4.13 Proving Identities

Learning Objectives

- Prove identities using several techniques.

Working with Trigonometric Identities

During the course, you will see complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process for showing two trigonometric expressions to be equivalent (regardless of the value of the angle) is known as validating or proving trigonometric identities.

There are several options a student can use when proving a trigonometric identity.

Option One: Often one of the steps for proving identities is to change each term into their sine and cosine equivalents:

Example 1: Prove the identity: $\csc\theta \times \tan\theta = \sec\theta$

Solution: Reducing each side separately. It might be helpful to put a line down, through the equals sign. Because we are proving this identity, we don't know if the two sides are equal, so wait until the end to include the equality.

$$\begin{array}{c|c} \csc x \times \tan x & \sec x \\ \frac{1}{\sin x} \times \frac{\sin x}{\cos x} & \frac{1}{\cos x} \\ \frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x} & \frac{1}{\cos x} \\ \frac{1}{\cos x} & \frac{1}{\cos x} \end{array}$$

At the end we ended up with the same thing, so we know that this is a valid identity.

Notice when working with identities, unlike equations, conversions and mathematical operations are performed only on one side of the identity. In more complex identities sometimes both sides of the identity are simplified or expanded. The thought process for establishing identities is to view each side of the identity separately, and at the end to show that both sides do in fact transform into identical mathematical statements.

Option Two: Use the Trigonometric Pythagorean Theorem and other Fundamental Identities.

Example 2: Prove the identity: $(1-\cos^2 x)(1+\cot^2 x) = 1$

Solution: Use the Pythagorean Identity and its alternate form. Manipulate $\sin^2\theta + \cos^2\theta = 1$ to be $\sin^2\theta = 1 - \cos^2\theta$. Also substitute $\csc^2 x$ for $1+\cot^2 x$, then cross-cancel.

$$\begin{array}{c|c} (1-\cos^2 x)(1+\cot^2 x) & 1 \\ \sin^2 x \cdot \csc^2 x & 1 \\ \sin^2 x \cdot \frac{1}{\sin^2 x} & 1 \\ 1 & 1 \end{array}$$

Option Three: When working with identities where there are fractions- combine using algebraic techniques for adding expressions with unlike denominators:

Example 3: Prove the identity: $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\csc\theta$.

Solution: Combine the two fractions on the left side of the equation by finding the common denominator: $(1+\cos\theta)\times\sin\theta$, and the change the right side into terms of sine.

$$\begin{array}{c|c}
\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} & 2\csc\theta \\
\frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} & 2\csc\theta \\
\frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} & 2\csc\theta
\end{array}$$

Now, we need to apply another algebraic technique, FOIL. (FOIL is a memory device that describes the process for multiplying two binomials, meaning multiplying the First two terms, the Outer two terms, the Inner two terms, and then the Last two terms, and then summing the four products.) Always leave the denominator factored, because you might be able to cancel something out at the end.

$$\begin{array}{c|c}
\frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta
\end{array}$$

Using the second option, substitute $\sin^2\theta+\cos^2\theta=1$ and simplify.

$$\begin{array}{c|c}
\frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\
\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\
\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\
\frac{2}{\sin\theta} & \frac{2}{\sin\theta}
\end{array}$$

Option Four: If possible, factor trigonometric expressions. Actually procedure four was used in the above example: $\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}=2\csc\theta$ can be *factored* to $\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}=2\csc\theta$ and in this situation, the factors cancel each other.

Example 4: Prove the identity: $\frac{1+\tan\theta}{(1+\cot\theta)}=\tan\theta$.

Solution: Change $\cot\theta$ to $\frac{1}{\tan\theta}$ and find a common denominator.

$$\frac{1+\tan\theta}{\left(1+\frac{1}{\tan\theta}\right)}=\tan\theta$$

$$\frac{1+\tan\theta}{\left(\frac{\tan\theta}{\tan\theta}+\frac{1}{\tan\theta}\right)}=\tan\theta \qquad \text{or} \qquad \frac{1+\tan\theta}{\frac{\tan\theta+1}{\tan\theta}}=\tan\theta$$

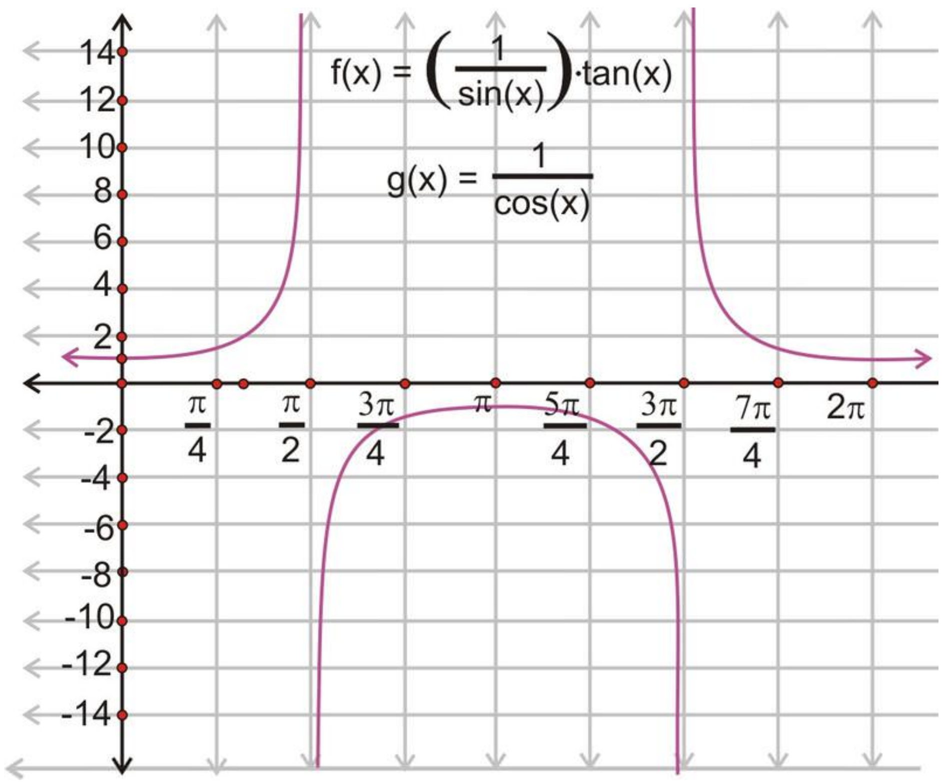
Now invert the denominator and multiply.

$$\frac{\tan\theta(1+\tan\theta)}{\tan\theta+1}=\tan\theta$$

$$\tan\theta=\tan\theta$$

Technology Note

A graphing calculator can help provide the correctness of an identity. For example looking at: $\csc x \times \tan x = \sec x$, first graph $y = \csc x \times \tan x$, and then graph $y = \sec x$. Examining the viewing screen for each demonstrates that the results produce the same graph.



To summarize, when verifying a trigonometric identity, use the following tips:

1. Work on one side of the identity- usually the more complicated looking side.
2. Try rewriting all given expressions in terms of sine and cosine.
3. If there are fractions involved, combine them.
4. After combining fractions, if the resulting fraction can be reduced, reduce it.
5. The goal is to make one side look exactly like the other—so as you change one side of the identity, look at the other side for a potential hint to what to do next. If you are stumped, work with the other side. Don't limit yourself to working only on the left side, a problem might require you to work on the right.

Points to Consider

- Are there other techniques that you could use to prove identities?
- What else, besides what is listed in this section, do you think would be useful in proving identities?

Review Questions

Prove the following identities true:

1. $\sin x \tan x + \cos x = \sec x$
2. $\cos x - \cos x \sin^2 x = \cos^3 x$
3. $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$
4. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
5. $\frac{1}{1+\cos a} + \frac{1}{1-\cos a} = 2 + 2\cot^2 a$
6. $\cos^4 b - \sin^4 b = 1 - 2\sin^2 b$
7. $\frac{\sin y+\cos y}{\sin y} - \frac{\cos y-\sin y}{\cos y} = \sec y \csc y$
8. $(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$
9. Show that $2\sin x \cos x = \sin 2x$ is true using $\frac{5\pi}{6}$.
10. Use the trig identities to prove $\sec x \cot x = \csc x$

4.14 Solving Trigonometric Equations

Learning Objectives

- Use the fundamental identities to solve trigonometric equations.
- Express trigonometric expressions in simplest form.
- Solve trigonometric equations by factoring.
- Solve trigonometric equations by using the Quadratic Formula.

By now we have seen trigonometric functions represented in many ways: Ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with graphs. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

Simplifying Trigonometric Expressions

Example 1: Simplify the following expressions using the basic trigonometric identities:

a. $\frac{1+\tan^2 x}{\csc^2 x}$

b. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$

c. $\cos x - \cos^3 x$

Solution:

a.

$$\begin{aligned}
\frac{1+\tan^2 x}{\csc^2 x} & \ldots (1+\tan^2 x = \sec^2 x)\text{Pythagorean Identity} \\
\frac{\sec^2 x}{\csc^2 x} & \ldots (\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x})\text{Reciprocal Identity} \\
\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} & = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right) \\
\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{\sin^2 x}{1}\right) & = \frac{\sin^2 x}{\cos^2 x} \\
& = \tan^2 x \rightarrow \text{Quotient Identity}
\end{aligned}$$

b.

$$\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x} \ldots (\sin^2 x + \cos^2 x = 1) \text{Pythagorean Identity}$$
$$\frac{1 + \tan^2 x}{\sec x} \ldots (1 + \tan^2 x = \sec^2 x) \text{Pythagorean Identity}$$
$$\frac{\sec^2 x}{\sec x} = \sec x$$

c.

$$\begin{aligned}
&\cos x - \cos^3 x \\
&\cos x(1 - \cos^2 x) \qquad \ldots \text{Factor out } \cos x \text{ and } \sin^2 x = 1 - \cos^2 x \\
&\cos x(\sin^2 x)
\end{aligned}$$

In the above examples, the given expressions were simplified by applying the patterns of the basic trigonometric identities. We can also apply the fundamental identities to trigonometric equations to solve for x. When solving trig equations, restrictions on x (or θ) must be provided, or else there would be infinitely many possible answers (because of the periodicity of trig functions).

Solving Trigonometric Equations

Example 2: Without the use of technology, find all solutions $\tan^2(x) = 3$, such that $0 \le x \le 2\pi$.

Solution:

$$\begin{aligned}
\tan^2 x &= 3 \\
\sqrt{\tan^2 x} &= \sqrt{3} \\
\tan x &= \pm\sqrt{3}
\end{aligned}$$

This means that there are four answers for x, because tangent is positive in the first and third quadrants and negative in the second and fourth. Combine that with the values that we know would generate $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}, x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

Example 3: Solve $2\cos x \sin x - \cos x = 0$ for all values of x between $[0, 2\pi]$.

Solution:

$$\cos x \;\; (2\sin x - 1) = 0 \rightarrow \text{set each factor equal to zero and solve them separately}$$
$$\downarrow \qquad\qquad \searrow$$
$$\cos x = 0 \qquad\qquad 2\sin x = 1$$
$$x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2} \qquad \sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}$$

In the above examples, exact values were obtained for the solutions of the equations. These solutions were within the domain that was specified.

Example 4: Solve $2\sin^2 x - \cos x - 1 = 0$ for all values of x.

Solution: The equation now has two functions –sine and cosine. Study the equation carefully and decide in which function to rewrite the equation. $\sin^2 x$ can be expressed in terms of cosine by manipulating the Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
2\sin^2 x - \cos x - 1 &= 0\\
2(1-\cos^2 x) - \cos x - 1 &= 0\\
2 - 2\cos^2 x - \cos x - 1 &= 0\\
-2\cos^2 x - \cos x + 1 &= 0\\
2\cos^2 x + \cos x - 1 &= 0\\
(2\cos x - 1)(\cos x + 1) &= 0\\
\swarrow \qquad\qquad &\searrow\\
2\cos x - 1 = 0 \quad \text{or} \quad & \cos x + 1 = 0\\
\cos x = \frac{1}{2} \qquad\qquad & \cos x = -1\\
x = \frac{\pi}{3} + 2\pi k, k\varepsilon Z \qquad & x = \pi + 2\pi k, k\varepsilon Z\\
x = \frac{5\pi}{3} + 2\pi k, k\varepsilon Z &
\end{aligned}$$

Solving Trigonometric Equations Using Factoring

Algebraic skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers.

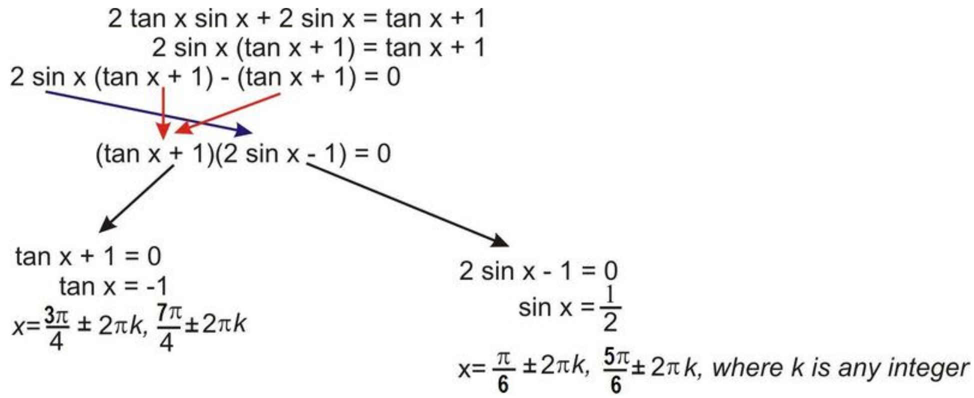
Example 5: Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 < x \leq 2\pi$.

Solution:

$$\begin{aligned}
2\sin^2 x - 3\sin x + 1 &= 0 \quad \text{Factor this like a quadratic equation}\\
(2\sin x - 1)(\sin x - 1) &= 0\\
\downarrow \qquad\qquad &\searrow\\
2\sin x - 1 = 0 \quad \text{or} \quad & \sin x - 1 = 0\\
2\sin x = 1 \qquad & \sin x = 1\\
\sin x = \frac{1}{2} \qquad & x = \frac{\pi}{2}\\
x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} &
\end{aligned}$$

Example 6: Solve $2\tan x\sin x + 2\sin x = \tan x + 1$ for all values of x.

Solution:



Pull out $\sin x$

There is a common factor of $(\tan x + 1)$

Think of the $-(\tan x + 1)$ as $(-1)(\tan x + 1)$, which is why there is a -1 behind the $2\sin x$.

Example 7: Solve $2\sin^2 x + 3\sin x - 2 = 0$ for all x, $[0, \pi]$.

Solution:

$$
\begin{array}{lll}
 & 2\sin^2 x + 3\sin x - 2 = 0 \rightarrow \text{Factor like a quadratic} & \\
 & (2\sin x - 1)(\sin x + 2) = 0 & \\
 & \swarrow \qquad\qquad \searrow & \\
 & 2\sin x - 1 = 0 \qquad \sin x + 2 = 0 & \\
 & \sin x = \frac{1}{2} \qquad\qquad \sin x = -2 & \\
x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} & \text{There is no solution because the range of } \sin x \text{ is } [-1, 1]. &
\end{array}
$$

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Example 8: Solve $4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$ for x in the interval $[0, 2\pi]$.

Solution: Even though this does not look like a factoring problem, it is. We are going to use factoring by grouping, from Algebra II. First group together the first two terms and the last two terms. Then find the greatest common factor for each pair.

$$
\begin{gathered}
\underbrace{4\sin^3 x + 2\sin^2 x} \qquad \underbrace{-2\sin x - 1} = 0 \\
2\sin^2 x(2\sin x + 1) - 1(2\sin x + 1)
\end{gathered}
$$

Notice we have gone from four terms to two. These new two terms have a common factor of $2\sin x + 1$. We can pull this common factor out and reduce our number of terms from two to one, comprised of two factors.

$$
\begin{gathered}
2\sin^2 x(2\sin x + 1) - 1(2\sin x + 1) = 0 \\
\searrow \qquad \swarrow \\
(2\sin x + 1)(2\sin^2 x - 1) = 0
\end{gathered}
$$

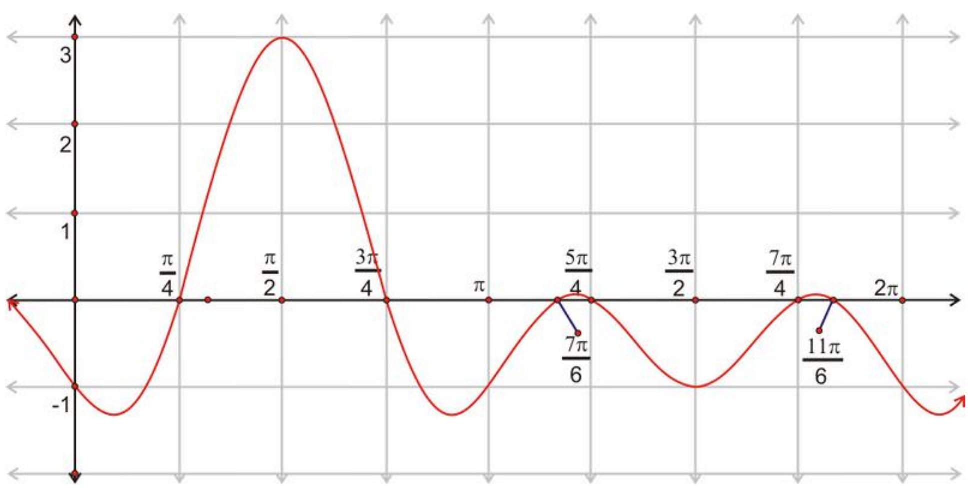
We can take this one step further because $2\sin^2 x - 1$ can factor again.

$$(2\sin x + 1)\left(\sqrt{2}\sin x - 1\right)\left(\sqrt{2}\sin x + 1\right) = 0$$

Set each factor equal to zero and solve.

$$\begin{aligned} 2\sin x + 1 &= 0 \\ 2\sin x &= -1 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \sqrt{2}\sin x + 1 &= 0 \\ \sqrt{2}\sin x &= -1 \\ \sin x &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ x &= \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \sqrt{2}\sin x - 1 &= 0 \\ \sqrt{2}\sin x &= 1 \\ \sin x &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ x &= \frac{\pi}{4}, \frac{3\pi}{4} \end{aligned}$$

Notice there are six solutions for x. Graphing the original function would show that the equation crosses the x−axis six times in the interval $[0, 2\pi]$.



Solving Trigonometric Equations Using the Quadratic Formula

When solving quadratic equations that do not factor, the quadratic formula is often used. The same can be applied when solving trigonometric equations that do not factor. The values for a is the numerical coefficient of the function's squared term, b is the numerical coefficient of the function term that is to the first power and c is a constant. The formula will result in two answers and both will have to be evaluated within the designated interval.

Example 9: Solve $3\ \cot^2 x - 3\ \cot x = 1$ for exact values of x over the interval $[0, 2\pi]$.

Solution:

$$\begin{aligned} 3\cot^2 x - 3\cot x &= 1 \\ 3\cot^2 x - 3\cot x - 1 &= 0 \end{aligned}$$

The equation will not factor. Use the quadratic formula for $\cot x$, $a = 3, b = -3, c = -1$.

$$\cot x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2-4(3)(-1)}}{2(3)}$$
$$\cot x = \frac{3 \pm \sqrt{9+12}}{6}$$
$$\cot x = \frac{3+\sqrt{21}}{6} \qquad \text{or} \qquad \cot x = \frac{3-\sqrt{21}}{6}$$
$$\cot x = \frac{3+4.5826}{6} \qquad \cot x = \frac{3-4.5826}{6}$$
$$\cot x = 1.2638 \qquad \cot x = -0.2638$$
$$\tan x = \frac{1}{1.2638} \qquad \tan x = \frac{1}{-0.2638}$$
$$x = 0.6694, 3.81099 \qquad x = 1.8287, 4.9703$$

Example 10: Solve $-5\cos^2 x+9\sin x+3=0$ for values of x over the interval $[0,2\pi]$.

Solution: Change $\cos^2 x$ to $1-\sin^2 x$ from the Pythagorean Identity.

$$\begin{aligned}
-5\cos^2 x+9\sin x+3&=0\\
-5(1-\sin^2 x)+9\sin x+3&=0\\
-5+5\sin^2 x+9\sin x+3&=0\\
5\sin^2 x+9\sin x-2&=0
\end{aligned}$$

$$\sin x = \frac{-9 \pm \sqrt{9^2-4(5)(-2)}}{2(5)}$$
$$\sin x = \frac{-9 \pm \sqrt{81+40}}{10}$$
$$\sin x = \frac{-9 \pm \sqrt{121}}{10}$$
$$\sin x = \frac{-9+11}{10} \text{ and } \sin x = \frac{-9-11}{10}$$
$$\sin x = \frac{1}{5} \text{ and } -2$$
$$\sin^{-1}(0.2) \text{ and } \sin^{-1}(-2)$$

$x \approx .201$ *rad* and $\pi - .201 \approx 2.941$

This is the only solutions for x since -2 is not in the range of values.

To summarize, to solve a trigonometric equation, you can use the following techniques:

1. Simplify expressions with the fundamental identities.
2. Factor, pull out common factors, use factoring by grouping.
3. The Quadratic Formula.
4. Be aware of the intervals for x. Make sure your final answer is in the specified domain.

Points to Consider

- Are there other methods for solving equations that can be adapted to solving trigonometric equations?
- Will any of the trigonometric equations involve solving quadratic equations?
- Is there a way to solve a trigonometric equation that will not factor?
- Is substitution of a function with an identity a feasible approach to solving a trigonometric equation?

Review Questions

1. Solve the equation $\sin 2\theta = 0.6$ for $0 \leq \theta < 2\pi$.
2. Solve the equation $\cos^2 x = \frac{1}{16}$ over the interval $[0, 2\pi]$
3. Solve the trigonometric equation $\tan^2 x = 1$ for all values of θ such that $0 \leq \theta \leq 2\pi$
4. Solve the trigonometric equation $4\sin x \cos x + 2\cos x - 2\sin x - 1 = 0$ such that $0 \leq x < 2\pi$.
5. Solve $\sin^2 x - 2\sin x - 3 = 0$ for x over $[0, \pi]$.
6. Solve $\tan^2 x = 3\tan x$ for x over $[0, \pi]$.
7. Find all the solutions for the trigonometric equation $2\sin^2 \frac{x}{4} - 3\cos \frac{x}{4} = 0$ over the interval $[0, 2\pi)$.
8. Solve the trigonometric equation $3 - 3\sin^2 x = 8\sin x$ over the interval $[0, 2\pi]$.
9. Solve $2\sin x \tan x = \tan x + \sec x$ for all values of $x\, \varepsilon[0, 2\pi]$.
10. Solve the trigonometric equation $2\cos^2 x + 3\sin x - 3 = 0$ over the interval $[0, 2\pi]$.
11. Solve $\tan^2 x + \tan x - 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
12. Solve the trigonometric equation such that $5\cos^2 \theta - 6\sin \theta = 0$ over the interval $[0, 2\pi]$.

4.15 Sum and Difference Identities

Learning Objectives

- Use and identify the sum and difference identities.
- Apply the sum and difference identities to solve trigonometric equations.
- Find the exact value of a trigonometric function for certain angles.

In this section we are going to explore $\cos(a \pm b), \sin(a \pm b)$, and $\tan(a \pm b)$. These identities have very useful expansions and can help to solve identities and equations.

Sum and Difference Formulas: Cosine

Is $\cos 15^\circ = \cos(45^\circ - 30^\circ)$? Upon appearance, yes, it is. This section explores how to find an expression that would equal $\cos(45^\circ - 30^\circ)$. To simplify this, let the two given angles be a and b where $0 < b < a < 2\pi$.

Begin with the unit circle and place the angles a and b in standard position as shown in Figure A. Point Pt1 lies on the terminal side of b, so its coordinates are $(\cos b, \sin b)$ and Point Pt2 lies on the terminal side of a so its coordinates are $(\cos a, \sin a)$. Place the $a - b$ in standard position, as shown in Figure B. The point A has coordinates $(1, 0)$ and the Pt3 is on the terminal side of the angle $a - b$, so its coordinates are $(\cos[a - b], \sin[a - b])$.

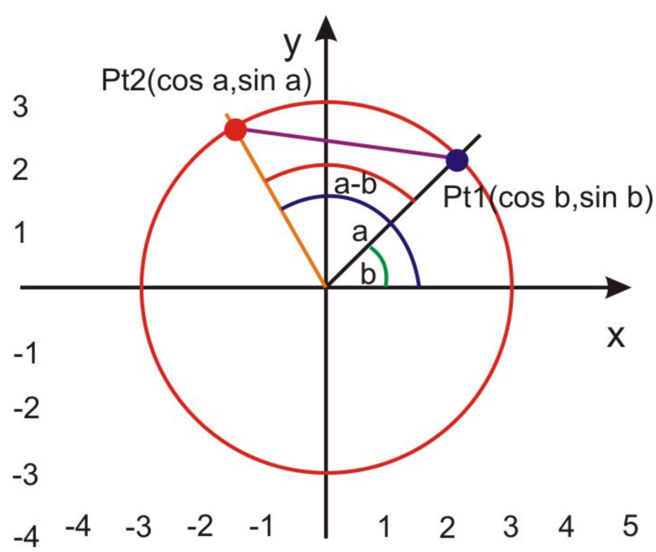


Figure A

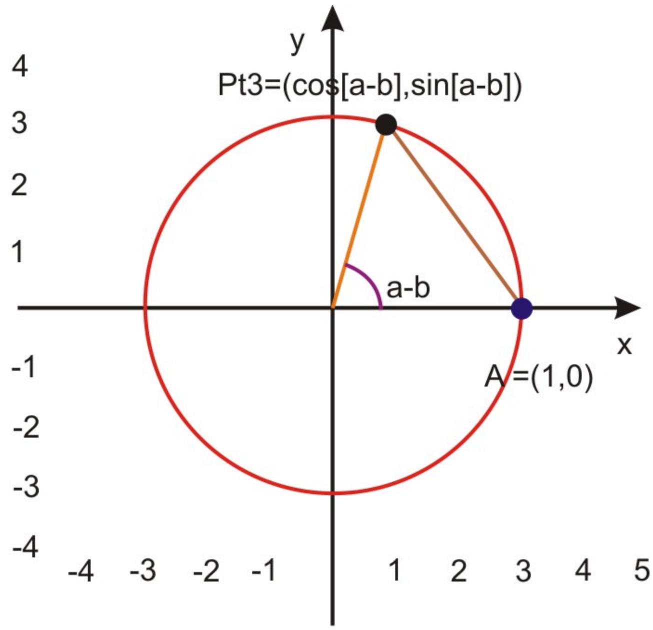


Figure B

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, $a-b$, are equal). Therefore the unknown side of each triangle must also be equal. That is: $d\ (A,P_3) = d\ (P_1,P_2)$

Applying the distance formula to the triangles in Figures A and B and setting them equal to each other:

$$\sqrt{[\cos(a-b)-1]^2+[\sin(a-b)-0]^2} = \sqrt{(\cos a-\cos b)^2+(\sin a-\sin b)^2}$$

Square both sides to eliminate the square root.

$$[\cos(a-b)-1]^2+[\sin(a-b)-0]^2 = (\cos a-\cos b)^2+(\sin a-\sin b)^2$$

FOIL all four squared expressions and simplify.

$$\begin{aligned}
\cos^2(a-b)-2\cos(a-b)+1+\sin^2(a-b) &= \cos^2 a-2\cos a\cos b+\cos^2 b+\sin^2 a-2\sin a\sin b+\sin^2 b\\
\underbrace{\sin^2(a-b)+\cos^2(a-b)}-2\cos(a-b)+1 &= \underbrace{\sin^2 a+\cos^2 a}-2\cos a\cos b+\underbrace{\sin^2 b+\cos^2 b}-2\sin a\sin b\\
1-2\cos(a-b)+1 &= 1-2\cos a\cos b+1-2\sin a\sin b\\
2-2\cos(a-b) &= 2-2\cos a\cos b-2\sin a\sin b\\
-2\cos(a-b) &= -2\cos a\cos b-2\sin a\sin b\\
\cos(a-b) &= \cos a\cos b+\sin a\sin b
\end{aligned}$$

In $\cos(a-b) = \cos a\cos b+\sin a\sin b$, the *difference* formula for cosine, you can substitute $a-(-b) = a+b$ to obtain: $\cos(a+b) = \cos[a-(-b)]$ or $\cos a\cos(-b)+\sin a\sin(-b)$. since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$, then $\cos(a+b) = \cos a\cos b-\sin a\sin b$, which is the *sum* formula for cosine.

Using the Sum and Difference Identities of Cosine

The sum/difference formulas for cosine can be used to establish other identities:

Example 1: Find an equivalent form of $\cos\left(\frac{\pi}{2}-\theta\right)$ using the cosine difference formula.

Solution:

$$\begin{aligned}
\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta \\
\cos\left(\frac{\pi}{2}-\theta\right) &= 0\times\cos\theta+1\times\sin\theta,\ \text{substitute } \cos\frac{\pi}{2}=0 \text{ and } \sin\frac{\pi}{2}=1 \\
\cos\left(\frac{\pi}{2}-\theta\right) &= \sin\theta
\end{aligned}$$

We know that is a true identity because of our understanding of the sine and cosine curves, which are a phase shift of $\frac{\pi}{2}$ off from each other.

The cosine formulas can also be used to find exact values of cosine that we weren't able to find before, such as $15^\circ=(45^\circ-30^\circ), 75^\circ=(45^\circ+30^\circ)$, among others.

Example 2: Find the exact value of $\cos 15^\circ$

Solution: Use the difference formula where $a=45^\circ$ and $b=30^\circ$.

$$\begin{aligned}
\cos(45^\circ-30^\circ) &= \cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ \\
\cos 15^\circ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2} \\
\cos 15^\circ &= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Example 3: Find the exact value of $\cos 105^\circ$.

Solution: There may be more than one pair of key angles that can add up (or subtract to) 105°. Both pairs, $45^\circ+60^\circ$ and $150^\circ-45^\circ$, will yield the correct answer.

1.

$$\begin{aligned}
\cos 105^\circ &= \cos(45^\circ+60^\circ) \\
&= \cos 45^\circ\cos 60^\circ-\sin 45^\circ\sin 60^\circ,\ \text{substitute in the known values} \\
&= \frac{\sqrt{2}}{2}\times\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

2.

$$\begin{aligned}
\cos 105^\circ &= \cos(150^\circ-45^\circ) \\
&= \cos 150^\circ\cos 45^\circ+\sin 150^\circ\sin 45^\circ \\
&= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

You do not need to do the problem multiple ways, just the one that seems easiest to you.

Example 4: Find the exact value of $\cos\frac{5\pi}{12}$, in radians.

Solution: $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$, notice that $\frac{\pi}{4} = \frac{3\pi}{12}$ and $\frac{\pi}{6} = \frac{2\pi}{12}$

$$\begin{aligned}
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\
\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Sum and Difference Identities: Sine

To find $\sin(a+b)$, use Example 1, from above:

$$\begin{aligned}
\sin(a+b) &= \cos\left[\frac{\pi}{2}-(a+b)\right] && \text{Set } \theta = a+b \\
&= \cos\left[\left(\frac{\pi}{2}-a\right)-b\right] && \text{Distribute the negative} \\
&= \cos\left(\frac{\pi}{2}-a\right)\cos b + \sin\left(\frac{\pi}{2}-a\right)\sin b && \text{Difference Formula for cosines} \\
&= \sin a\cos b + \cos a\sin b && \text{Co-function Identities}
\end{aligned}$$

In conclusion, $\sin(a+b) = \sin a\cos b + \cos a\sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a-b)$:

$$\begin{aligned}
\sin(a-b) &= \sin[a+(-b)] \\
&= \sin a\cos(-b) + \cos a\sin(-b) && \text{Use the sine sum formula} \\
\sin(a-b) &= \sin a\cos b - \cos a\sin b && \text{Use } \cos(-b) = \cos b, \text{ and } \sin(-b) = -\sin b
\end{aligned}$$

In conclusion, $\sin(a-b) = \sin a\cos b - \cos a\sin b$, so, this is the *difference* formula for sine.

Example 5: Find the exact value of $\sin\frac{5\pi}{12}$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}
\sin\frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right) \\
&= \sin\frac{3\pi}{12}\cos\frac{2\pi}{12} + \cos\frac{3\pi}{12}\sin\frac{2\pi}{12} \\
\sin\frac{5\pi}{12} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\times\frac{1}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Example 6: Given $\sin\alpha = \frac{12}{13}$, where α is in Quadrant II, and $\sin\beta = \frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha+\beta)$.

Solution: To find the exact value of $\sin(\alpha+\beta)$, here we use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. The values of $\sin\alpha$ and $\sin\beta$ are known, however the values of $\cos\alpha$ and $\cos\beta$ need to be found.

Use $\sin^2\alpha + \cos^2\alpha = 1$, to find the values of each of the missing cosine values.

For $\cos a : \sin^2\alpha + \cos^2\alpha = 1$, substituting $\sin\alpha = \frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2 + \cos^2\alpha = \frac{144}{169} + \cos^2\alpha = 1$ or $\cos^2\alpha = \frac{25}{169}\cos\alpha = \pm\frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos\alpha = -\frac{5}{13}$.

For $\cos\beta$ use $\sin^2\beta + \cos^2\beta = 1$ and substitute $\sin\beta = \frac{3}{5}$, $\left(\frac{3}{5}\right)^2 + \cos^2\beta = \frac{9}{25} + \cos^2\beta = 1$ or $\cos^2\beta = \frac{16}{25}$ and $\cos\beta = \pm\frac{4}{5}$ and since β is in Quadrant I, $\cos\beta = \frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{12}{13} \times \frac{4}{5} + \left(-\frac{5}{13}\right) \times \frac{3}{5} \text{ or } \frac{48}{65} - \frac{15}{65} \\
\sin(\alpha+\beta) &= \frac{33}{65}
\end{aligned}$$

Sum and Difference Identities: Tangent

To find the sum formula for tangent:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} && \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} && \text{Substituting the sum formulas for sine and cosine} \\
&= \frac{\frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b - \sin a\sin b}{\cos a\cos b}} && \text{Divide both the numerator and the denominator by } \cos a\cos b \\
&= \frac{\frac{\sin a\cos b}{\cos a\cos b} + \frac{\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b} - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Reduce each of the fractions} \\
&= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\tan b} && \text{Sum formula for tangent}
\end{aligned}$$

In conclusion, $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Example 7: Find the exact value of $\tan 285^\circ$.

Solution: Use the difference formula for tangent, with $285^\circ = 330^\circ - 45^\circ$

$$
\begin{aligned}
\tan(330^\circ - 45^\circ) &= \frac{\tan 330^\circ - \tan 45^\circ}{1 + \tan 330^\circ \tan 45^\circ} \\
&= \frac{-\frac{\sqrt{3}}{3} - 1}{1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \\
&= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \\
&= \frac{-9 - 6\sqrt{3} - 3}{9 - 3} \\
&= \frac{-12 - 6\sqrt{3}}{6} \\
&= -2 - \sqrt{3}
\end{aligned}
$$

To verify this on the calculator, $\tan 285^\circ = -3.732$ and $-2 - \sqrt{3} = -3.732$.

Using the Sum and Difference Identities to Verify Other Identities

Example 8: Verify the identity $\frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

$$
\begin{aligned}
\cot x \cot y + 1 &= \frac{\cos(x-y)}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} && \text{Expand using the cosine difference formula.} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + 1 \\
\cot x \cot y + 1 &= \cot x \cot y + 1 && \text{cotangent equals cosine over sine}
\end{aligned}
$$

Example 9: Show $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

Solution: First, expand left hand side using the sum and difference formulas:

$$
\begin{aligned}
\cos(a+b)\cos(a-b) &= (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b) \\
&= \cos^2 a \cos^2 b - \sin^2 a \sin^2 b \rightarrow \text{FOIL, middle terms cancel out}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Substitute}(1 - \sin^2 b)\text{for} \cos^2 b \text{ and}(1 - \cos^2 a)\text{for} \sin^2 a \text{ and simplify.} \\
&\cos^2 a(1 - \sin^2 b) - \sin^2 b(1 - \cos^2 a) \\
&\cos^2 a - \cos^2 a \sin^2 b - \sin^2 b + \cos^2 a \sin^2 b \\
&\cos^2 a - \sin^2 b
\end{aligned}
$$

Solving Equations with the Sum and Difference Formulas

Just like the section before, we can incorporate all of the sum and difference formulas into equations and solve for values of x. In general, you will apply the formula *before* solving for the variable. Typically, the goal will be to isolate $\sin x, \cos x$, or $\tan x$ and then apply the inverse. Remember, that you may have to use the identities in addition to the formulas seen in this section to solve an equation.

Example 10: Solve $3\sin(x-\pi) = 3$ in the interval $[0, 2\pi)$.

Solution: First, get $\sin(x-\pi)$ by itself, by dividing both sides by 3.

$$\begin{aligned} \frac{3\sin(x-\pi)}{3} &= \frac{3}{3} \\ \sin(x-\pi) &= 1 \end{aligned}$$

Now, expand the left side using the sine difference formula.

$$\begin{aligned} \sin x \cos \pi - \cos x \sin \pi &= 1 \\ \sin x(-1) - \cos x(0) &= 1 \\ -\sin x &= 1 \\ \sin x &= -1 \end{aligned}$$

The $\sin x = -1$ when x is $\frac{3\pi}{2}$.

Example 11: Find all the solutions for $2\cos^2\left(x+\frac{\pi}{2}\right) = 1$ in the interval $[0, 2\pi)$.

Solution: Get the $\cos^2\left(x+\frac{\pi}{2}\right)$ by itself and then take the square root.

$$\begin{aligned} 2\cos^2\left(x+\frac{\pi}{2}\right) &= 1 \\ \cos^2\left(x+\frac{\pi}{2}\right) &= \frac{1}{2} \\ \cos\left(x+\frac{\pi}{2}\right) &= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$

Now, use the cosine sum formula to expand and solve.

$$\begin{aligned} \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2} &= \frac{\sqrt{2}}{2} \\ \cos x(0) - \sin x(1) &= \frac{\sqrt{2}}{2} \\ -\sin x &= \frac{\sqrt{2}}{2} \\ \sin x &= -\frac{\sqrt{2}}{2} \end{aligned}$$

The $\sin x = -\frac{\sqrt{2}}{2}$ is in Quadrants III and IV, so $x = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Points to Consider

- What are the angles that have 15° and 75° as reference angles?

- Are the only angles that we can find the exact sine, cosine, or tangent values for, multiples of $\frac{\pi}{12}$? (Recall that $\frac{\pi}{2}$ would be $6 \cdot \frac{\pi}{12}$, making it a multiple of $\frac{\pi}{12}$)

Review Questions

1. Find the exact value for:
   a. $\cos \frac{5\pi}{12}$
   b. $\cos \frac{7\pi}{12}$
   c. $\sin 345^\circ$
   d. $\tan 75^\circ$
   e. $\cos 345^\circ$
   f. $\sin \frac{17\pi}{12}$
2. If $\sin y = \frac{12}{13}$, y is in quad II, and $\sin z = \frac{3}{5}$, z is in quad I find $\cos(y-z)$
3. If $\sin y = -\frac{5}{13}$, y is in quad III, and $\sin z = \frac{4}{5}$, z is in quad II find $\sin(y+z)$
4. Simplify:
   a. $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$
   b. $\sin 25^\circ \cos 5^\circ + \cos 25^\circ \sin 5^\circ$
5. Prove the identity: $\frac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$
6. Simplify $\cos(\pi + \theta) = -\cos\theta$
7. Verify the identity: $\sin(a+b)\sin(a-b) = \cos^2 b - \cos^2 a$
8. Simplify $\tan(\pi + \theta)$
9. Verify that $\sin\frac{\pi}{2} = 1$, using the sine sum formula.
10. Reduce the following to a single term: $\cos(x+y)\cos y + \sin(x+y)\sin y$.
11. Prove $\frac{\cos(c+d)}{\cos(c-d)} = \frac{1-\tan c \tan d}{1+\tan c \tan d}$
12. Find all solutions to $2\cos^2\left(x+\frac{\pi}{2}\right) = 1$, when x is between $[0, 2\pi)$.
13. Solve for all values of x between $[0, 2\pi)$ for $2\tan^2\left(x+\frac{\pi}{6}\right)+1 = 7$.
14. Find all solutions to $\sin\left(x+\frac{\pi}{6}\right) = \sin\left(x-\frac{\pi}{4}\right)$, when x is between $[0, 2\pi)$.

4.16 Double Angle Identities

Learning Objectives

- Use the double angle identities to solve other identities.
- Use the double angle identities to solve equations.

Deriving the Double Angle Identities

One of the formulas for calculating the sum of two angles is:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\sin(\alpha+\alpha)&=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha\\ \sin 2\alpha&=2\sin\alpha\cos\alpha\end{aligned}$$

This is the double angle formula for the sine function. The same procedure can be used in the sum formula for cosine, start with the sum angle formula:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\cos(\alpha+\alpha)&=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha\\ \cos 2\alpha&=\cos^2\alpha-\sin^2\alpha\end{aligned}$$

This is one of the double angle formulas for the cosine function. Two more formulas can be derived by using the Pythagorean Identity, $\sin^2\alpha+\cos^2\alpha=1$.

$\sin^2\alpha=1-\cos^2\alpha$ and likewise $\cos^2\alpha=1-\sin^2\alpha$

$$\begin{aligned}&\text{Using } \sin^2\alpha=1-\cos^2\alpha:\\ &\cos 2\alpha=\cos^2\alpha-\sin^2\alpha\\ &\qquad=\cos^2\alpha-(1-\cos^2\alpha)\\ &\qquad=\cos^2\alpha-1+\cos^2\alpha\\ &\qquad=2\cos^2\alpha-1\end{aligned}$$

$$\begin{aligned}&\text{Using } \cos^2\alpha=1-\sin^2\alpha:\\ &\cos 2\alpha=\cos^2\alpha-\sin^2\alpha\\ &\qquad=(1-\sin^2\alpha)-\sin^2\alpha\\ &\qquad=1-\sin^2\alpha-\sin^2\alpha\\ &\qquad=1-2\sin^2\alpha\end{aligned}$$

Therefore, the double angle formulas for $\cos 2a$ are:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ \cos 2\alpha &= 2\cos^2\alpha - 1\\ \cos 2\alpha &= 1 - 2\sin^2\alpha\end{aligned}$$

Finally, we can calculate the double angle formula for tangent, using the tangent sum formula:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\tan(\alpha+\alpha) &= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}\\ \tan 2\alpha &= \frac{2\tan\alpha}{1 - \tan^2\alpha}\end{aligned}$$

Applying the Double Angle Identities

Example 1: If $\sin a = \frac{5}{13}$ and a is in Quadrant II, find $\sin 2a$, $\cos 2a$, and $\tan 2a$.

Solution: To use $\sin 2a = 2\sin a\cos a$, the value of $\cos a$ must be found first.

$$\begin{aligned}&= \cos^2 a + \sin^2 a = 1\\ &= \cos^2 a + \left(\frac{5}{13}\right)^2 = 1\\ &= \cos^2 a + \frac{25}{169} = 1\\ &= \cos^2 a = \frac{144}{169}, \cos a = \pm\frac{12}{13}\end{aligned}$$

.

However since a is in Quadrant II, $\cos a$ is negative or $\cos a = -\frac{12}{13}$.

$$\sin 2a = 2\sin a\cos a = 2\left(\frac{5}{13}\right)\times\left(-\frac{12}{13}\right) = \sin 2a = -\frac{120}{169}$$

For $\cos 2a$, use $\cos(2a) = \cos^2 a - \sin^2 a$

$$\begin{aligned}\cos(2a) &= \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \text{ or } \frac{144-25}{169}\\ \cos(2a) &= \frac{119}{169}\end{aligned}$$

For $\tan 2a$, use $\tan 2a = \frac{2\tan a}{1-\tan^2 a}$. From above, $\tan a = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$.

$$\tan(2a) = \frac{2 \cdot \frac{-5}{12}}{1-\left(\frac{-5}{12}\right)^2} = \frac{\frac{-5}{6}}{1-\frac{25}{144}} = \frac{\frac{-5}{6}}{\frac{119}{144}} = -\frac{5}{6} \cdot \frac{144}{119} = -\frac{120}{119}$$

Example 2: Find $\cos 4\theta$.

Solution: Think of $\cos 4\theta$ as $\cos(2\theta + 2\theta)$.

$$\cos 4\theta = \cos(2\theta+2\theta) = \cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta = \cos^2 2\theta - \sin^2 2\theta$$

Now, use the double angle formulas for both sine and cosine. For cosine, you can pick which formula you would like to use. In general, because we are proving a cosine identity, stay with cosine.

$$\begin{aligned}
&= (2\cos^2\theta - 1)^2 - (2\sin\theta\cos\theta)^2 \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\sin^2\theta\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4(1-\cos^2\theta)\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\cos^2\theta + 4\cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Example 3: If $\cot x = \frac{4}{3}$ and x is an acute angle, find the exact value of $\tan 2x$.

Solution: Cotangent and tangent are reciprocal functions, $\tan x = \frac{1}{\cot x}$ and $\tan x = \frac{3}{4}$.

$$\begin{aligned}
\tan 2x &= \frac{2\tan x}{1-\tan^2 x} \\
&= \frac{2 \cdot \frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} \\
&= \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} \\
&= \frac{3}{2} \cdot \frac{16}{7} = \frac{24}{7}
\end{aligned}$$

Example 4: Given $\sin(2x) = \frac{2}{3}$ and x is in Quadrant I, find the value of $\sin x$.

Solution: Using the double angle formula, $\sin 2x = 2\sin x \cos x$. Because we do not know $\cos x$, we need to solve for $\cos x$ in the Pythagorean Identity, $\cos x = \sqrt{1-\sin^2 x}$. Substitute this into our formula and solve for $\sin x$.

$$\begin{aligned}
\sin 2x &= 2\sin x \cos x \\
\frac{2}{3} &= 2\sin x\sqrt{1-\sin^2 x} \\
\left(\frac{2}{3}\right)^2 &= \left(2\sin x\sqrt{1-\sin^2 x}\right)^2 \\
\frac{4}{9} &= 4\sin^2 x(1-\sin^2 x) \\
\frac{4}{9} &= 4\sin^2 x - 4\sin^4 x
\end{aligned}$$

At this point we need to get rid of the fraction, so multiply both sides by the reciprocal.

$$\begin{aligned}
\frac{9}{4}\left(\frac{4}{9}\right. &= 4\sin^2 x - 4\sin^4 x\left.\right) \\
1 &= 9\sin^2 x - 9\sin^4 x \\
0 &= 9\sin^4 x - 9\sin^2 x + 1
\end{aligned}$$

Now, this is in the form of a quadratic equation, even though it is a quartic. Set $a = \sin^2 x$, making the equation $9a^2 - 9a + 1 = 0$. Once we have solved for a, then we can substitute $\sin^2 x$ back in and solve for x. In the Quadratic Formula, $a = 9, b = -9, c = 1$.

$$\frac{9 \pm \sqrt{(-9)^2 - 4(9)(1)}}{2(9)} = \frac{9 \pm \sqrt{81 - 36}}{18} = \frac{9 \pm \sqrt{45}}{18} = \frac{9 \pm 3\sqrt{5}}{18} = \frac{3 \pm \sqrt{5}}{6}$$

So, $a = \frac{3+\sqrt{5}}{6} \approx 0.873$ or $\frac{3-\sqrt{5}}{6} \approx .1273$. This means that $\sin^2 x \approx 0.873$ or $.1273$ so $\sin x \approx 0.934$ or $\sin x \approx .357$.

Example 5: Prove $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$

Solution: Substitute in the double angle formulas. Use $\cos 2\theta = 1 - 2\sin^2\theta$, since it will produce only one term in the numerator.

$$\begin{aligned}
\tan\theta &= \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} \\
&= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta
\end{aligned}$$

Solving Equations with Double Angle Identities

Much like the previous sections, these problems all involve similar steps to solve for the variable. Isolate the trigonometric function, using any of the identities and formulas you have accumulated thus far.

Example 6: Find all solutions to the equation $\sin 2x = \cos x$ in the interval $[0, 2\pi]$

Solution: Apply the double angle formula $\sin 2x = 2\sin x\cos x$

$$
\begin{aligned}
2\sin x\cos x &= \cos x\\
2\sin x\cos x-\cos x &= \cos x-\cos x\\
2\sin x\cos x-\cos x &= 0\\
\cos x(2\sin x-1) &= 0 \text{ Factor out } \cos x\\
\text{Then } \cos x &= 0 \text{ or } 2\sin x-1=0\\
\cos x &= 0 \text{ or } 2\sin x-1+1=0+1\\
\frac{2}{2}\sin x &= \frac{1}{2}\\
\sin x &= \frac{1}{2}
\end{aligned}
$$

The values for $\cos x = 0$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ and the values for $\sin x = \frac{1}{2}$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Thus, there are four solutions.

Example 7: Solve the trigonometric equation $\sin 2x = \sin x$ such that $(-\pi \le x < \pi)$

Solution: Using the sine double angle formula:

$$
\begin{aligned}
\sin 2x &= \sin x\\
2\sin x\cos x &= \sin x\\
2\sin x\cos x-\sin x &= 0\\
\sin x(2\cos x-1) &= 0
\end{aligned}
$$

$$\downarrow \qquad \searrow$$

$$
\begin{aligned}
2\cos x-1 &= 0\\
2\cos x &= 1
\end{aligned}
$$

$$\sin x = 0$$

$$x = 0, -\pi \qquad \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

Example 8: Find the exact value of $\cos 2x$ given $\cos x = -\frac{13}{14}$ if x is in the second quadrant.

Solution: Use the double-angle formula with cosine only.

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 2\left(-\frac{13}{14}\right)^2 - 1$$
$$\cos 2x = 2\left(\frac{169}{196}\right) - 1$$
$$\cos 2x = \left(\frac{338}{196}\right) - 1$$
$$\cos 2x = \frac{338}{196} - \frac{196}{196}$$
$$\cos 2x = \frac{142}{196} = \frac{71}{98}$$

Example 9: Solve the trigonometric equation $4\sin\theta\cos\theta = \sqrt{3}$ over the interval $[0, 2\pi)$.

Solution: Pull out a 2 from the left-hand side and this is the formula for $\sin 2x$.

$$4\sin\theta\cos\theta = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = 2\sin 2\theta$$
$$2\sin 2\theta = \sqrt{3}$$
$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

The solutions for 2θ are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$, dividing each of these by 2, we get the solutions for θ, which are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{8\pi}{6}$.

Points to Consider

- Are there similar formulas that can be derived for other angles?
- Can technology be used to either solve these trigonometric equations or to confirm the solutions?

Review Questions

1. If $\sin x = \frac{4}{5}$ and x is in Quad II, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
2. Find the exact value of $\cos^2 15^\circ - \sin^2 15^\circ$
3. Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
4. Verify the identity: $\sin 2t - \tan t = \tan t \cos 2t$
5. If $\sin x = -\frac{9}{41}$ and x is in Quad III, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
6. Find all solutions to $\sin 2x + \sin x = 0$ if $0 \leq x < 2\pi$
7. Find all solutions to $\cos^2 x - \cos 2x = 0$ if $0 \leq x < 2\pi$
8. If $\tan x = \frac{3}{4}$ and $0^\circ < x < 90^\circ$, use the double angle formulas to determine each of the following:
   a. $\tan 2x$
   b. $\sin 2x$

   c. $\cos 2x$

9. Use the double angle formulas to prove that the following equations are identities.
   a. $2\csc 2x = \csc^2 x \tan x$
   b. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$
   c. $\frac{\sin 2x}{1+\cos 2x} = \tan x$
10. Solve the trigonometric equation $\cos 2x - 1 = \sin^2 x$ such that $[0, 2\pi)$
11. Solve the trigonometric equation $\cos 2x = \cos x$ such that $0 \leq x < \pi$
12. Prove $2\csc 2x \tan x = \sec^2 x$.
13. Solve $\sin 2x - \cos 2x = 1$ for x in the interval $[0, 2\pi)$.
14. Solve the trigonometric equation $\sin^2 x - 2 = \cos 2x$ such that $0 \leq x < 2\pi$

4.17 Half-Angle Identities

Learning Objectives

- Apply the half angle identities to expressions, equations and other identities.
- Use the half angle identities to find the exact value of trigonometric functions for certain angles.

Just as there are double angle identities, there are also half angle identities. For example: $\sin \frac{1}{2}a$ can be found in terms of the angle "a''". Recall that $\frac{1}{2}a$ and $\frac{a}{2}$ are the same thing and will be used interchangeably throughout this section.

Deriving the Half Angle Formulas

In the previous lesson, one of the formulas that was derived for the cosine of a double angle is: $\cos 2\theta = 1 - 2\sin^2 \theta$. Set $\theta = \frac{\alpha}{2}$, so the equation above becomes $\cos 2\frac{\alpha}{2} = 1 - 2\sin^2 \frac{\alpha}{2}$.

Solving this for $\sin \frac{\alpha}{2}$, we get:

$$\begin{aligned}
\cos 2\frac{\alpha}{2} &= 1 - 2\sin^2 \frac{\alpha}{2} \\
\cos \alpha &= 1 - 2\sin^2 \frac{\alpha}{2} \\
2\sin^2 \frac{\alpha}{2} &= 1 - \cos \alpha \\
\sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\
\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}}
\end{aligned}$$

$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin \frac{\alpha}{2} = -\sqrt{\frac{1 - \cos \alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

Example 1: Determine the exact value of $\sin 15^\circ$.

Solution: Using the half angle identity, $\alpha = 30^\circ$, and 15° is located in the first quadrant. Therefore, $\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}$.

$$\begin{aligned}
\sin 15^\circ &= \sqrt{\frac{1-\cos 30^\circ}{2}} \\
&= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}
\end{aligned}$$

Plugging this into a calculator, $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.2588$. Using the sine function on your calculator will validate that this answer is correct.

Example 2: Use the half angle identity to find exact value of $\sin 112.5^\circ$

Solution: since $\sin \frac{225^\circ}{2} = \sin 112.5^\circ$, use the half angle formula for sine, where $\alpha = 225^\circ$. In this example, the angle 112.5° is a second quadrant angle, and the sin of a second quadrant angle is positive.

$$\begin{aligned}
\sin 112.5^\circ &= \sin \frac{225^\circ}{2} \\
&= \pm\sqrt{\frac{1-\cos 225^\circ}{2}} \\
&= +\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}} \\
&= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}}
\end{aligned}$$

One of the other formulas that was derived for the cosine of a double angle is:

$\cos 2\theta = 2\cos^2\theta - 1$. Set $\theta = \frac{\alpha}{2}$, so the equation becomes $\cos 2\frac{\alpha}{2} = -1 + 2\cos^2\frac{\alpha}{2}$. Solving this for $\cos\frac{\alpha}{2}$, we get:

$$\begin{aligned}
\cos 2\frac{\alpha}{2} &= 2\cos^2\frac{\alpha}{2} - 1 \\
\cos\alpha &= 2\cos^2\frac{\alpha}{2} - 1 \\
2\cos^2\frac{\alpha}{2} &= 1+\cos\alpha \\
\cos^2\frac{\alpha}{2} &= \frac{1+\cos\alpha}{2} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}$$

$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

Example 3: Given that the $\cos\theta = \frac{3}{4}$, and that θ is a fourth quadrant angle, find $\cos\frac{1}{2}\theta$

Solution: Because θ is in the fourth quadrant, the half angle would be in the second quadrant, making the cosine of the half angle negative.

$$\begin{aligned}
\cos\frac{\theta}{2} &= -\sqrt{\frac{1+\cos\theta}{2}} \\
&= -\sqrt{\frac{1+\frac{3}{4}}{2}} \\
&= -\sqrt{\frac{\frac{7}{4}}{2}} \\
&= -\sqrt{\frac{7}{8}} = -\frac{\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14}}{4}
\end{aligned}$$

Example 4: Use the half angle formula for the cosine function to prove that the following expression is an identity: $2\cos^2\frac{x}{2} - \cos x = 1$

Solution: Use the formula $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ and substitute it on the left-hand side of the expression.

$$\begin{aligned}
2\left(\sqrt{\frac{1+\cos\theta}{2}}\right)^2 - \cos\theta &= 1 \\
2\left(\frac{1+\cos\theta}{2}\right) - \cos\theta &= 1 \\
1+\cos\theta - \cos\theta &= 1 \\
1 &= 1
\end{aligned}$$

The half angle identity for the tangent function begins with the reciprocal identity for tangent.

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

The half angle formulas for sine and cosine are then substituted into the identity.

$$\begin{aligned}
\tan\frac{\alpha}{2} &= \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} \\
&= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}
\end{aligned}$$

At this point, you can multiply by either $\frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}}$ or $\frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}}$. We will show both, because they produce different answers.

$$= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}} \qquad = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

$$= \frac{1-\cos\alpha}{\sqrt{1-\cos^2\alpha}} \qquad \text{or} \qquad = \frac{\sqrt{1-\cos^2\alpha}}{1+\cos\alpha}$$

$$= \frac{1-\cos\alpha}{\sqrt{\sin^2\alpha}} \qquad = \frac{\sqrt{\sin^2\alpha}}{1+\cos\alpha}$$

$$= \frac{1-\cos\alpha}{\sin\alpha} \qquad = \frac{\sin\alpha}{1+\cos\alpha}$$

So, the two half angle identities for tangent are $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ and $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.

Example 5: Use the half-angle identity for tangent to determine an exact value for $\tan\frac{7\pi}{12}$.

Solution:

$$\begin{aligned}
\tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} \\
\tan\frac{7\pi}{12} &= \frac{1-\cos\frac{7\pi}{6}}{\sin\frac{7\pi}{6}} \\
\tan\frac{7\pi}{12} &= \frac{1+\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\
\tan\frac{7\pi}{12} &= -2-\sqrt{3}
\end{aligned}$$

Example 6: Prove the following identity: $\tan x = \frac{1-\cos 2x}{\sin 2x}$

Solution: Substitute the double angle formulas for $\cos 2x$ and $\sin 2x$.

$$\begin{aligned}
\tan x &= \frac{1-\cos 2x}{\sin 2x} \\
&= \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} \\
&= \frac{1-1+2\sin^2 x}{2\sin x\cos x} \\
&= \frac{2\sin^2 x}{2\sin x\cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x
\end{aligned}$$

Solving Trigonometric Equations Using Half Angle Formulas

Example 7: Solve the trigonometric equation $\sin^2\theta = 2\sin^2\frac{\theta}{2}$ over the interval $[0, 2\pi)$.

Solution:

$$
\begin{aligned}
\sin^2\theta &= 2\sin^2\frac{\theta}{2} & \\
\sin^2\theta &= 2\left(\frac{1-\cos\theta}{2}\right) & \text{Half angle identity} \\
1-\cos^2\theta &= 1-\cos\theta & \text{Pythagorean identity} \\
\cos\theta-\cos^2\theta &= 0 & \\
\cos\theta(1-\cos\theta) &= 0 &
\end{aligned}
$$

Then $\cos\theta = 0$ or $1-\cos\theta = 0$, which is $\cos\theta = 1$.

$\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, or 2π.

Points to Consider

- Can you derive a third or fourth angle formula?
- How do $\frac{1}{2}\sin x$ and $\sin\frac{1}{2}x$ differ? Is there a formula for $\frac{1}{2}\sin x$?

Review Questions

1. Find the exact value of:
   a. $\cos 112.5^\circ$
   b. $\sin 105^\circ$
   c. $\tan\frac{7\pi}{8}$
   d. $\tan\frac{\pi}{8}$
   e. $\sin 67.5^\circ$
   f. $\tan 165^\circ$
2. If $\sin\theta = \frac{7}{25}$ and θ is in Quad II, find $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, \tan\frac{\theta}{2}$
3. Prove the identity: $\tan\frac{b}{2} = \frac{\sec b}{\sec b\csc b+\csc b}$
4. Verify the identity: $\cot\frac{c}{2} = \frac{\sin c}{1-\cos c}$
5. Prove that $\sin x\tan\frac{x}{2}+2\cos x = 2\cos^2\frac{x}{2}$
6. If $\sin u = -\frac{8}{13}$, find $\cos\frac{u}{2}$
7. Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$
8. Solve $\tan\frac{a}{2} = 4$ for $0^\circ \le a < 360^\circ$
9. Solve the trigonometric equation $\cos\frac{x}{2} = 1+\cos x$ such that $0 \le x < 2\pi$.
10. Prove $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$.

4.18 Basic Inverse Trigonometric Functions

Introduction

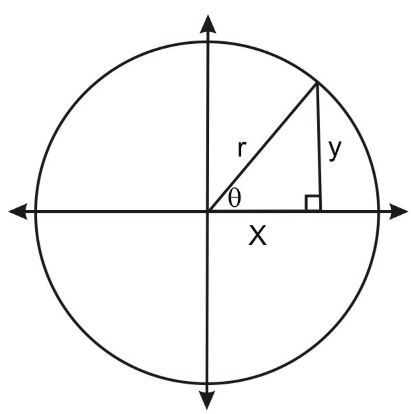
Recall that an inverse function is a reflection of the function over the line $y = x$. In order to find the inverse of a function, you must switch the x and y values and then solve for y. A function has an inverse if and only if it has exactly one output for every input and exactly one input for every output. All of the trig functions fit these criteria over a specific range. In this chapter, we will explore inverse trig functions and equations.

Learning Objectives

- Understand and evaluate inverse trigonometric functions.
- Extend the inverse trigonometric functions to include the $\csc^{-1}$, $\sec^{-1}$ and $\cot^{-1}$ functions.
- Apply inverse trigonometric functions to the critical values on the unit circle.

Defining the Inverse of the Trigonometric Ratios

Recall from Chapter 1, the ratios of the six trig functions and their inverses, with regard to the unit circle.

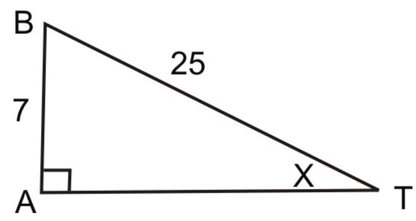


$$\sin\theta = \frac{y}{r} \rightarrow \sin^{-1}\frac{y}{r} = \theta \qquad \cos\theta = \frac{x}{r} \rightarrow \cos^{-1}\frac{x}{r} = \theta$$
$$\tan\theta = \frac{y}{x} \rightarrow \tan^{-1}\frac{y}{x} = \theta \qquad \cot\theta = \frac{x}{y} \rightarrow \cot^{-1}\frac{x}{y} = \theta$$
$$\csc\theta = \frac{r}{y} \rightarrow \csc^{-1}\frac{r}{y} = \theta \qquad \sec\theta = \frac{r}{x} \rightarrow \sec^{-1}\frac{r}{x} = \theta$$

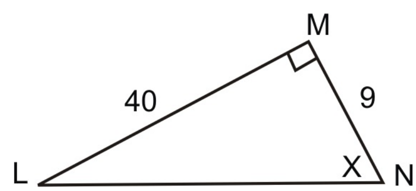
These ratios can be used to find any θ in standard position or in a triangle.

Example 1: Find the measure of the angles below.

a.



b.



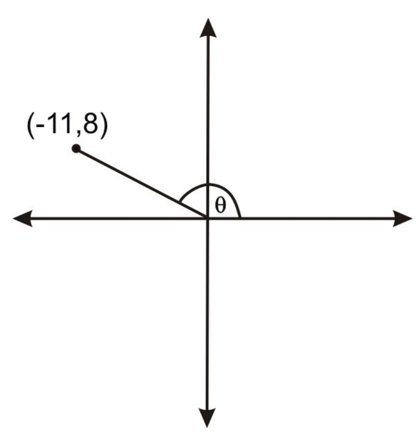
Solution: For part a, you need to use the sine function and part b utilizes the tangent function. Because both problems require you to solve for an angle, the inverse of each must be used.

a. $\sin x = \frac{7}{25} \rightarrow \sin^{-1} \frac{7}{25} = x \rightarrow x = 16.26^\circ$

b. $\tan x = \frac{40}{9} \rightarrow \tan^{-1} \frac{40}{9} = x \rightarrow x = 77.32^\circ$

The trigonometric value $\tan\theta = \frac{40}{9}$ of the angle is known, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. (Directions for how to find inverse function values in the graphing calculator are in Chapter 1). The inverse of the tangent function is read "tangent inverse" and is also called the arctangent relation. The inverse of the cosine function is read "cosine inverse" and is also called the arccosine relation. The inverse of the sine function is read "sine inverse" and is also called the arcsine relation.

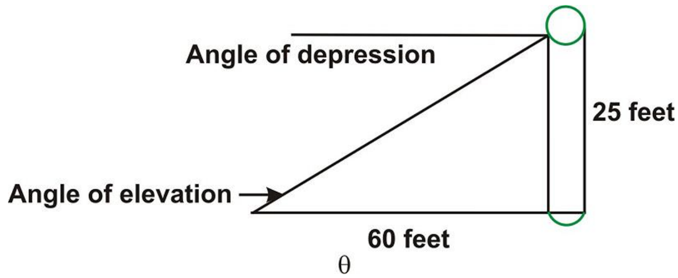
Example 2: Find the angle, θ, in standard position.



Solution: The $\tan\theta = \frac{y}{x}$ or, in this case, $\tan\theta = \frac{8}{-11}$. Using the inverse tangent, you get $\tan^{-1} -\frac{8}{11} = -36.03^\circ$. This means that the reference angle is 36.03°. This value of 36.03° is the angle you also see if you move counterclockwise from the -x axis. To find the corresponding angle in the second quadrant (which is the same as though you started at the +x axis and moved counterclockwise), subtract 36.03° from 180°, yielding 143.97°.

Recall that inverse trigonometric functions are also used to find the angle of depression or elevation.

Example 3: A new outdoor skating rink has just been installed outside a local community center. A light is mounted on a pole 25 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the skating rink. If the end of the rink is 60 feet from the pole, at what angle of depression should the light be installed?



Solution: In this diagram, the angle of depression, which is located outside of the triangle, is not known. Recall, the angle of depression equals the angle of elevation. For the angle of elevation, the pole where the light is located is the opposite and is 25 feet high. The length of the rink is the adjacent side and is 60 feet in length. To calculate the measure of the angle of elevation the trigonometric ratio for tangent can be applied.

$$\begin{aligned}
\tan\theta &= \frac{25}{60} \\
\tan\theta &= 0.4166 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(0.4166) \\
\theta &= 22.6^\circ
\end{aligned}$$

The angle of depression at which the light must be placed to light the rink is 22.6°

Exact Values for Inverse Sine, Cosine, and Tangent

Recall the unit circle and the critical values. With the inverse trigonometric functions, you can find the angle value (in either radians or degrees) when given the ratio and function. Make sure that you find all solutions within the given interval.

Example 4: Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\tan^{-1}\sqrt{3}$

Solution: These are all values from the special right triangles and the unit circle.

a. Recall that $-\frac{\sqrt{3}}{2}$ is from the $30-60-90$ triangle. The reference angle for sin and $\frac{\sqrt{3}}{2}$ would be 60°. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

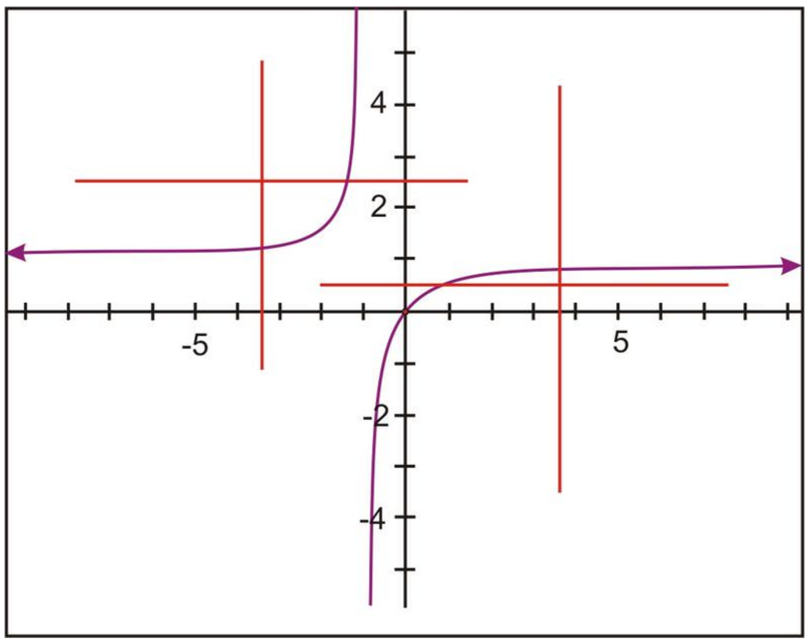
b. $-\frac{\sqrt{2}}{2}$ is from an isosceles right triangle. The reference angle is then 45°. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

c. $\sqrt{3}$ is also from a $30-60-90$ triangle. Tangent is $\sqrt{3}$ for the reference angle 60°. Tangent is positive in the first and third quadrants, so the answer would be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.

Notice how each one of these examples yield two answers. This poses a problem when finding a singular inverse for each of the trig functions. Therefore, we need to restrict the domain in which the inverses can be found, which will be addressed in the next section. Unless otherwise stated, all angles are in radians.

Finding Inverses Algebraically

In the Prerequisite Chapter, you learned that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function is one-to-one when its graph passes both the vertical and the horizontal line test. This means that every vertical and horizontal line will intersect the graph in exactly one place.



This is the graph of $f(x) = \frac{x}{x+1}$. The graph suggests that f is one-to-one because it passes both the vertical and the horizontal line tests. To find the inverse of f, switch **the** x **and** y **and solve for** y.

First, switch x and y.

$$x = \frac{y}{y+1}$$

Next, multiply both sides by $(y+1)$.

$$\begin{aligned}(y+1)x &= \frac{y}{y+1}(y+1)\\ x(y+1) &= y\end{aligned}$$

Then, apply the distributive property and put all the y terms on one side so you can pull out the y.

$$\begin{aligned} xy + x &= y \\ xy - y &= -x \\ y(x-1) &= -x \end{aligned}$$

Divide by $(x-1)$ to get y by itself.

$$y = \frac{-x}{x-1}$$

Finally, multiply the right side by $\frac{-1}{-1}$.

$$y = \frac{x}{1-x}$$

Therefore the inverse of f is $f^{-1}(x) = \frac{x}{1-x}$.

The symbol f^{-1} is read "f inverse" and is not the reciprocal of f.

Example 5: Find the inverse of $f(x) = \frac{1}{x-5}$ algebraically.

Solution: To find the inverse algebraically, switch $f(x)$ to y and then switch x and y.

$$\begin{aligned} y &= \frac{1}{x-5} \\ x &= \frac{1}{y-5} \\ x(y-5) &= 1 \\ xy - 5x &= 1 \\ xy &= 5x + 1 \\ y &= \frac{5x+1}{x} \end{aligned}$$

Example 6: Find the inverse of $f(x) = 5\sin^{-1}\left(\frac{2}{x-3}\right)$

Solution:

a.

$$\begin{aligned}
f(x) &= 5\sin^{-1}\left(\frac{2}{x-3}\right)\\
x &= 5\sin^{-1}\left(\frac{2}{y-3}\right)\\
\frac{x}{5} &= \sin^{-1}\left(\frac{2}{y-3}\right)\\
\sin\frac{x}{5} &= \left(\frac{2}{y-3}\right)\\
(y-3)\sin\frac{x}{5} &= 2\\
(y-3) &= \frac{2}{\sin\frac{x}{5}}\\
y &= \frac{2}{\sin\frac{x}{5}}+3
\end{aligned}$$

Example 7: Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x+4)$

Solution:

$$\begin{aligned}
x &= 4\tan^{-1}(3y+4)\\
\frac{x}{4} &= \tan^{-1}(3y+4)\\
\tan\frac{x}{4} &= 3y+4\\
\tan\frac{x}{4}-4 &= 3y\\
\frac{\tan\frac{x}{4}-4}{3} &= y\\
f^{-1}(x) &= \frac{\tan\frac{x}{4}-4}{3}
\end{aligned}$$

Points to Consider

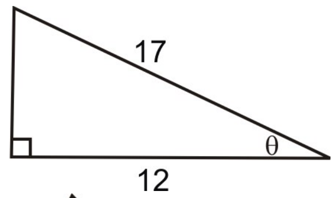
- What is the difference between an inverse and a reciprocal?
- Considering that most graphing calculators do not have csc, sec or cot buttons, how would you find the inverse of each of these?
- Besides algebraically, is there another way to find the inverse?

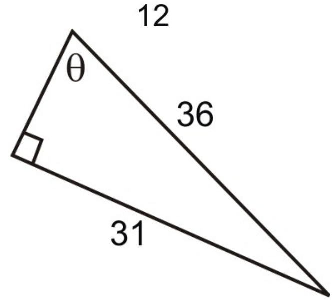
Review Questions

1. Use the special triangles or the unit circle to evaluate each of the following:
   a. $\cos 120^\circ$
   b. $\csc\frac{3\pi}{4}$
   c. $\tan\frac{5\pi}{3}$

2. Find the exact value of each inverse function, without a calculator in $[0, 2\pi)$:
   a. $\cos^{-1}(0)$
   b. $\tan^{-1}\left(-\sqrt{3}\right)$
   c. $\sin^{-1}\left(-\frac{1}{2}\right)$

Find the value of the missing angle.

3. 

4. 

5. What is the value of the angle with its terminal side passing through (-14, -23)?
6. A 9-foot ladder is leaning against a wall. If the foot of the ladder is 4 feet from the base of the wall, what angle does the ladder make with the floor?

Find the inverse of the following functions.

7. $f(x) = 2x^3 - 5$
8. $y = \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right)$
9. $g(x) = 2\sin(x-1) + 4$
10. $h(x) = 5 - \cos^{-1}(2x+3)$

4.19 Inverse Trigonometric Properties

Learning Objectives

- Relate the concept of inverse functions to trigonometric functions.
- Reduce the composite function to an algebraic expression involving no trigonometric functions.
- Use the inverse reciprocal properties.
- Compose each of the six basic trigonometric functions with $\tan^{-1} x$.

Composing Trig Functions and their Inverses

In the Prerequisite Chapter, you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x$$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig. function of a trig. function, will only be true for values of x within the restricted domains.

$$\sin^{-1}(\sin(x)) = x \qquad \cos^{-1}(\cos(x)) = x \qquad \tan^{-1}(\tan(x)) = x$$

These equations are better known as composite functions and are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.

Example 1: Find $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$.

Solution: We know that $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, within the defined restricted domain. Then, we need to find $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$. So, the above properties allow for a short cut. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$, think of it like the sine and sine inverse cancel each other out and all that is left is the $\frac{\sqrt{2}}{2}$.

Composing Trigonometric Functions

Besides composing trig functions with their own inverses, you can also compose any trig functions with any inverse. When solving these types of problems, start with the function that is composed inside of the other and work your way out. Use the following examples as a guideline.

Example 2: Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

c. $\cos(\tan^{-1}(-1))$

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

Solution: For all of these types of problems, the answer is restricted to the inverse functions' ranges.

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$: First find $\tan^{-1}\sqrt{3}$, which is $\frac{\pi}{3}$. Then find $\cos\frac{\pi}{3}$. Your final answer is $\frac{1}{2}$. Therefore, $\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$.

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

c. $\cos(\tan^{-1}(-1)) = \cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Inverse Reciprocal Functions

We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$\begin{aligned} y &= \sin^{-1}x \\ x &= \sin y \\ \frac{1}{x} &= \csc y \\ \csc^{-1}\frac{1}{x} &= y \\ \csc^{-1}\frac{1}{x} &= \sin^{-1}x \end{aligned}$$

Because cosecant and secant are inverses, $\sin^{-1}\frac{1}{x} = \csc^{-1}x$ is also true.

The inverse reciprocal identity for cosine and secant can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid.

$$\sec^{-1}\frac{1}{x} = \cos^{-1}x \leftrightarrow \cos^{-1}\frac{1}{x} = \sec^{-1}x$$

Tangent and cotangent have a slightly different relationship. Recall that the graph of cotangent differs from tangent by a reflection over the $y-$axis and a shift of $\frac{\pi}{2}$. As an equation, this can be written as $\cot x = -\tan\left(x - \frac{\pi}{2}\right)$. Taking the inverse of this function will show the inverse reciprocal relationship between arccotangent and arctangent.

$$\begin{aligned}
y &= \cot^{-1} x \\
y &= -\tan^{-1}\left(x - \frac{\pi}{2}\right) \\
x &= -\tan\left(y - \frac{\pi}{2}\right) \\
-x &= \tan\left(y - \frac{\pi}{2}\right) \\
\tan^{-1}(-x) &= y - \frac{\pi}{2} \\
\frac{\pi}{2} + \tan^{-1}(-x) &= y \\
\frac{\pi}{2} - \tan^{-1} x &= y
\end{aligned}$$

Remember that tangent is an odd function, so that $\tan(-x) = -\tan(x)$. Because tangent is odd, its inverse is also odd. So, this tells us that $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$. You will determine the domain and range of all of these functions when you graph them in the exercises for this section. To graph arcsecant, arccosecant, and arccotangent in your calculator you will use these conversion identities: $\sec^{-1} x = \cos^{-1} \frac{1}{x}, \csc^{-1} x = \sin^{-1} \frac{1}{x}, \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$. Note: It is also true that $\cot^{-1} x = \tan^{-1} \frac{1}{x}$.

Now, let's apply these identities to some problems that will give us an insight into how they work.

Example 3: Evaluate $\sec^{-1} \sqrt{2}$

Solution: Use the inverse reciprocal property. $\sec^{-1} x = \cos^{-1} \frac{1}{x} \rightarrow \sec^{-1} \sqrt{2} = \cos^{-1} \frac{1}{\sqrt{2}}$. Recall that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$, and we know that $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$.

Example 4: Find the exact value of each expression within the restricted domain, without a calculator.

a. $\sec^{-1} \sqrt{2}$

b. $\cot^{-1}\left(-\sqrt{3}\right)$

c. $\csc^{-1} \frac{2\sqrt{3}}{3}$

Solution: For each of these problems, first find the reciprocal and then determine the angle from that.

a. $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$ From the unit circle, we know that the answer is $\frac{\pi}{4}$.

b. $\cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\sqrt{3}\right)$ From the unit circle, the answer is $\frac{5\pi}{6}$.

c. $\csc^{-1} \frac{2\sqrt{3}}{3} = \sin^{-1} \frac{\sqrt{3}}{2}$ Within our interval, there are is one answer, $\frac{\pi}{3}$.

Example 5: Using technology, find the value in radian measure, of each of the following:

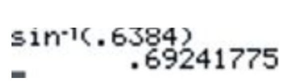
a. $\arcsin 0.6384$

b. $\arccos(-0.8126)$

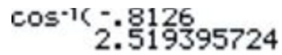
c. $\arctan(-1.9249)$

Solution:

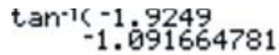
a.



b.



c.



Make sure that your calculator's MODE is RAD (radians).

Composing Inverse Reciprocal Trig Functions

In this subsection, we will combine what was learned in the previous two sections. Here are a few examples:

Example 6: Without a calculator, find $\cos\left(\cot^{-1}\sqrt{3}\right)$.

Solution: First, find $\cot^{-1}\sqrt{3}$, which is also $\tan^{-1}\frac{\sqrt{3}}{3}$. This is $\frac{\pi}{6}$. Now, find $\cos\frac{\pi}{6}$, which is $\frac{\sqrt{3}}{2}$. So, our answer is $\frac{\sqrt{3}}{2}$.

Example 7: Without a calculator, find $\sec^{-1}\left(\csc\frac{\pi}{3}\right)$.

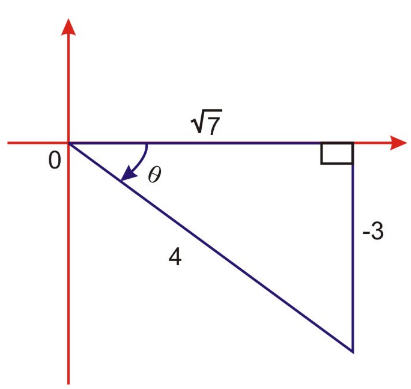
Solution: First, $\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$. Then $\sec^{-1}\frac{2\sqrt{3}}{3}=\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$.

Example 8: Evaluate $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Solution: Even though this problem is not a critical value, it can still be done without a calculator. Recall that sine is the opposite side over the hypotenuse of a triangle. So, 3 is the opposite side and 5 is the hypotenuse. This is a Pythagorean Triple, and thus, the adjacent side is 4. To continue, let $\theta=\sin^{-1}\frac{3}{5}$ or $\sin\theta=\frac{3}{5}$, which means θ is in the Quadrant 1 (from our restricted domain, it cannot also be in Quadrant II). Substituting in θ we get $\cos\left(\sin^{-1}\frac{3}{5}\right)=\cos\theta$ and $\cos\theta=\frac{4}{5}$.

Example 9: Evaluate $\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

Solution: Even though $\frac{3}{4}$ does not represent two lengths from a Pythagorean Triple, you can still use the Pythagorean Theorem to find the missing side. $(-3)^2+b^2=4^2$, so $b=\sqrt{16-9}=\sqrt{7}$. From the restricted domain, sine inverse is negative in the 4^{th} Quadrant. To illustrate:

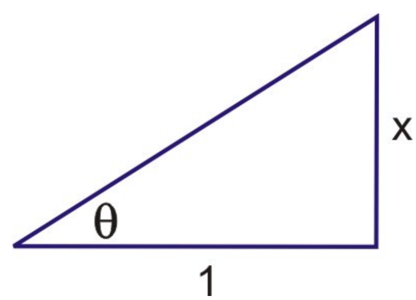


Let

$$\begin{aligned}\theta &= \sin^{-1}\left(-\frac{3}{4}\right)\\ \sin\theta &= -\frac{3}{4}\\ \tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right) &= \tan\theta\\ \tan\theta &= \frac{-3}{\sqrt{7}} \text{ or } \frac{-3\sqrt{7}}{7}\end{aligned}$$

Trigonometry in Terms of Algebra

All of the trigonometric functions can be rewritten in terms of only x, when using one of the inverse trigonometric functions. Starting with tangent, we draw a triangle where the opposite side (from θ) is defined as x and the adjacent side is 1. The hypotenuse, from the Pythagorean Theorem would be $\sqrt{x^2+1}$. Substituting $\tan^{-1} x$ for θ, we get:



$$\begin{aligned}\tan\theta &= \frac{x}{1}\\ \tan\theta &= x\\ \theta &= \tan^{-1} x\end{aligned} \qquad hypotenuse = \sqrt{x^2+1}$$

$$\begin{aligned}\sin(\tan^{-1} x) &= \sin\theta = \frac{x}{\sqrt{x^2+1}} & \csc(\tan^{-1} x) &= \csc\theta = \frac{\sqrt{x^2+1}}{x}\\ \cos(\tan^{-1} x) &= \cos\theta = \frac{1}{\sqrt{x^2+1}} & \sec(\tan^{-1} x) &= \sec\theta = \sqrt{x^2+1}\\ \tan(\tan^{-1} x) &= \tan\theta = x & \cot(\tan^{-1} x) &= \cot\theta = \frac{1}{x}\end{aligned}$$

Example 10: Find $\sin(\tan^{-1} 3x)$.

Solution: Instead of using x in the ratios above, use $3x$.

$$\sin(\tan^{-1} 3x) = \sin\theta = \frac{3x}{\sqrt{(3x)^2+1}} = \frac{3x}{\sqrt{9x^2+1}}$$

Example 11: Find $\sec^2(\tan^{-1} x)$.

Solution: This problem might be better written as $[\sec(\tan^{-1} x)]^2$. Therefore, all you need to do is square the ratio above.

$$[\sec(\tan^{-1} x)]^2 = \left(\sqrt{x^2+1}\right)^2 = x^2+1$$

You can also write the all of the trig functions in terms of arcsine and arccosine. However, for each inverse function, there is a different triangle. You will derive these formulas in the exercise for this section.

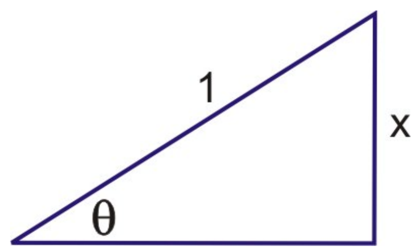
Points to Consider

- Is it possible to graph these composite functions? What happens when you graph $y = \sin(\cos^{-1} x)$ in your calculator?
- Do exact values of functions of inverse functions exist if any value is used?

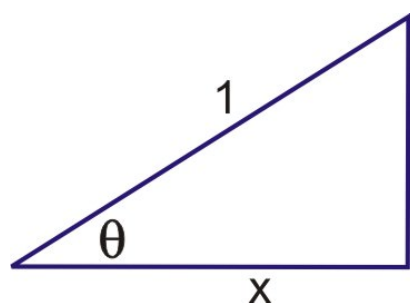
Review Questions

1. Find the exact value of the functions, without a calculator, over their restricted domains.
   a. $\cos^{-1} \frac{\sqrt{3}}{2}$
   b. $\sec^{-1} \sqrt{2}$
   c. $\sec^{-1}\left(-\sqrt{2}\right)$
   d. $\sec^{-1}(-2)$
   e. $\cot^{-1}(-1)$
   f. $\csc^{-1}\left(\sqrt{2}\right)$
2. Use your calculator to find:
   a. $\arccos(-0.923)$
   b. $\arcsin 0.368$
   c. $\arctan 5.698$
3. Find the exact value of the functions, without a calculator, over their restricted domains.
   a. $\csc\left(\cos^{-1} \frac{\sqrt{3}}{2}\right)$
   b. $\sec^{-1}(\tan(\cot^{-1} 1))$
   c. $\tan^{-1}\left(\cos \frac{\pi}{2}\right)$
   d. $\cot\left(\sec^{-1} \frac{2\sqrt{3}}{3}\right)$
4. Using your graphing calculator, graph $y = \sec^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
5. Using your graphing calculator, graph $y = \csc^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)

6. Using your graphing calculator, graph $y = \cot^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
7. Evaluate:
   a. $\sin\left(\cos^{-1} \frac{5}{13}\right)$
   b. $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right)$
   c. $\cos\left(\csc^{-1} \frac{25}{7}\right)$
8. Express each of the following functions as an algebraic expression involving no trigonometric functions.
   a. $\cos^2(\tan^{-1} x)$
   b. $\cot(\tan^{-1} x^2)$
9. To find trigonometric functions in terms of sine inverse, use the following triangle.



   a. Determine the sine, cosine and tangent in terms of arcsine.
   b. Find $\tan(\sin^{-1} 2x^3)$.
10. To find the trigonometric functions in terms of cosine inverse, use the following triangle.



   a. Determine the sine, cosine and tangent in terms of arccosine.
   b. Find $\sin^2\left(\cos^{-1} \frac{1}{2}x\right)$.

4.20 Using Trigonometric Identities

Objective

Here you'll learn how to use trigonometric identities to solve and prove trigonometric statements and problems.

Review Queue

Find the exact value of the trig functions below.

1. $\sin \frac{\pi}{3}$
2. $\cos \left(-\frac{5\pi}{6}\right)$
3. $\sin 4\pi$
4. $\tan \frac{7\pi}{4}$
5. $\cos \frac{2\pi}{3}$
6. $\tan \left(-\frac{2\pi}{3}\right)$

Introduction to Trig Identities

Objective

Here you'll learn about the basic trigonometric identities and how to use them.

Guidance

Trigonometric identities are true for any value of x (as long as the value is in the domain). In the *Reciprocal Trigonometric Functions* concept from the previous chapter, you learned about secant, cosecant, and cotangent, which are all reciprocal functions of sine, cosine and tangent. These functions can be rewritten as the Reciprocal Identities because they are always true.

Reciprocal Identities: $\csc\theta = \frac{1}{\sin\theta}$ $\quad \sec\theta = \frac{1}{\cos\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta}$

Other identities involve the tangent, variations on the Pythagorean Theorem, phase shifts, and negative angles. We will discover them in this concept.

Example A

$\tan\theta = \frac{opposite}{adjacent}$. Show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.This is the **Tangent Identity**.

Solution: Whenever we are trying to verify, or prove, an identity, we start with the statement we are trying to prove and work towards the desired answer. In this case, we will start with $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and show that it is equivalent to $\tan\theta = \frac{opposite}{adjacent}$. First, rewrite sine and cosine in terms of the ratios of the sides.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}}\end{aligned}$$

Then, rewrite the complex fraction as a division problem and simplify.

$$\begin{aligned}
&= \frac{opposite}{hypotenuse} \div \frac{adjacent}{hypotenuse} \\
&= \frac{opposite}{\cancel{hypotenuse}} \cdot \frac{\cancel{hypotenuse}}{adjacent} \\
&= \frac{opposite}{adjacent}
\end{aligned}$$

We now have what we wanted to prove and we are done. ***Once you verify an identity, you may use it to verify other identities.***

Example B

Show that $\sin^2\theta + \cos^2\theta = 1$ is a true identity.

Solution: Change the sine and cosine in the equation into the ratios. In this example, we will use y as the opposite side, x is the adjacent side, and r is the hypotenuse (or radius), as in the unit circle.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 &= 1 \\
\frac{y^2}{r^2} + \frac{x^2}{r^2} &= 1 \\
\frac{y^2 + x^2}{r^2} &= 1
\end{aligned}$$

Now, $x^2 + y^2 = r^2$ from the Pythagorean Theorem. Substitute this in for the numerator of the fraction.

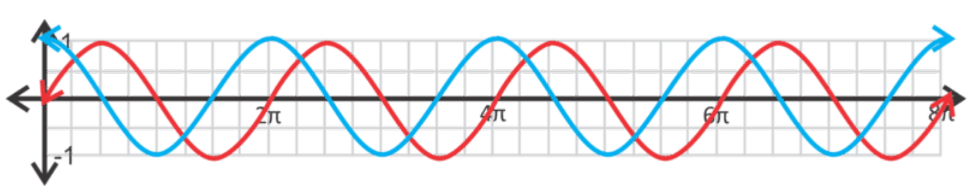
$$\frac{r^2}{r^2} = 1$$

This is one of the **Pythagorean Identities** and very useful.

Example C

Verify that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ by using the graphs of the functions.

Solution: The function $y = \sin\left(\frac{\pi}{2} - x\right)$ is a phase shift of $\frac{\pi}{2}$ of the sine curve.



The red function above is $y = \sin x$ and the blue is $y = \cos x$. If we were to shift the sine curve $\frac{\pi}{2}$, it would overlap perfectly with the cosine curve, thus proving this **Cofunction Identity**.

Guided Practice

1. Prove the Pythagorean Identity: $1 + \tan^2\theta = \sec^2\theta$

2. Without graphing, show that $\sin(-\theta) = -\sin\theta$.

Answers

1. First, let's use the Tangent Identity and the Reciprocal Identity to change tangent and secant in terms of sine and cosine.

$$1+\tan^2\theta=\sec^2\theta$$
$$1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

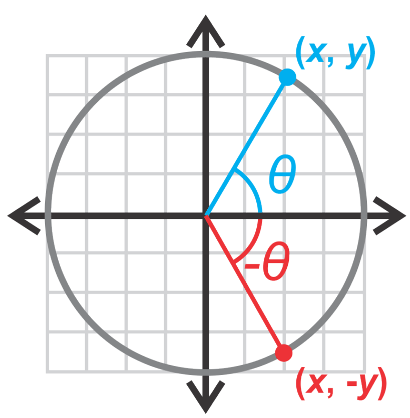
Now, change the 1 into a fraction with a base of $\cos^2\theta$ and simplify.

$$1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\frac{1}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

In the second to last step, we arrived at the original Pythagorean Identity $\sin^2\theta+\cos^2\theta$ in the numerator of the left-hand side. Therefore, we can substitute in 1 for this and the two sides of the equation are the same.

2. First, recall that $\sin\theta=y$, where (x,y) is the endpoint of the terminal side of θ on the unit circle.

Now, if we have $\sin(-\theta)$, what is it's endpoint? Well, the negative sign tells us that the angle is rotated in a clockwise direction, rather than the usual counter-clockwise. By looking at the picture, we see that $\sin(-\theta)=-y$. Therefore, if $\sin\theta=y$, then $-\sin\theta=-y$ and combining the equations, we have $\sin(-\theta)=-\sin\theta$.



Vocabulary

Trigonometric Identity
A trigonometric equation that is true for any x value (within the domain).

Verify
To prove a trigonometric identity.

Reciprocal Identities
$\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

Tangent Identity
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Cotangent Identity
$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities
$\sin^2\theta + \cos^2\theta = 1, 1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$

Cofunction Identities
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, and $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

Negative Angle Identities
$\sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta$, and $\tan(-\theta) = -\tan\theta$

Problem Set

1. Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$ by following the steps from Example A.
2. Show that $\tan\theta = \frac{\sec\theta}{\csc\theta}$. Refer to Example A to help you.
3. Show that $1 + \cot^2\theta = \csc^2\theta$ by following the steps from #1 in the Guided Practice.
4. Explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ by using the graphs of the two functions.
5. Following the steps from #2 in the Guided Practice, show that $\cos(-\theta) = \cos\theta$.
6. Explain why $\tan(-\theta) = -\tan\theta$ is true, using the Tangent Identity and the other Negative Angle Identities.

Verify the following identities.

7. $\cot\theta\sec\theta = \csc\theta$
8. $\frac{\cos\theta}{\cot\theta} = \frac{\tan\theta}{\sec\theta}$
9. $\sin\theta\csc\theta = 1$
10. $\cot(-\theta) = -\cot\theta$
11. $\tan x \csc x \cos x = 1$
12. $\frac{\sin^2(-x)}{\tan^2 x} = \cos^2 x$

Show that $\sin^2\theta + \cos^2\theta = 1$ is true for the following values of θ.

13. $\frac{\pi}{4}$
14. $\frac{2\pi}{3}$
15. $-\frac{7\pi}{6}$
16. Recall that a function is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$. Which of the six trigonometric functions are odd? Which are even?

Using Trig Identities to Find Exact Trig Values

Objective

Here you'll use the basic trig identities to find exact trig values of angles other than the critical angles.

Guidance

You can use the Pythagorean, Tangent and Reciprocal Identities to find all six trigonometric values for certain angles. Let's walk through a few examples so that you understand how to do this.

Example A

Given that $\cos\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\sin\theta$.

Solution: Use the Pythagorean Identity to find $\sin\theta$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1\\ \sin^2\theta &= 1 - \frac{9}{25}\\ \sin^2\theta &= \frac{16}{25}\\ \sin\theta &= \pm\frac{4}{5}\end{aligned}$$

Because θ is in the first quadrant, we know that sine will be positive. $\sin\theta = \frac{4}{5}$

Example B

Find $\tan\theta$ of θ from Example A.

Solution: Use the Tangent Identity to find $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

Example C

Find the other three trigonometric functions of θ from Example.

Solution: To find secant, cosecant, and cotangent use the Reciprocal Identities.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4} \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Guided Practice

Find the values of the other five trigonometric functions.

1. $\tan\theta = -\frac{5}{12}, \frac{\pi}{2} < \theta < \pi$

2. $\csc\theta = -8, \pi < \theta < \frac{3\pi}{2}$

Answers

1. First, we know that θ is in the second quadrant, making sine positive and cosine negative. For this problem, we will use the Pythagorean Identity $1 + \tan^2\theta = \sec^2\theta$ to find secant.

$$1+\left(-\frac{5}{12}\right)^2=\sec^2\theta$$
$$1+\frac{25}{144}=\sec^2\theta$$
$$\frac{169}{144}=\sec^2\theta$$
$$\pm\frac{13}{12}=\sec\theta$$
$$-\frac{13}{12}=\sec\theta$$

If $\sec\theta=-\frac{13}{12}$, then $\cos\theta=-\frac{12}{13}$. $\sin\theta=\frac{5}{13}$ because the numerator value of tangent is the sine and it has the same denominator value as cosine. $\csc\theta=\frac{13}{5}$ and $\cot\theta=-\frac{12}{5}$ from the Reciprocal Identities.

2. θ is in the third quadrant, so both sine and cosine are negative. The reciprocal of $\csc\theta=-8$, will give us $\sin\theta=-\frac{1}{8}$. Now, use the Pythagorean Identity $\sin^2\theta+\cos^2\theta=1$ to find cosine.

$$\left(-\frac{1}{8}\right)^2+\cos^2\theta=1$$
$$\cos^2\theta=1-\frac{1}{64}$$
$$\cos^2\theta=\frac{63}{64}$$
$$\cos\theta=\pm\frac{3\sqrt{7}}{8}$$
$$\cos\theta=-\frac{3\sqrt{7}}{8}$$

$\sec\theta=-\frac{8}{3\sqrt{7}}=-\frac{8\sqrt{7}}{21}, \tan\theta=\frac{1}{3\sqrt{7}}=\frac{\sqrt{7}}{21}$, and $\cot\theta=3\sqrt{7}$

Problem Set

1. In which quadrants is the sine value positive? Negative?
2. In which quadrants is the cosine value positive? Negative?
3. In which quadrants is the tangent value positive? Negative?

Find the values of the other five trigonometric functions of θ.

4. $\sin\theta=\frac{8}{17}, 0<\theta<\frac{\pi}{2}$
5. $\cos\theta=-\frac{5}{6}, \frac{\pi}{2}<\theta<\pi$
6. $\tan\theta=\frac{\sqrt{3}}{4}, 0<\theta<\frac{\pi}{2}$
7. $\sec\theta=-\frac{41}{9}, \pi<\theta<\frac{3\pi}{2}$
8. $\sin\theta=-\frac{11}{14}, \frac{3\pi}{2}<\theta<2\pi$
9. $\cos\theta=\frac{2\sqrt{2}}{4}, 0<\theta<\frac{\pi}{2}$
10. $\cot\theta=-\sqrt{5}, \pi<\theta<\frac{3\pi}{2}$
11. $\csc\theta=4, \frac{\pi}{2}<\theta<\pi$

12. $\tan\theta = -\frac{7}{10}, \frac{3\pi}{2} < \theta < 2\pi$
13. Aside from using the identities, how else can you find the values of the other five trigonometric functions?
14. Given that $\cos\theta = \frac{6}{11}$ and θ is in the 2^{nd} quadrant, what is $\sin(-\theta)$?
15. Given that $\tan\theta = -\frac{5}{8}$ and θ is in the 4^{th} quadrant, what is $\sec(-\theta)$?

Simplifying Trigonometric Expressions

Objective

Here you'll use the basic trig identities to simplify more complicated expressions.

Guidance

Now that you are more familiar with trig identities, we can use them to simplify expressions. Remember, that you can use any of the identities in the *Introduction to Trig Identities* concept. Here is a list of the identities again:

Reciprocal Identities: $\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

Tangent and Cotangent Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1, 1+\tan^2\theta = \sec^2\theta$, and $1+\cot^2\theta = \csc^2\theta$

Cofunction Identities: $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta, \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, and $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

Negative Angle Identities: $\sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta$, and $\tan(-\theta) = -\tan\theta$

Example A

Simplify $\frac{\sec x}{\sec x - 1}$.

Solution: When simplifying trigonometric expressions, one approach is to change everything into sine or cosine. First, we can change secant to cosine using the Reciprocal Identity.

$$\frac{\sec x}{\sec x - 1} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - 1}$$

Now, combine the denominator into one fraction by multiplying 1 by $\frac{\cos x}{\cos x}$.

$$\frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - 1} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} - \frac{\cos x}{\cos x}} \rightarrow \frac{\frac{1}{\cos x}}{\frac{1-\cos x}{\cos x}}$$

Change this problem into a division problem and simplify.

$$\begin{aligned}\frac{\frac{1}{\cos x}}{\frac{1-\cos x}{\cos x}} \rightarrow &\frac{1}{\cos x} \div \frac{1-\cos x}{\cos x}\\ &\frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1-\cos x}\\ &\frac{1}{1-\cos x}\end{aligned}$$

Example B

Simplify $\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x}$.

Solution: With this problem, we need to factor the numerator and denominator and see if anything cancels. The rules of factoring a quadratic and the special quadratic formulas can be used in this scenario.

$$\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} \rightarrow \frac{\cancel{(\sin^2 x - \cos^2 x)}\left(\sin^2 x + \cos^2 x\right)}{\cancel{(\sin^2 x - \cos^2 x)}} \rightarrow \sin^2 x + \cos^2 x \rightarrow 1$$

In the last step, we simplified to the left hand side of the Pythagorean Identity. Therefore, this expression simplifies to 1.

Example C

Simplify $\sec\theta\tan^2\theta + \sec\theta$.

Solution: First, pull out the GCF.

$$\sec\theta\tan^2\theta + \sec\theta \rightarrow \sec\theta(\tan^2\theta + 1)$$

Now, $\tan^2\theta + 1 = \sec^2\theta$ from the Pythagorean Identities, so simplify further.

$$\sec\theta(\tan^2\theta + 1) \rightarrow \sec\theta \cdot \sec^2\theta \rightarrow \sec^3\theta$$

Guided Practice

Simplify the following trigonometric expressions.

1. $\cos\left(\frac{\pi}{2} - x\right)\cot x$

2. $\frac{\sin(-x)\cos x}{\tan x}$

3. $\frac{\cot x \cos x}{\tan(-x)\sin\left(\frac{\pi}{2} - x\right)}$

Answers

1. Use the Cotangent Identity and the Cofunction Identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

$\cos\left(\frac{\pi}{2} - x\right)\cot x \rightarrow \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} \rightarrow \cos x$

2. Use the Negative Angle Identity and the Tangent Identity.

$\frac{\sin(-x)\cos x}{\tan x} \rightarrow \frac{-\sin x \cos x}{\frac{\sin x}{\cos x}} \rightarrow -\cancel{\sin x}\cos x \cdot \frac{\cos x}{\cancel{\sin x}} \rightarrow -\cos^2 x$

3. In this problem, you will use several identities.

$\frac{\cot x \cos x}{\tan(-x)\sin\left(\frac{\pi}{2} - x\right)} \rightarrow \frac{\frac{\cos x}{\sin x} \cdot \cos x}{-\frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x}} \rightarrow \frac{\frac{\cos^2 x}{\sin x}}{-\sin x} \rightarrow \frac{\cos^2 x}{\sin x} \cdot -\frac{1}{\sin x} \rightarrow -\frac{\cos^2 x}{\sin^2 x} \rightarrow -\cot^2 x$

Problem Set

Simplify the following expressions.

1. $\cot x \sin x$
2. $\cos^2 x \tan(-x)$
3. $\frac{\cos(-x)}{\sin(-x)}$
4. $\sec x \cos(-x) - \sin^2 x$
5. $\sin x(1 + \cot^2 x)$
6. $1 - \sin^2\left(\frac{\pi}{2} - x\right)$

7. $1-\cos^2\left(\frac{\pi}{2}-x\right)$
8. $\frac{\tan\left(\frac{\pi}{2}-x\right)\sec x}{1-\csc^2 x}$
9. $\frac{\cos^2 x\tan^2(-x)-1}{\cos^2 x}$
10. $\cot^2 x+\sin^2 x+\cos^2(-x)$
11. $\frac{\sec x\sin x+\cos\left(\frac{\pi}{2}-x\right)}{1+\sin(-x)}$
12. $\frac{\cos(-x)}{1+\sin(-x)}$
13. $\frac{\sin^2(-x)}{\tan^2 x}$
14. $\tan\left(\frac{\pi}{2}-x\right)\cot x-\csc^2 x$
15. $\frac{\csc x\left(1-\cos^2 x\right)}{\sin x\cos x}$

Verifying a Trigonometry Identity

Objective

Here you'll use the basic trig identities to prove other identities.

Guidance

This concept continues where the previous one left off. Now that you are comfortable simplifying expressions, we will extend the idea to verifying entire identities. Here are a few helpful hints to verify an identity:

- Change everything into terms of sine and cosine.
- Use the identities when you can.
- Start with simplifying the left-hand side of the equation, then, once you get stuck, simplify the right-hand side. As long as the two sides end up with the same final expression, the identity is true.

Example A

Verify the identity $\frac{\cot^2 x}{\csc x}=\csc x-\sin x$.

Solution: Rather than have an equal sign between the two sides of the equation, we will draw a vertical line so that it is easier to see what we do to each side of the equation. Start with changing everything into sine and cosine.

$$\begin{array}{c|l} \frac{\cot^2 x}{\csc x} & \csc x-\sin x \\ \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin x}} & \frac{1}{\sin x}-\sin x \\ \frac{\cos^2 x}{\sin x} & \end{array}$$

Now, it looks like we are at an impasse with the left-hand side. Let's combine the right-hand side by giving them same denominator.

$$\begin{array}{|c} \frac{1}{\sin x}-\frac{\sin^2 x}{\sin x} \\ \frac{1-\sin^2 x}{\sin x} \\ \frac{\cos^2 x}{\sin x} \end{array}$$

The two sides reduce to the same expression, so we can conclude this is a valid identity. In the last step, we used the Pythagorean Identity, $\sin^2\theta+\cos^2\theta=1$, and isolated the $\cos^2 x=1-\sin^2 x$.

There are usually more than one way to verify a trig identity. When proving this identity in the first step, rather than changing the cotangent to $\frac{\cos^2 x}{\sin^2 x}$, we could have also substituted the identity $\cot^2 x=\csc^2 x-1$.

Example B

Verify the identity $\frac{\sin x}{1-\cos x}=\frac{1+\cos x}{\sin x}$.

Solution: Multiply the left-hand side of the equation by $\frac{1+\cos x}{1+\cos x}$.

$$\begin{aligned}
\frac{\sin x}{1-\cos x} &= \frac{1+\cos x}{\sin x} \\
\frac{1+\cos x}{1+\cos x} \cdot \frac{\sin x}{1-\cos x} &= \\
\frac{\sin(1+\cos x)}{1-\cos^2 x} &= \\
\frac{\sin(1+\cos x)}{\sin^2 x} &= \\
\frac{1+\cos x}{\sin x} &=
\end{aligned}$$

The two sides are the same, so we are done.

Example C

Verify the identity $\sec(-x) = \sec x$.

Solution: Change secant to cosine.

$$\sec(-x) = \frac{1}{\cos(-x)}$$

From the Negative Angle Identities, we know that $\cos(-x) = \cos x$.

$$\begin{aligned}
&= \frac{1}{\cos x} \\
&= \sec x
\end{aligned}$$

Guided Practice

Verify the following identities.

1. $\cos x \sec x = 1$

2. $2 - \sec^2 x = 1 - \tan^2 x$

3. $\frac{\cos(-x)}{1+\sin(-x)} = \sec x + \tan x$

Answers

1. Change secant to cosine.

$$\begin{aligned}
\cos x \sec x &= \cos \cdot \frac{1}{\cos x} \\
&= 1
\end{aligned}$$

2. Use the identity $1 + \tan^2 \theta = \sec^2 \theta$.

$$\begin{aligned}
2 - \sec^2 x &= 2 - (1 + \tan^2 x) \\
&= 2 - 1 - \tan^2 x \\
&= 1 - \tan^2 x
\end{aligned}$$

3. Here, start with the Negative Angle Identities and multiply the top and bottom by $\frac{1+\sin x}{1+\sin x}$ to make the denominator a monomial.

$$\begin{aligned}
\frac{\cos(-x)}{1+\sin(-x)} &= \frac{\cos x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\\
&= \frac{\cos x(1+\sin x)}{1-\sin^2 x}\\
&= \frac{\cos x(1+\sin x)}{\cos^2 x}\\
&= \frac{1+\sin x}{\cos x}\\
&= \frac{1}{\cos x}+\frac{\sin x}{\cos x}\\
&= \sec x+\tan x
\end{aligned}$$

Problem Set

Verify the following identities.

1. $\cot(-x) = -\cot x$
2. $\csc(-x) = -\csc x$
3. $\tan x \csc x \cos x = 1$
4. $\sin x + \cos x \cot x = \csc x$
5. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$
6. $\tan\left(\frac{\pi}{2} - x\right) = \tan x$
7. $\frac{\csc x}{\sin x} - \frac{\cot x}{\tan x} = 1$
8. $\frac{\tan^2 x}{\tan^2 x+1} = \sin^2 x$
9. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$
10. $\sin x - \sin x \cos^2 x = \sin^3 x$
11. $\tan^2 x + 1 + \tan x \sec x = \frac{1+\sin x}{\cos^2 x}$
12. $\cos^2 x = \frac{\csc x \cos x}{\tan x + \cot x}$
13. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x \sec x$
14. $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$
15. $(\sin x - \tan x)(\cos x - \cot x) = (\sin x - 1)(\cos x - 1)$

4.21 Sum and Difference Formulas

Objective

To use and derive the sum and difference formulas.

Review Queue

Using your calculator, find the value of each trig function below. Round your answer to 4 decimal places.

1. $\sin 75^\circ$

2. $\cos 15^\circ$

3. $\tan 105^\circ$

4. $\cos 255^\circ$

Finding Exact Trig Values using Sum and Difference Formulas

Objective

Here you'll use the sum and difference formulas to find exact values of angles other than the critical angles.

Guidance

You know that $\sin 30^\circ = \frac{1}{2}, \cos 135^\circ = -\frac{\sqrt{2}}{2}, \tan 300^\circ = -\sqrt{3}$, etc... from the special right triangles. In this concept, we will learn how to find the exact values of the trig functions for angles other than these multiples of $30^\circ, 45^\circ$, and 60°. Using the Sum and Difference Formulas, we can find these exact trig values.

Sum and Difference Formulas

$$\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}
\end{aligned}$$

Example A

Find the exact value of $\sin 75^\circ$.

Solution: This is an example of where we can use the sine sum formula from above, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, where $a = 45^\circ$ and $b = 30^\circ$.

$$\begin{aligned}
\sin 75^\circ &= \sin(45^\circ + 30^\circ) \\
&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

In general, $\sin(a+b) \neq \sin a + \sin b$ and similar statements can be made for the other sum and difference formulas.

Example B

Find the exact value of $\cos \frac{11\pi}{12}$.

Solution: For this example, we could use either the sum or difference cosine formula, $\frac{11\pi}{12} = \frac{2\pi}{3} + \frac{\pi}{4}$ or $\frac{11\pi}{12} = \frac{7\pi}{6} - \frac{\pi}{4}$. Let's use the sum formula.

$$\begin{aligned}
\cos \frac{11\pi}{12} &= \cos\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) \\
&= \cos \frac{2\pi}{3} \cos \frac{\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{\pi}{4} \\
&= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
&= -\frac{\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

Example C

Find the exact value of $\tan\left(-\frac{\pi}{12}\right)$.

Solution: This angle is the difference between $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

$$\begin{aligned}
\tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) &= \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{3}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{3}} \\
&= \frac{1 - \sqrt{3}}{1 + \sqrt{3}}
\end{aligned}$$

This angle is also the same as $\frac{23\pi}{12}$. You could have also used this value and done $\tan\left(\frac{\pi}{4} + \frac{5\pi}{3}\right)$ and arrived at the same answer.

Guided Practice

Find the exact values of:

1. $\cos 15^\circ$

2. $\tan 255^\circ$

Answers

1.

$$\begin{aligned}
\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

2.

$$\begin{aligned}
\tan(210^\circ + 45^\circ) &= \frac{\tan 210^\circ + \tan 45^\circ}{1 - \tan 210^\circ \tan 45^\circ} \\
&= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}} = \frac{\sqrt{3} + 3}{3 - \sqrt{3}}
\end{aligned}$$

Vocabulary

Sum and Difference Formulas

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Problem Set

Find the exact value of the following trig functions.

1. $\sin 15^\circ$
2. $\cos \frac{5\pi}{12}$
3. $\tan 345^\circ$
4. $\cos(-255^\circ)$
5. $\sin \frac{13\pi}{12}$
6. $\sin \frac{17\pi}{12}$
7. $\cos 15^\circ$
8. $\tan(-15^\circ)$
9. $\sin 345^\circ$
10. Now, use $\sin 15^\circ$ from #1, and find $\sin 345^\circ$. Do you arrive at the same answer? Why or why not?
11. Using $\cos 15^\circ$ from #7, find $\cos 165^\circ$. What is another way you could find $\cos 165^\circ$?
12. Describe any patterns you see between the sine, cosine, and tangent of these "new" angles.
13. Using your calculator, find the $\sin 142^\circ$. Now, use the sum formula and your calculator to find the $\sin 142^\circ$ using 83° and 59°.
14. Use the sine difference formula to find $\sin 142^\circ$ with any two angles you choose. Do you arrive at the same answer? Why or why not?
15. **Challenge** Using $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$, show that $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$.

Simplifying Trig Expressions using Sum and Difference Formulas

Objective

Here you'll use the sum and difference formulas to simplify expressions.

Guidance

We can also use the sum and difference formulas to simplify trigonometric expressions.

Example A

The $\sin a = -\frac{3}{5}$ and $\cos b = \frac{12}{13}$. a is in the 3^{rd} quadrant and b is in the 1^{st}. Find $\sin(a+b)$.

Solution: First, we need to find $\cos a$ and $\sin b$. Using the Pythagorean Theorem, missing lengths are 4 and 5, respectively. So, $\cos a = -\frac{4}{5}$ because it is in the 3^{rd} quadrant and $\sin b = \frac{5}{13}$. Now, use the appropriate formulas.

$$\begin{aligned}\sin(a+b) &= \sin a\cos b+\cos a\sin b\\ &= -\frac{3}{5}\cdot\frac{12}{13}+-\frac{4}{5}\cdot\frac{5}{13}\\ &= -\frac{56}{65}\end{aligned}$$

Example B

Using the information from Example A, find $\tan(a+b)$.

Solution: From the cosine and sine of a and b, we know that $\tan a=\frac{3}{4}$ and $\tan b=\frac{5}{12}$.

$$\begin{aligned}\tan(a+b) &= \frac{\tan a+\tan b}{1-\tan a\tan b}\\ &= \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}\\ &= \frac{\frac{14}{12}}{\frac{11}{16}}=\frac{56}{33}\end{aligned}$$

Example C

Simplify $\cos(\pi-x)$.

Solution: Expand this using the difference formula and then simplify.

$$\begin{aligned}\cos(\pi-x) &= \cos\pi\cos x+\sin\pi\sin x\\ &= -1\cdot\cos x+0\cdot\sin x\\ &= -\cos x\end{aligned}$$

Guided Practice

1. Using the information from Example A, find $\cos(a-b)$.

2. Simplify $\tan(x+\pi)$.

Answers

1.

$$\begin{aligned}\cos(a-b) &= \cos a\cos b+\sin a\sin b\\ &= -\frac{4}{5}\cdot\frac{12}{13}+-\frac{3}{5}\cdot\frac{5}{13}\\ &= -\frac{63}{65}\end{aligned}$$

2.

$$\begin{aligned}\tan(x+\pi) &= \frac{\tan x+\tan\pi}{1-\tan x\tan\pi}\\ &= \frac{\tan x+0}{1-\tan 0}\\ &= \tan x\end{aligned}$$

Problem Set

$\sin a = -\frac{8}{17}, \pi \le a < \frac{3\pi}{2}$ and $\sin b = -\frac{1}{2}, \frac{3\pi}{2} \le b < 2\pi$. Find the exact trig values of:

1. $\sin(a+b)$
2. $\cos(a+b)$
3. $\sin(a-b)$
4. $\tan(a+b)$
5. $\cos(a-b)$
6. $\tan(a-b)$

Simplify the following expressions.

7. $\sin(2\pi - x)$
8. $\sin\left(\frac{\pi}{2} + x\right)$
9. $\cos(x+\pi)$
10. $\cos\left(\frac{3\pi}{2} - x\right)$
11. $\tan(x+2\pi)$
12. $\tan(x-\pi)$
13. $\sin\left(\frac{\pi}{6} - x\right)$
14. $\tan\left(\frac{\pi}{4} + x\right)$
15. $\cos\left(x - \frac{\pi}{3}\right)$

Determine if the following trig statements are true or false.

16. $\sin(\pi - x) = \sin(x - \pi)$
17. $\cos(\pi - x) = \cos(x - \pi)$
18. $\tan(\pi - x) = \tan(x - \pi)$

Solving Trig Equations using Sum and Difference Formulas

Objective

Here you'll solve trig equations using the sum and difference formulas.

Guidance

Lastly, we can use the sum and difference formulas to solve trigonometric equations. For this concept, we will only find solutions in the interval $0 \le x < 2\pi$.

Example A

Solve $\cos(x-\pi) = \frac{\sqrt{2}}{2}$.

Solution: Use the formula to simplify the left-hand side and then solve for x.

$$\begin{aligned}
\cos(x-\pi) &= \frac{\sqrt{2}}{2} \\
\cos x \cos \pi + \sin x \sin \pi &= \frac{\sqrt{2}}{2} \\
-\cos x &= \frac{\sqrt{2}}{2} \\
\cos x &= -\frac{\sqrt{2}}{2}
\end{aligned}$$

The cosine negative in the 2^{nd} and 3^{rd} quadrants. $x = \frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

Example B

Solve $\sin\left(x+\frac{\pi}{4}\right)+1 = \sin\left(\frac{\pi}{4}-x\right)$.

Solution:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{4}\right)+1 &= \sin\left(\frac{\pi}{4}-x\right) \\
\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} + 1 &= \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x \\
\sin x \cdot \frac{\sqrt{2}}{2} + \cos x \cdot \frac{\sqrt{2}}{2} + 1 &= \frac{\sqrt{2}}{2} . \cos x - \frac{\sqrt{2}}{2} \cdot \sin x \\
\sqrt{2}\sin x &= -1 \\
\sin x &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\end{aligned}$$

In the interval, $x = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Example C

Solve $2\sin\left(x+\frac{\pi}{3}\right) = \tan\frac{\pi}{3}$.

Solution:

$$\begin{aligned}
2\sin\left(x+\frac{\pi}{3}\right) &= \tan\frac{\pi}{3} \\
2\left(\sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}\right) &= \sqrt{3} \\
2\sin x \cdot \frac{1}{2} + 2\cos x \cdot \frac{\sqrt{3}}{2} &= \sqrt{3} \\
\sin x + \sqrt{3}\cos x &= \sqrt{3} \\
\sin x &= \sqrt{3}(1-\cos x) \\
\sin^2 x &= 3(1-2\cos x+\cos^2 x) && \text{square both sides} \\
1-\cos^2 x &= 3-6\cos x+3\cos^2 x && \text{substitute } \sin^2 x = 1-\cos^2 x \\
0 &= 4\cos^2 x - 6\cos x + 2 \\
0 &= 2\cos^2 x - 3\cos x + 1
\end{aligned}$$

At this point, we can factor the equation to be $(2\cos x - 1)(\cos x - 1) = 0$. $\cos x = \frac{1}{2}$, and 1, so $x = 0, \frac{\pi}{3}, \frac{5\pi}{3}$. Be careful with these answers. When we check these solutions it turns out that $\frac{5\pi}{3}$ does not work.

$$\begin{aligned} 2\sin\left(\frac{5\pi}{3}+\frac{\pi}{3}\right) &= \tan\frac{\pi}{3} \\ 2\sin 2\pi &= \sqrt{3} \\ 0 &\neq \sqrt{3} \end{aligned}$$

Therefore, $\frac{5\pi}{3}$ is an extraneous solution.

Guided Practice

Solve the following equations in the interval $0 \leq x < 2\pi$.

1. $\cos(2\pi - x) = \frac{1}{2}$
2. $\sin\left(\frac{\pi}{6} - x\right) + 1 = \sin\left(x + \frac{\pi}{6}\right)$
3. $\cos\left(\frac{\pi}{2} + x\right) = \tan\frac{\pi}{4}$

Answers

1.

$$\begin{aligned} \cos(2\pi - x) &= \frac{1}{2} \\ \cos 2\pi \cos x + \sin 2\pi \sin x &= \frac{1}{2} \\ \cos x &= \frac{1}{2} \\ x &= \frac{\pi}{3} \textit{ and } \frac{5\pi}{3} \end{aligned}$$

2.

$$\begin{aligned} \sin\left(\frac{\pi}{6} - x\right) + 1 &= \sin\left(x + \frac{\pi}{6}\right) \\ \sin\frac{\pi}{6}\cos x - \cos\frac{\pi}{6}\sin x + 1 &= \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} \\ \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x + 1 &= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x \\ 1 &= \sqrt{3}\sin x \\ \frac{1}{\sqrt{3}} &= \sin x \\ x &= \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.6155 \textit{ and } 2.5261 \text{ rad} \end{aligned}$$

3.

$$\begin{aligned} \cos\left(\frac{\pi}{2} + x\right) &= \tan\frac{\pi}{4} \\ \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x &= 1 \\ -\sin x &= 1 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2} \end{aligned}$$

Problem Set

Solve the following trig equations in the interval $0 \leq x < 2\pi$.

1. $\sin(x-\pi) = -\frac{\sqrt{2}}{2}$
2. $\cos(2\pi + x) = -1$
3. $\tan\left(x+\frac{\pi}{4}\right) = 1$
4. $\sin\left(\frac{\pi}{2} - x\right) = \frac{1}{2}$
5. $\sin\left(x+\frac{3\pi}{4}\right) + \sin\left(x-\frac{3\pi}{4}\right) = 1$
6. $\sin\left(x+\frac{\pi}{6}\right) = -\sin\left(x-\frac{\pi}{6}\right)$
7. $\cos\left(x+\frac{\pi}{6}\right) = \cos\left(x-\frac{\pi}{6}\right) + 1$
8. $\cos\left(x+\frac{\pi}{3}\right) + \cos\left(x-\frac{\pi}{3}\right) = 1$
9. $\tan(x+\pi) + 2\sin(x+\pi) = 0$
10. $\tan(x+\pi) + \cos\left(x+\frac{\pi}{2}\right) = 0$
11. $\tan\left(x+\frac{\pi}{6}\right) = \tan\left(x+\frac{\pi}{4}\right)$
12. $\sin\left(x-\frac{5\pi}{3}\right) - 2\sin\left(x-\frac{2\pi}{3}\right) = 0$
13. $4\sin(x+\pi) - 2 = 2\cos\left(x+\frac{\pi}{2}\right)$
14. $1 + 2\cos(x-\pi) + \cos x = 0$
15. **Real Life Application** The height, h (in feet), of two people in different seats on a Ferris wheel can be modeled by $h_1 = 50\cos 6t + 46$ and $h_2 = 50\cos 6\left(t-\frac{\pi}{3}\right) + 46$ where t is the time (in minutes). When are the two people at the same height?

4.22 Double and Half Angle Formulas

Objective

You'll learn how to use the double and half angle formulas.

Review Queue

Use your calculator to find the value of the trig functions below. Round your answers to 4 decimal places.

1. $\sin 22.5^\circ$

2. $\tan 157.5$

Find the exact values of the trig expressions below.

3. $\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$

4. $\sin\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)$

Finding Exact Trig Values using Double and Half Angle Formulas

Objective

Here you'll use the half and double angle formulas to find exact values of angles other than the critical angles.

Guidance

In the previous concept, we added two different angles together to find the exact values of trig functions. In this concept, we will learn how to find the exact values of the trig functions for angles that are half or double of other angles. Here we will introduce the Double-Angle $(2a)$ and Half-Angle $\left(\frac{a}{2}\right)$ Formulas.

Double-Angle and Half-Angle Formulas

$$\begin{aligned} \cos 2a &= \cos^2 a - \sin^2 a \\ &= 2\cos^2 a - 1 \\ &= 1 - \sin^2 a \\ \sin\frac{a}{2} &= \pm\sqrt{\frac{1-\cos a}{2}} \\ \cos\frac{a}{2} &= \pm\sqrt{\frac{1+\cos a}{2}} \end{aligned}$$

$$\begin{aligned} \sin 2a &= 2\sin a\cos a \\ \tan 2a &= \frac{2\tan a}{1-\tan^2 a} \\ \tan\frac{a}{2} &= \frac{1-\cos a}{\sin a} \\ &= \frac{\sin a}{1+\cos a} \end{aligned}$$

The signs of $\sin\frac{a}{2}$ and $\cos\frac{a}{2}$ depend on which quadrant $\frac{a}{2}$ lies in. For $\cos 2a$ and $\tan\frac{a}{2}$ any formula can be used to solve for the exact value.

Example A

Find the exact value of $\cos\frac{\pi}{8}$.

Solution: $\frac{\pi}{8}$ is half of $\frac{\pi}{4}$ and in the first quadrant.

$$\begin{aligned}\cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) &= \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}\\ &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}\\ &= \sqrt{\frac{1}{2}\cdot\frac{2+\sqrt{2}}{2}}\\ &= \frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

Example B

Find the exact value of $\sin 2a$ if $\cos a = -\frac{4}{5}$ and $\frac{3\pi}{2} \le a < 2\pi$.

Solution: To use the sine double-angle formula, we also need to find $\sin a$, which would be $\frac{3}{5}$ because a is in the 4^{th} quadrant.

$$\begin{aligned}\sin 2a &= 2\sin a\cos a\\ &= 2\cdot\frac{3}{5}\cdot -\frac{4}{5}\\ &= -\frac{24}{25}\end{aligned}$$

Example C

Find the exact value of $\tan 2a$ for a from Example B.

Solution: Use $\tan a = \frac{\sin a}{\cos a} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$ to solve for $\tan 2a$.

$$\tan 2a = \frac{2\cdot -\frac{3}{4}}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{3}{2}\cdot\frac{16}{7} = -\frac{24}{7}$$

Guided Practice

1. Find the exact value of $\cos\left(-\frac{5\pi}{8}\right)$.

2. $\cos a = \frac{4}{7}$ and $0 \le a < \frac{\pi}{2}$. Find:

a) $\sin 2a$

b) $\tan\frac{a}{2}$

Answers

1. $-\frac{5\pi}{8}$ is in the 3^{rd} quadrant.

$-\frac{5\pi}{8} = \frac{1}{2}\left(-\frac{5\pi}{4}\right) \rightarrow \cos\frac{1}{2}\left(-\frac{5\pi}{4}\right) = -\sqrt{\frac{1+\cos\left(-\frac{5\pi}{4}\right)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{1}{2}\cdot\frac{2-\sqrt{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

2. First, find $\sin a$. $4^2+y^2=7^2 \rightarrow y=\sqrt{33}$, so $\sin a = \frac{\sqrt{33}}{7}$

a) $\sin 2a = 2\cdot\frac{\sqrt{33}}{7}\cdot\frac{4}{7} = \frac{8\sqrt{33}}{49}$

b) You can use either $\tan \frac{a}{2}$ formula.

$$\tan \frac{a}{2} = \frac{1-\frac{4}{7}}{\frac{\sqrt{33}}{7}} = \frac{3}{7} \cdot \frac{7}{\sqrt{33}} = \frac{3}{\sqrt{33}} = \frac{\sqrt{33}}{11}$$

Vocabulary

Double-Angle and Half-Angle Formulas

$$\begin{aligned} \cos 2a &= \cos^2 a - \sin^2 a \\ &= 2\cos^2 a - 1 \\ &= 1 - \sin^2 a \\ \sin \frac{a}{2} &= \pm \sqrt{\frac{1-\cos a}{2}} \\ \cos \frac{a}{2} &= \pm \sqrt{\frac{1+\cos a}{2}} \end{aligned}$$

$$\begin{aligned} \sin 2a &= 2\sin a \cos a \\ \tan 2a &= \frac{2\tan a}{1-\tan^2 a} \\ \tan \frac{a}{2} &= \frac{1-\cos a}{\sin a} \\ &= \frac{\sin a}{1+\cos a} \end{aligned}$$

Problem Set

Find the exact value of the following angles.

1. $\sin 105^\circ$
2. $\tan \frac{\pi}{8}$
3. $\cos \frac{5\pi}{12}$
4. $\cos 165^\circ$
5. $\sin \frac{3\pi}{8}$
6. $\tan \left(-\frac{\pi}{12}\right)$
7. $\sin \frac{11\pi}{8}$
8. $\cos \frac{19\pi}{12}$

The $\cos a = -\frac{5}{13}$ and $\frac{3\pi}{2} \le a < 2\pi$. Find:

9. $\sin 2a$
10. $\cos \frac{a}{2}$
11. $\tan \frac{a}{2}$
12. $\cos 2a$

The $\sin a = \frac{8}{11}$ and $\frac{\pi}{2} \le a < \pi$. Find:

13. $\tan 2a$
14. $\sin \frac{a}{2}$
15. $\cos \frac{a}{2}$
16. $\sin 2a$

Simplifying Trig Expressions using Double and Half Angle Formulas

Objective

Here you'll use the half and double angle formulas to simplify more complicated expressions.

Guidance

We can also use the double-angle and half-angle formulas to simplify trigonometric expressions.

Example A

Simplify $\frac{\cos 2x}{\sin x \cos x}$.

Solution: Use $\cos 2a = \cos^2 a - \sin^2 a$ and then factor.

$$\begin{aligned}
\frac{\cos 2x}{\sin x \cos x} &= \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x} \\
&= \frac{(\cos x - \sin x)\cancel{(\cos x + \sin x)}}{\cancel{\sin x + \cos x}} \\
&= \cos x - \sin x
\end{aligned}$$

Example B

Find the formula for $\sin 3x$.

Solution: You will need to use the sum formula and the double-angle formula. $\sin 3x = \sin(2x + x)$

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= 2 \sin x \cos x \cos x + \sin x(2\cos^2 x - 1) \\
&= 2 \sin x \cos^2 x + 2 \sin x \cos^2 x - \sin x \\
&= 4 \sin x \cos^2 x - \sin x \\
&= \sin x(4\cos^2 x - 1)
\end{aligned}$$

We will explore other possibilities for the $\sin 3x$ because of the different formulas for $\cos 2a$ in the Problem Set.

Example C

Verify the identity $\cos x + 2\sin^2 \frac{x}{2} = 1$.

Solution: Simplify the left-hand side use the half-angle formula.

$$\begin{aligned}
&\cos x + 2\sin^2 \frac{x}{2} \\
&\cos x + 2\left(\sqrt{\frac{1 - \cos x}{2}}\right)^2 \\
&\cos x + 2 \cdot \frac{1 - \cos x}{2} \\
&\cos x + 1 - \cos x \\
&1
\end{aligned}$$

Guided Practice

1. Simplify $\frac{\sin 2x}{\sin x}$.

2. Verify $\cos x + 2\cos^2 \frac{x}{2} = 1 + 2\cos x$.

Answers

1. $\frac{\sin 2x}{\sin x} = \frac{2\sin x\cos x}{\sin x} = 2\cos x$

2.

$$\begin{aligned} \cos x + 2\cos^2\frac{x}{2} &= 1 + 2\cos x \\ \cos x + 2\sqrt{\frac{1+\cos x}{2}}^2 &= \\ \cos x + 1 + \cos x &= \\ 1 + 2\cos x &= \end{aligned}$$

Problem Set

Simplify the following expressions.

1. $\sqrt{2+2\cos x}\left(\cos\frac{x}{2}\right)$
2. $\frac{\cos 2x}{\cos^2 x}$
3. $\tan 2x(1+\tan x)$
4. $\cos 2x - 3\sin^2 x$
5. $\frac{1+\cos 2x}{\cot x}$
6. $(1+\cos x)^2\tan\frac{x}{2}$

Verify the following identities.

7. $\cot\frac{x}{2} = \frac{\sin x}{1-\cos x}$
8. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
9. $\frac{\sin 2x}{1+\cos 2x} = \tan x$
10. $(\sin x + \cos x)^2 = 1 + \sin 2x$
11. $\sin x\tan\frac{x}{2} + 2\cos x = 2\cos^2\frac{x}{2}$
12. $\cot x + \tan x = 2\csc 2x$
13. $\cos 3x = 4\cos^3 x - 3\cos x$
14. $\cos 3x = \cos^3 x - 3\sin^2 x\cos x$
15. $\sin 2x - \tan x = \tan x\cos 2x$
16. $\cos^4 x - \sin^4 x = \cos 2x$

Solving Trig Equations using Double and Half Angle Formulas

Objective

Here you'll solve trig equations using the half and double angle formulas.

Guidance

Lastly, we can use the half and double angle formulas to solve trigonometric equations.

Example A

Solve $\tan 2x + \tan x = 0$ when $0 \leq x < 2\pi$.

Solution: Change $\tan 2x$ and simplify.

$$\begin{aligned}
\tan 2x + \tan x &= 0 \\
\frac{2\tan x}{1-\tan^2 x} + \tan x &= 0 \\
2\tan x + \tan x(1-\tan^2 x) &= 0 && \rightarrow \text{Multiply everything by } 1-\tan^2 x \text{ to eliminate denominator.} \\
2\tan x + \tan x - \tan^3 x &= 0 \\
3\tan x - \tan^3 x &= 0 \\
\tan x(3-\tan^2 x) &= 0
\end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned}
&& && 3-\tan^2 x &= 0 \\
&& && -\tan^2 x &= -3 \\
\tan x &= 0 && \textit{and} & \tan^2 x &= 3 \\
x &= 0 \textit{ and } \pi && & \tan x &= \pm\sqrt{3} \\
&& && x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}
\end{aligned}$$

Example B

Solve $2\cos\frac{x}{2}+1=0$ when $0 \le x < 2\pi$.

Solution: In this case, you do not have to use the half-angle formula. Solve for $\frac{x}{2}$.

$$\begin{aligned}
2\cos\frac{x}{2}+1 &= 0 \\
2\cos\frac{x}{2} &= -1 \\
\cos\frac{x}{2} &= -\frac{1}{2}
\end{aligned}$$

Now, let's find $\cos a = -\frac{1}{2}$ and then solve for x by dividing by 2.

$$\begin{aligned}
\frac{x}{2} &= \frac{2\pi}{3}, \frac{4\pi}{3} \\
&= \frac{4\pi}{3}, \frac{8\pi}{3}
\end{aligned}$$

Now, the second solution is not in our range, so the only solution is $x = \frac{4\pi}{3}$.

Example C

Solve $4\sin x\cos x = \sqrt{3}$ for $0 \le x < 2\pi$.

Solution: Pull a 2 out of the left-hand side and use the $\sin 2x$ formula.

$$\begin{aligned}
4\sin x\cos x &= \sqrt{3} \\
2\cdot 2\sin x\cos x &= \sqrt{3} \\
2\cdot \sin 2x &= \sqrt{3} \\
\sin 2x &= \frac{\sqrt{3}}{2} \\
2x &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\
x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

Guided Practice

Solve the following equations for $0 \le x < 2\pi$.

1. $\sin \frac{x}{2} = -1$

2. $\cos 2x - \cos x = 0$

Answers

1.

$$\begin{aligned}
\sin \frac{x}{2} &= -1 \\
\frac{x}{2} &= \frac{3\pi}{2} \\
x &= 3\pi
\end{aligned}$$

From this we can see that there are no solutions within our interval.

2.

$$\begin{aligned}
\cos 2x - \cos x &= 0 \\
2\cos^2 x - \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0
\end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned}
2\cos x - 1 &= 0 & & & & \\
2\cos x &= 1 & & & \cos x + 1 &= 0 \\
\cos x &= \frac{1}{2} & & \textit{and} & \cos x &= -1 \\
x &= \frac{\pi}{3}, \frac{5\pi}{3} & & & x &= \pi
\end{aligned}$$

Problem Set

Solve the following equations for $0 \le x < 2\pi$.

1. $\cos x - \cos \frac{1}{2}x = 0$

2. $\sin 2x \cos x = \sin x$
3. $\cos 3x = \cos^3 x = 3\sin^2 x \cos x$
4. $\tan 2x - \tan x = 0$
5. $\cos 2x - \cos x = 0$
6. $2\cos^2 \frac{x}{2} = 1$
7. $\tan \frac{x}{2} = 4$
8. $\cos \frac{x}{2} = 1 + \cos x$
9. $\sin 2x + \sin x = 0$
10. $\cos^2 x = \cos 2x = 0$
11. $\frac{\cos 2x}{\cos^2 x} = 1$
12. $\cos 2x - 1 = \sin^2 x$
13. $\cos 2x = \cos x$
14. $\sin 2x - \cos 2x = 1$
15. $\sin^2 x - 2 = \cos 2x$
16. $\cot x + \tan x = 2\csc 2x$

CHAPTER 5 Applications of Trigonometry

Chapter Outline

5.1 Basic Properties of Vectors

Learning Objectives

Here you will find out what a vector is algebraically and graphically.

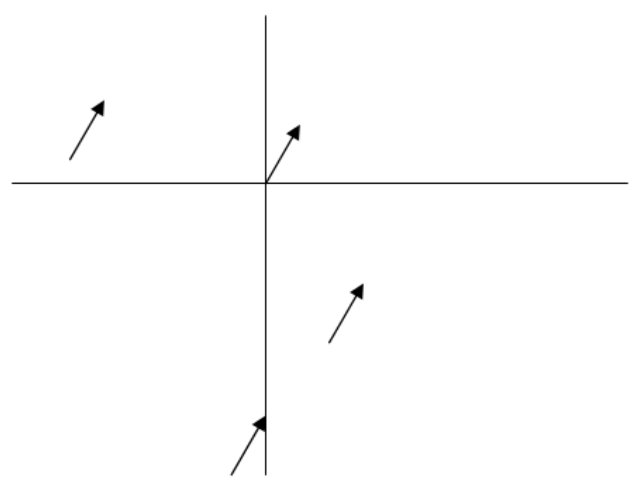
An airplane being pushed off course by wind and a swimmer's movement across a moving river are both examples of vectors in action. Points in the coordinate plane describe location. **Vectors**, on the other hand, have no location and indicate only direction and magnitude. Vectors can describe the strength of forces like gravity or speed and direction of a ship at sea. Vectors are extremely useful in modeling complex situations in the real world.

What are other differences between vectors and points?

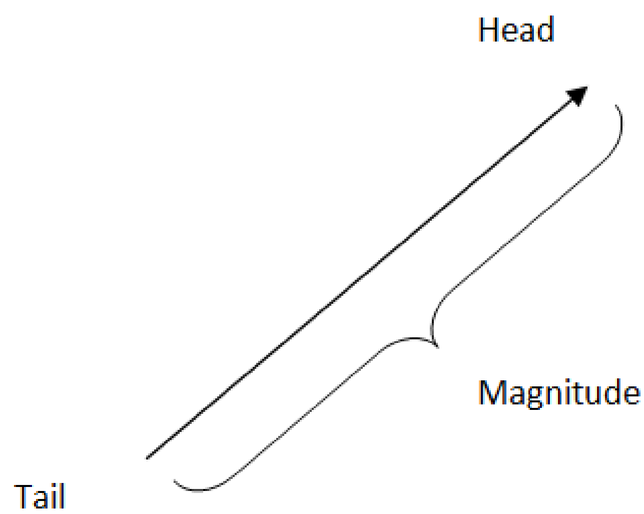
Properties of Vectors

A two dimensional vector is represented graphically as an arrow with a tail and a head. The **head** is the arrow and is also called the **terminal point.** When finding the vector between two points start with the terminal point and subtract the **initial** point (the tail).

The two defining characteristics of a vector are its magnitude and its direction. The magnitude is shown graphically by the length of the arrow and the direction is indicated by the angle that the arrow is pointing. Notice how the following vector is shown multiple times on the same coordinate plane. This emphasizes that the location on the coordinate plane does not matter and is not unique. Each representation of the vector has identical direction and magnitude.



One way to define a vector is as a line segment with a direction. Vectors are said to be equal if they have the same magnitude and the same direction. The absolute value of a vector is the same as the length of the line segment or the **magnitude** of the vector. Magnitude can be found by using the Pythagorean Theorem or the distance formula.

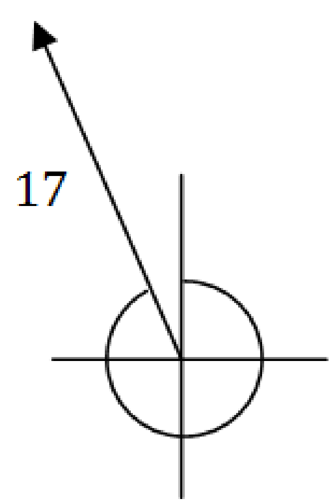


There are a few different ways to write a vector *v*.

$$v, \overrightarrow{v}, \vec{v}, \text{ or } v \text{ with a } \sim \text{ underneath}$$

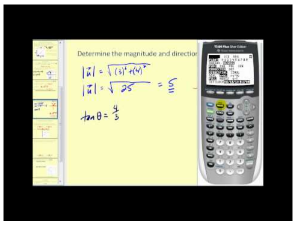
When you write about vectors algebraically there are a few ways to describe a specific vector. First, you could describe its angle and magnitude as r, θ. Second, you could describe it as an ordered pair: $< x, y >$. Notice that when discussing vectors you should use the brackets $<>$ instead of parentheses because it helps avoid confusion between a vector and a point. Vectors can be multidimensional.

Vectors are often used to describe the movement of objects. To describe the movement of a ship is traveling NNW at 17 knots (nautical mph) as a vector, notice that NNW is halfway between NW and N. When describing ships at sea, it is best to use bearing which has 0° as due North and 270° as due West. This makes NW equal to 315° and NNW equal to 337.5°.



When you see this picture, it turns into a basic trig question to find the x and y components of the vector. Note that the reference angle that the vector makes with the negative portion of the x axis is 67.5°.

$$\sin 67.5^\circ = \frac{y}{17}, \cos 67.5^\circ = \frac{x}{17}$$
$$< x,y > \approx < -6.5, 15.7 >$$



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Examples

Example 1

Earlier, you were asked what the differences between points and vectors are. There are many differences between points and vectors. Points are locations and vectors are made up of distance and angles. Parentheses are used for points and $<>$ are used for vectors. One relationship between vectors and points is that a point plus a vector will yield a new point. It is as if there is a starting place and then a vector tells you where to go from that point. Without the starting point, the vector could start from anywhere.

Example 2

Consider the points: $A(1,3), B(-4,-6), C(5,-13)$. Find the vectors in component form of $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{CB}$.

Remember that when finding the vector between two points, start with the terminal point and subtract the initial point.

$$\overrightarrow{AB} = < -5, -9 >$$
$$\overrightarrow{BA} = < 5, 9 >$$
$$\overrightarrow{AC} = < 4, -16 >$$
$$\overrightarrow{CB} = < -9, 7 >$$

Example 3

A father is pulling his daughter up a hill. The hill has a 20° incline. The daughter is on a sled which sits on the ground and has a rope that the father pulls as he walks. The rope makes a 39° angle with the slope. A force diagram is a collection of vectors that each represent a force like gravity or wind acting on an object. Draw a force diagram showing how these forces act on the daughter's center of gravity:

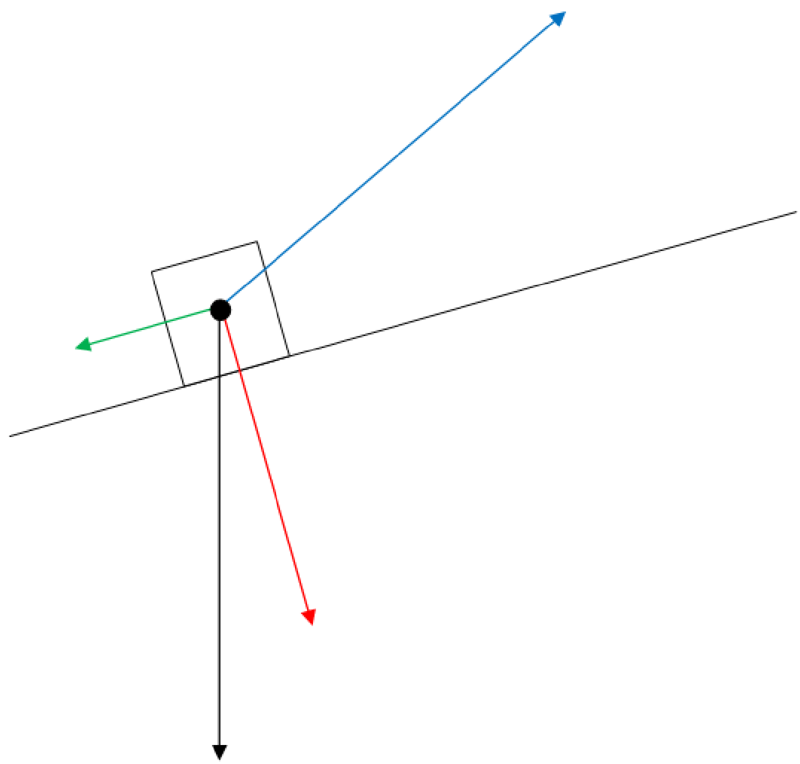
a. The force of gravity.

b. The force holding the daughter in the sled to the ground.

c. The force pulling the daughter backwards down the slope.

d. The force of the father pulling the daughter up the slope.

The girl's center of gravity is represented by the black dot. The force of gravity is the black arrow straight down. The green arrow is gravity's effect pulling the girl down the slope. The red arrow is gravity's effect pulling the girl straight into the slope. The blue arrow represents the force that the father is exerting as he pulls the girl up the hill.

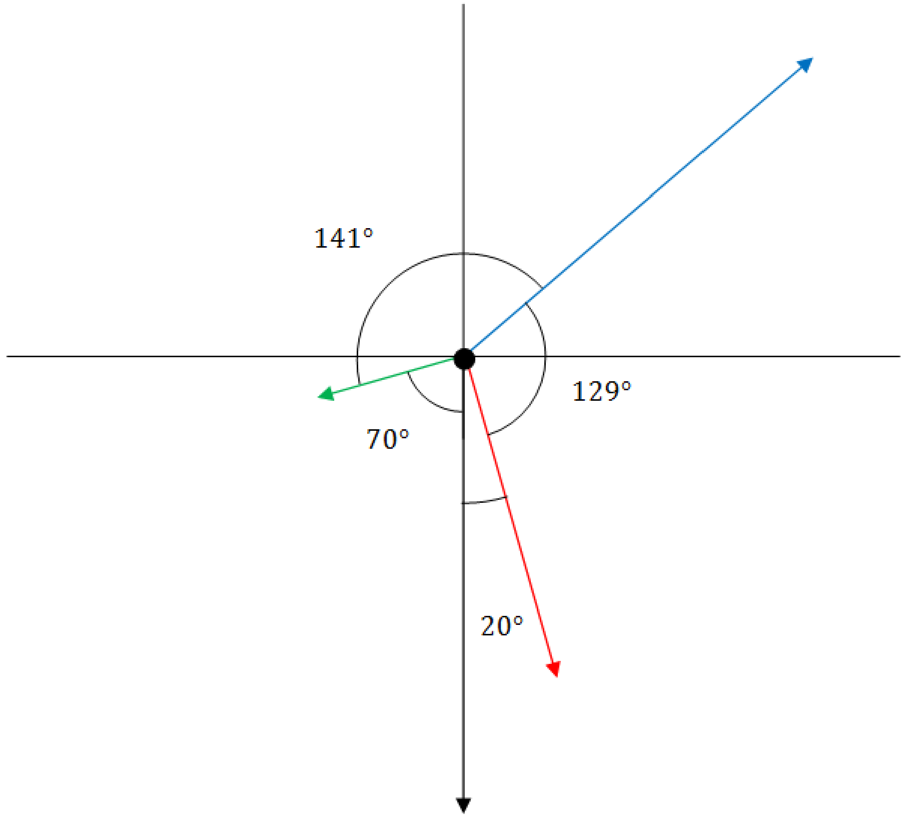


Notice that the father's force vector (blue) is longer than the force pulling the girl down the hill. This means that over time they will make progress and ascend the hill. Also note that the father is wasting some of his energy lifting rather than just pulling. If he could pull at an angle directly opposing the force pulling the girl down the hill, then he would be using all of his energy efficiently.

Example 4

Center the force diagram from the previous question into the origin and identify the angle between each consecutive force vector.

The x and y axis are included as reference and note that the gravity vector overlaps with the negative y axis. In order to find each angle, you must use your knowledge of supplementary, complementary and vertical angles and all the clues from the question. To check, see if all the angles sum to be 360°.



Example 5

Given the following vectors and point, compute the sum.

$A = (1,3), \overrightarrow{v} =< 4,8 >, \overrightarrow{u} =< -1,-5 >$

$A + \overrightarrow{v} + \overrightarrow{u} = ?$

$A = (1,3), \overrightarrow{v} =< 4,8 >, \overrightarrow{u} =< -1,-5 >$. $A + \overrightarrow{v} + \overrightarrow{u} = (4,6)$.

Review

1. Describe what a vector is and give a real-life example of something that a vector could model.

Consider the points: $A(3,5), B(-2,-4), C(1,-12), D(-5,7)$. Find the vectors in component form of:

2. $\overrightarrow{AB}$
3. $\overrightarrow{BA}$
4. $\overrightarrow{AC}$
5. $\overrightarrow{CB}$
6. $\overrightarrow{AD}$
7. $\overrightarrow{DA}$
8. What is $C + \overrightarrow{CB}$? Compute this algebraically and describe why the answer makes sense.
9. Use your answer to the previous problem to help you determine $D + \overrightarrow{DA}$ without doing any algebra.

10. A ship is traveling SSW at 13 knots. Describe this ship's movement in a vector.

11. A vector that describes a ships movement is $< 5\sqrt{2}, 5\sqrt{2} >$. What direction is the ship traveling in and what is its speed in knots?

For each of the following vectors, draw the vector on a coordinate plane starting at the origin and find its magnitude.

12. $< 3,7 >$

13. $< -3,4 >$

14. $< -5,10 >$

15. $< 6,-8 >$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.1.

5.2 Operations with Vectors

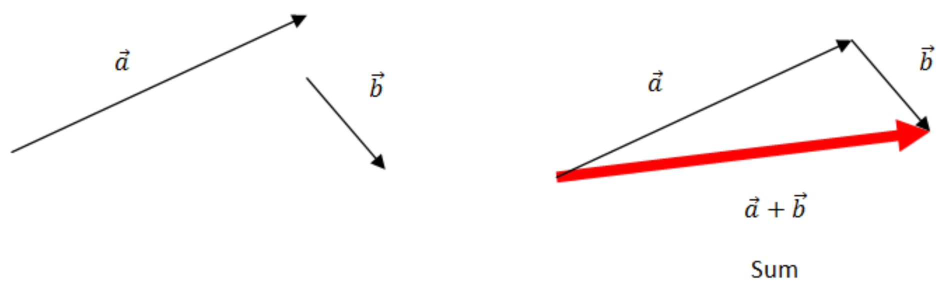
Learning Objectives

Here you will add and subtract vectors with vectors and vectors with points. When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 miles an hour in a headwind of 2 miles an hour only makes headway at a rate of 8 miles per hour. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing NE at 2 miles per hour. How far does the bird get in one hour?

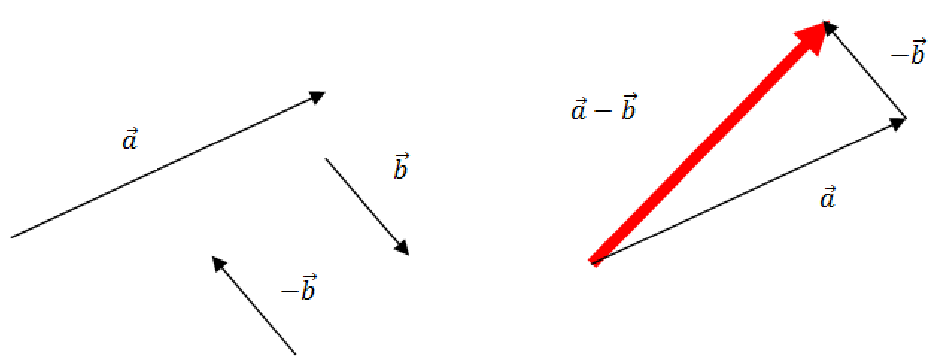
Basic Vector Operations

Scalar multiplication means to multiply a vector by a number. This changes the magnitude of the vector, but not its direction. If $\overrightarrow{v} =< 3,4 >$, then $2\overrightarrow{v} =< 6,8 >$. Scalar multiplication is fairly simple.

Adding and subtracting vectors is slightly more difficult. When adding vectors, place the tail of one vector at the head of the other. This is called the **tail-to-head rule**. The vector that is formed by joining the tail of the first vector with the head of the second is called the **resultant vector.**

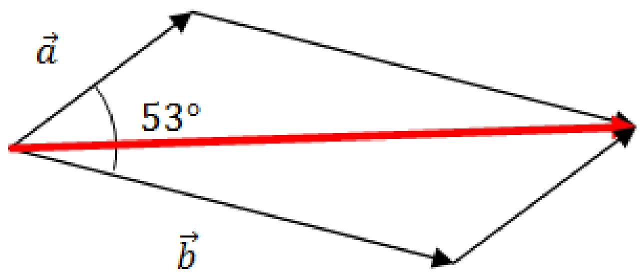


Vector subtraction reverses the direction of the second vector. $\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{a} + (-\overrightarrow{b})$:



Adding vectors can be done in either order (just like with regular numbers). Subtracting vectors must be done in a specific order or else the vector will be negative (just like with regular numbers).

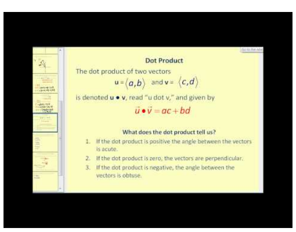
To find the length or magnitude of a resultant vector, you can use the law of cosines. To do this, you also need to know the angle between the two vectors. Say you were given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, have magnitudes of 5 and 9 respectively and that the angle between the vectors is 53°. To find the magnitude of $\vec{a}+\vec{b}$, which is written as $|\overrightarrow{a} + \overrightarrow{b}|$, notice that you have a parallelogram.



In order to fine the magnitude of the resulting vector in red, note that the triangle on the bottom has sides 9 and 5 with included angle 127° due to the properties of parallelograms. And, so applying the Law of Cosines, you get:

$$x^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127^\circ$$
$$x \approx 12.66$$

For this video, focus on scalar multiplication and adding and subtracting vectors:



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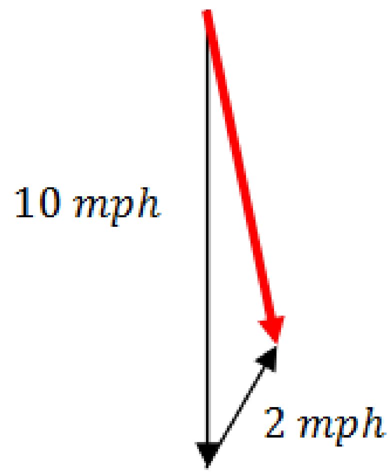
Examples

Example 1

Earlier, you were asked about how fast a bird was flying. A bird flying due south at 10 miles an hour with a cross headwind of 2 mph heading NE would have a force diagram that looks like this:



The angle between the bird's vector and the wind vector is 45° which means this is a perfect situation for the Law of Cosines. Let $x =$ the red vector.

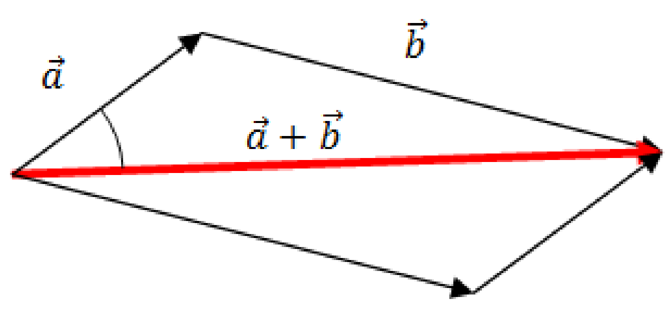
$$x^2 = 10^2 + 2^2 - 2 \cdot 10 \cdot 2 \cdot \cos 45^\circ$$
$$x \approx 8.7$$

The bird is blown slightly off track and travels only about 8.7 miles in one hour.

Example 2

Using the same vectors with magnitude 5 and 9 and angle of 53° from the main section, what is the angle that the sum $\overrightarrow{a} + \overrightarrow{b}$ makes with $\overrightarrow{a}$?

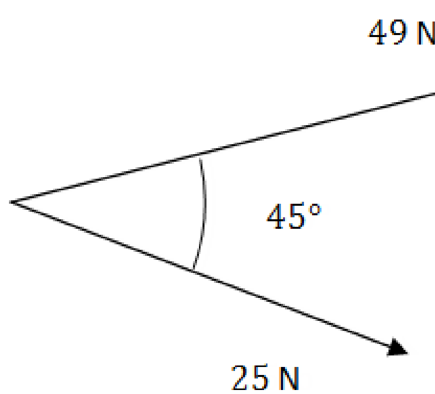
Start by drawing a good picture and labeling what you know. $|\overrightarrow{a}| = 5, |\overrightarrow{b}| = 9, |\overrightarrow{a} + \overrightarrow{b}| \approx 12.66$. Since you know three sides of the triangle and you need to find one angle, this is the SSS application of the Law of Cosines.



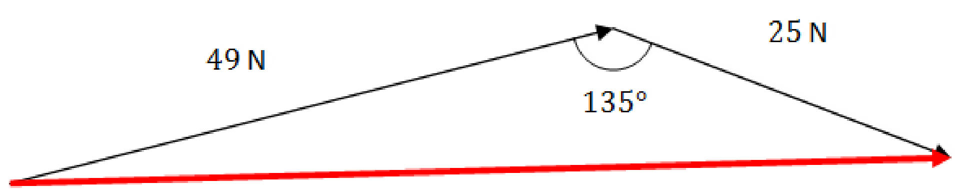
$$9^2 = 12.66^2 + 5^2 - 2 \cdot 12.66 \cdot 5 \cdot \cos \theta$$
$$\theta = 34.6^\circ$$

Example 3

Elaine started a dog walking business. She walks two dogs at a time named Elvis and Ruby. They each pull her in different directions at a 45° angle with different forces. Elvis pulls at a force of 25 N and Ruby pulls at a force of 49 N. How hard does Elaine need to pull so that she can stay balanced? Note that N stands for Newtons which is the standard unit of force.



Even though the two vectors are centered at Elaine, the forces are added which means that you need to use the tail-to-head rule to add the vectors together. Finding the angle between each component vector requires logical use of supplement angles.



$$x^2 = 49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ$$
$$x \approx 68.98\ N$$

In order for Elaine to stay balanced, she will need to counteract this force with an equivalent force of her own in the exact opposite direction.

Example 4

Consider vector $\overrightarrow{v} = < 2, 5 >$ and vector $\overrightarrow{u} = < -1, 9 >$. Determine the component form of the following: $3\overrightarrow{v} - 2\overrightarrow{u}$.

Do multiplication first for each term, followed by vector subtraction.

$$\begin{aligned} 3 \cdot \overrightarrow{v} - 2 \cdot \overrightarrow{u} &= 3 \cdot < 2, 5 > -2 \cdot < -1, 9 > \\ &= < 6, 15 > - < -2, 18 > \\ &= < 8, -3 > \end{aligned}$$

Example 5

An airplane is flying at a bearing of 270° at 400 mph. A wind is blowing due south at 30 mph. Does this cross wind affect the plane's speed?

Since the cross wind is perpendicular to the plane, it pushes the plane south as the plane tries to go directly east. As a result the plane still has an airspeed of 400 mph but the groundspeed (true speed) needs to be calculated.

$$400^2 + 30^2 = x^2$$
$$x \approx 401$$

Review

Consider vector $\overrightarrow{v} =< 1,3 >$ and vector $\overrightarrow{u} =< -2,4 >$.

1. Determine the component form of $5\overrightarrow{v} - 2\overrightarrow{u}$.
2. Determine the component form of $-2\overrightarrow{v} + 4\overrightarrow{u}$.
3. Determine the component form of $6\overrightarrow{v} + \overrightarrow{u}$.
4. Determine the component form of $3\overrightarrow{v} - 6\overrightarrow{u}$.
5. Find the magnitude of the resultant vector from #1.
6. Find the magnitude of the resultant vector from #2.
7. Find the magnitude of the resultant vector from #3.
8. Find the magnitude of the resultant vector from #4.
9. The vector $< 3,4 >$ starts at the origin. What is the direction of the vector?
10. The vector $< -1,2 >$ starts at the origin. What is the direction of the vector?
11. The vector $< 3,-4 >$ starts at the origin. What is the direction of the vector?
12. A bird flies due south at 8 miles an hour with a cross headwind blowing due east at 15 miles per hour. How far does the bird get in one hour?
13. What direction is the bird in the previous problem actually moving?
14. A football is thrown at 50 miles per hour due north. There is a wind blowing due east at 8 miles per hour. What is the actual speed of the football?
15. What direction is the football in the previous problem actually moving?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.2.

5.3 Resolution of Vectors into Components

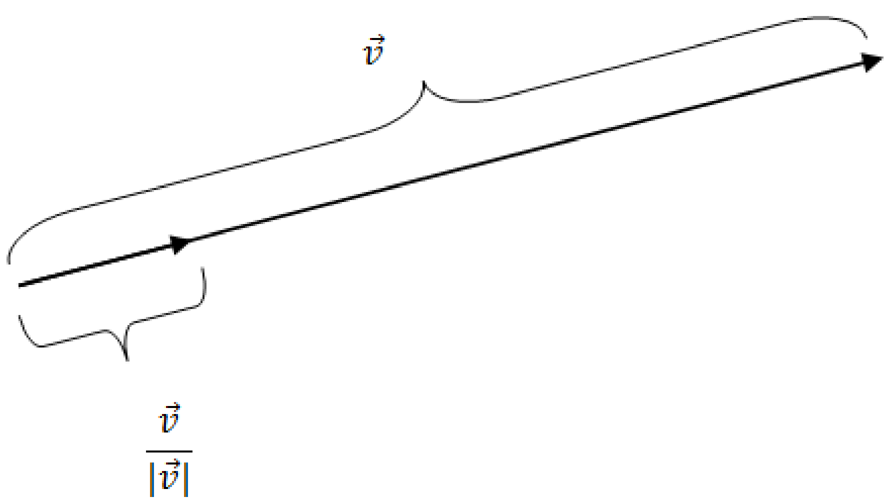
Learning Objectives

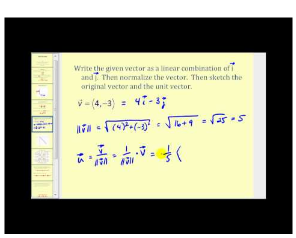
Here you will find unit vectors and you will convert vectors into linear combinations of standard unit vectors and component vectors. Sometimes working with horizontal and vertical components of a vector can be significantly easier than working with just an angle and a magnitude. This is especially true when combining several forces together.

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of 41°. Connie pulls with 10 lb of force at an angle of 100°. Cynthia pulls with 12 lb of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

The Unit Vector and Component Form

A **unit vector** is a vector of length one. Sometimes you might wish to scale a vector you already have so that it has a length of one. If the length was five, you would scale the vector by a factor of $\frac{1}{5}$ so that the resulting vector has magnitude of 1. Another way of saying this is that a **unit vector in the direction of vector** $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$.





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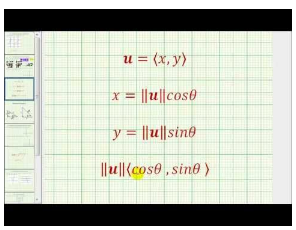
There are two standard unit vectors that make up all other vectors in the coordinate plane. They are $\overrightarrow{i}$ which is the vector $< 1,0 >$ and $\overrightarrow{j}$ which is the vector $< 0,1 >$. These two unit vectors are perpendicular to each other. A **linear combination** of $\overrightarrow{i}$ and $\overrightarrow{j}$ will allow you to uniquely describe any other vector in the coordinate plane in component form. For instance the vector $< 5,3 >$ is the same as $5\overrightarrow{i} + 3\overrightarrow{j}$.

Often vectors are initially described as an angle and a magnitude rather than in component form. Working with vectors written as an angle and magnitude requires extremely precise geometric reasoning and excellent pictures. One advantage of rewriting the vectors in component form is that much of this work is simplified. Remember that component form is the form $< x,y >$ and to translate from magnitude r and direction θ to component form, use the relationship $< r \cdot \cos\theta, r \cdot \sin\theta >=< x,y >$. In this situation r is the magnitude and θ is the direction.

Take a plane that has a bearing of 60° and is going 350 mph. To find the component form of the velocity of the airplane, note that a bearing of 60° is the same as a 30° on the unit circle. This corresponds to the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ which is the same as $(\cos 30, \sin 30)$. When written as a vector $< \frac{\sqrt{3}}{2}, \frac{1}{2} >$ is a unit vector because it has magnitude 1. Now you just need to scale by a factor of 350 and you get your answer of $< 175\sqrt{3}, 175 >$. This is the same as using the relationship $< r \cdot \cos\theta, r \cdot \sin\theta >=< x,y >$.



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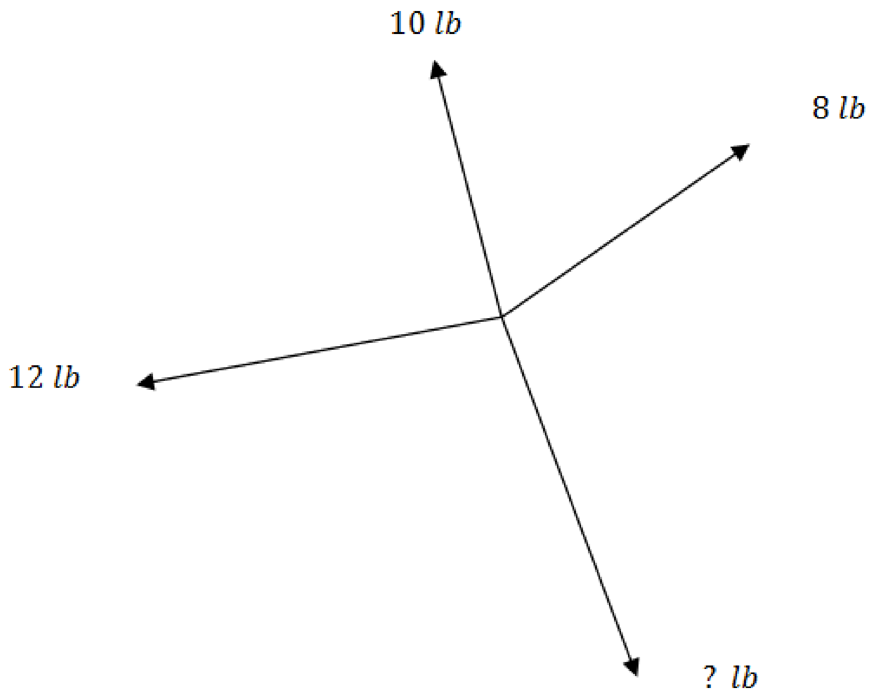
Examples

Example 1

Earlier, you were asked to consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of 41°. Connie pulls with 10 lb of force at an angle of 100°. Cynthia pulls with 12 lb of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?



To add the three vectors together would require several iterations of the Law of Cosines. Instead, write each vector in component form and set equal to a zero vector indicating that the candy does not move.

$$\overrightarrow{L} + \overrightarrow{CON} + \overrightarrow{CYN} + \overrightarrow{T} = < 0,0 >$$

$$< 8 \cdot \cos 41^\circ, 8 \cdot \sin 41^\circ > + < 10 \cdot \cos 100^\circ, 10 \cdot \sin 100^\circ >$$
$$+ < 12 \cdot \cos 200^\circ, 12 \cdot \sin 200^\circ > + \overrightarrow{T} = < 0,0 >$$

Use a calculator to add all the x components and bring them to the far side and the y components and then subtract from the far side to get:

$$\overrightarrow{T} \approx < 6.98, -10.99 >$$

Turning this component vector into an angle and magnitude yields how hard and in what direction he would have to pull. Terry will have to pull with about 13 lb of force at an angle of 302.4°.

Example 2

Consider the plane that has a bearing of60° and is going 350 mph. If there is wind blowing with the bearing of 300° at 45 mph, what is the component form of the total velocity of the airplane?

A bearing of 300° is the same as 150° on the unit circle which corresponds to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. You can now write and then scale the wind vector.

$45 \cdot < -\frac{\sqrt{3}}{2}, \frac{1}{2} > = < -\frac{45\sqrt{3}}{2}, \frac{45}{2} >$

Since both the wind vector and the velocity vector of the airplane are written in component form, you can simply sum them to find the component vector of the total velocity of the airplane.

$< 175\sqrt{3}, 175 > + < -\frac{45\sqrt{3}}{2}, \frac{45}{2} > = < \frac{305\sqrt{3}}{2}, \frac{395}{2} >$

Example 3

Consider the plane from Example 2 with the same wind and velocity. Find the actual ground speed and direction of the plane (as a bearing).

Since you already know the component vector of the total velocity of the airplane, you should remember that these components represent an x distance and a y distance and the question asks for the hypotenuse.

$$\left(\frac{305\sqrt{3}}{2}\right)^2 + \left(\frac{395}{2}\right)^2 = c^2$$
$$329.8 \approx c$$

The airplane is traveling at about 329.8 mph.

Since you know the x and y components, you can use tangent to find the angle. Then convert this angle into bearing.

$$\tan\theta = \frac{\left(\frac{395}{2}\right)}{\left(\frac{305\sqrt{3}}{2}\right)}$$
$$\theta \approx 36.8^\circ$$

An angle of 36.8° on the unit circle is equivalent to a bearing of 53.2°.

Note that you can do the entire problem in bearing by just switching sine and cosine, but it is best to truly understand what you are doing every step of the way and this will probably involve always going back to the unit circle.

For Examples 4 and 5, use the following information:

$\overrightarrow{v} = < 2, -5 >, \overrightarrow{u} = < -3, 2 >, \overrightarrow{t} = < -4, -3 >, \overrightarrow{r} = < 5, y >$

$B = (4, -5), P = (-3, 8)$

Example 4

Find the unit vectors in the same direction as $\overrightarrow{u}$ and $\overrightarrow{t}$.

To find a unit vector, divide each vector by its magnitude.

$\frac{\overrightarrow{u}}{|\overrightarrow{u}|} = < \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} >, \frac{\overrightarrow{t}}{|\overrightarrow{t}|} = < \frac{-4}{5}, \frac{-3}{5} >$

Example 5

Find the point 10 units away from B in the direction of P.

The vector $\overrightarrow{BP}$ is $< -7, 13 >$. First take the unit vector and then scale it so it has a magnitude of 10.

$$\frac{BP}{|BP|} =< \frac{-7}{\sqrt{218}}, \frac{13}{\sqrt{218}} >$$
$$10 \cdot \frac{BP}{|BP|} =< \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} >$$

You end up with a vector that is ten units long in the right direction. The question asked for a point from B which means you need to add this vector to point B.

$(4,-5)+< \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} > \approx (-0.74, 3.8)$

Review

$\overrightarrow{v} =< 1,-3 >, \overrightarrow{u} =< 2,5 >, \overrightarrow{t} =< 9,-1 >, \overrightarrow{r} =< 2,y >$

$A = (-3,2), B = (5,-2)$

1. Solve for y in vector $\overrightarrow{r}$ to make $\overrightarrow{r}$ perpendicular to $\overrightarrow{t}$.

2. Find the unit vector in the same direction as $\overrightarrow{u}$.

3. Find the unit vector in the same direction as $\overrightarrow{t}$.

4. Find the unit vector in the same direction as $\overrightarrow{v}$.

5. Find the unit vector in the same direction as $\overrightarrow{r}$.

6. Find the point exactly 3 units away from A in the direction of B.

7. Find the point exactly 6 units away from B in the direction of A.

8. Find the point exactly 5 units away from A in the direction of B.

9. Jack and Jill went up a hill to fetch a pail of water. When they got to the top of the hill, they were very thirsty so they each pulled on the bucket. Jill pulled at 30° with 20 lbs of force. Jack pulled at 45° with 28 lbs of force. What is the resulting vector for the bucket?

10. A plane is flying on a bearing of 60° at 400 mph. Find the component form of the velocity of the plane. What does the component form tell you?

11. A baseball is thrown at a 70° angle with the horizontal with an initial speed of 30 mph. Find the component form of the initial velocity.

12. A plane is flying on a bearing of 200° at 450 mph. Find the component form of the velocity of the plane.

13. A plane is flying on a bearing of 260° at 430 mph. At the same time, there is a wind blowing at a bearing of 30° at 60 mph. What is the component form of the velocity of the plane?

14. Use the information from the previous problem to find the actual ground speed and direction of the plane.

15. Wind is blowing at a magnitude of 40 mph with an angle of 25° with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.3.

5.4 Dot Product and Angle Between Two Vectors

Learning Objectives

Here you will compute the dot product between two vectors and interpret its meaning.

While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. You can study more about the cross product between two vectors when you take Linear Algebra. The second type of product is the **dot product** between two vectors which results in a regular number. Other names for the dot product include **inner product** and **scalar product**. This number represents how much of one vector goes in the direction of the other. In one sense, it indicates how much the two vectors agree with each other. This concept will focus on the dot product between two vectors.

What is the dot product between $< -1, 1 >$ and $< 4, 4 >$? What does the result mean?

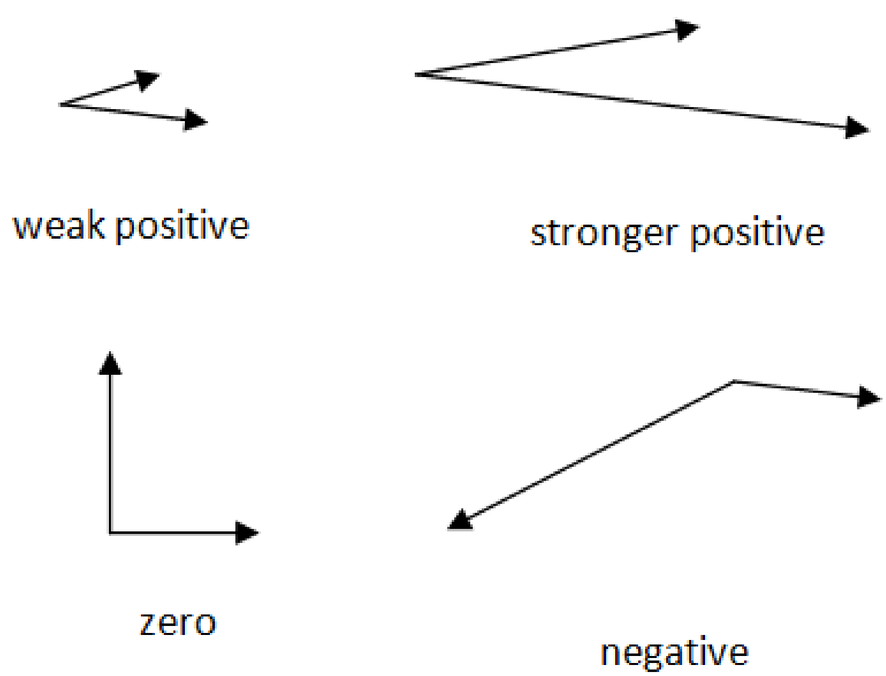
Properties of the Dot Product

The dot product is defined as:

$u \cdot v = < u_1, u_2 > \cdot < v_1, v_2 > = u_1 v_1 + u_2 v_2$

This procedure states that you multiply the corresponding values and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product.



Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than 90°.

There are a many important properties related to the dot product. The two most important are 1) what happens when a vector has a dot product with itself and 2) what is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = |v|^2$
- v and u are perpendicular if and only if $v \cdot u = 0$

The commutative property, $u \cdot v = v \cdot u$, holds for the dot product between two vectors. The following proof is for two dimensional vectors although it holds for any dimensional vectors.

Start with the vectors in component form.

$$\begin{aligned} u &= < u_1, u_2 > \\ v &= < v_1, v_2 > \end{aligned}$$

Then apply the definition of dot product and rearrange the terms. The commutative property is already known for regular numbers so we can use that.

$$\begin{aligned} u \cdot v &= < u_1, u_2 > \cdot < v_1, v_2 > \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \\ &= < v_1, v_2 > \cdot < u_1, u_2 > \end{aligned}$$
$$\& = v \cdot u$$

The distributive property, $u \cdot (v+w) = uv + uw$, also holds under the dot product**.** The following proof will work with two dimensional vectors although the property does hold in general.

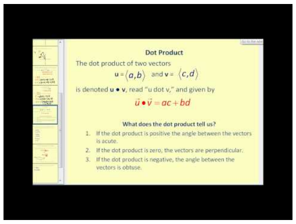
$u = < u_1, u_2 >, v = < v_1, v_2 >, w = < w_1, w_2 >$

$$\begin{aligned} u \cdot (v+w) &= u \cdot (< v_1, v_2 > + < w_1, w_2 >) \\ &= u \cdot < v_1 + w_1, v_2 + w_2 > \\ &= < u_1, u_2 > \cdot < v_1 + w_1, v_2 + w_2 > \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ &= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 \\ &= u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2 \\ &= u \cdot v + v \cdot w \end{aligned}$$

The dot product can help you determine the angle between two vectors using the following formula. Notice that in the numerator the dot product is required because each term is a vector. In the denominator only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

Watch the portion of this video focusing on the dot product:



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Examples

Example 1

Earlier, you were asked to find the dot product between the two vectors $< -1, 1 >$ and $< 4, 4 >$. It can be computed as:

$(-1)(4) + 1(4) = -4 + 4 = 0$

The result of zero makes sense because these two vectors are perpendicular to each other.

Example 2

Find the dot product between the following vectors: $< 3, 1 > \cdot < 5, -4 >$

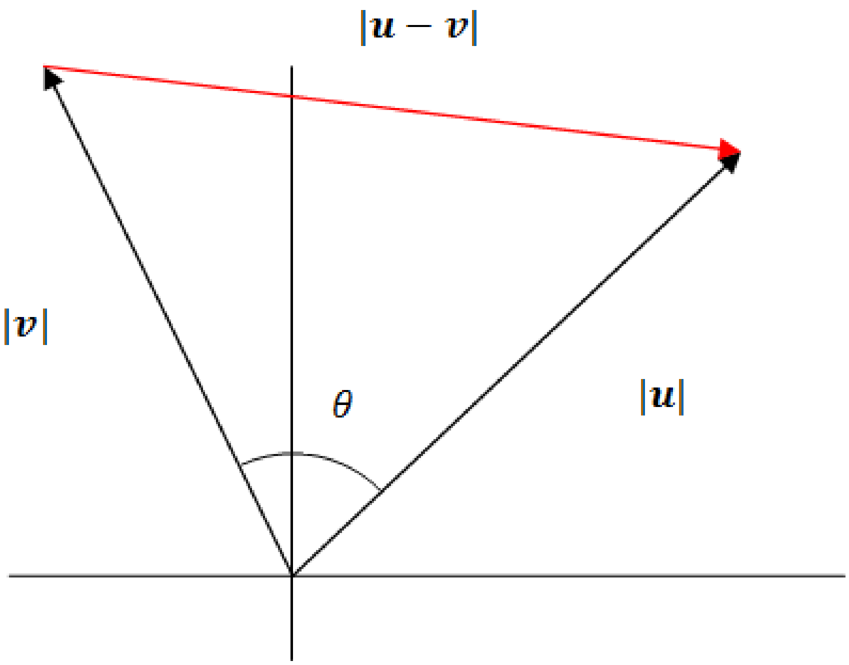
$< 3, 1 > \cdot < 5, -4 >= 3 \cdot 5 + 1 \cdot (-4) = 15 - 4 = 11$

Example 3

Prove the angle between two vectors formula:

$\cos \theta = \frac{u \cdot v}{|u||v|}$

Start with the law of cosines.



$$\begin{aligned}
|u-v|^2 &= |v|^2+|u|^2-2|v||u|\cos\theta \\
(u-v)\cdot(u-v) &= \\
u\cdot u-2u\cdot v+v\cdot v &= \\
|u|^2-2u\cdot v+|v|^2 &= \\
-2u\cdot v &= -2|v||u|\cos\theta \\
\frac{u\cdot v}{|u||v|} &= \cos\theta
\end{aligned}$$

Example 4

Find the dot product between the following vectors.

$(4i-2j)\cdot(3i-8j)$

The standard unit vectors can be written as component vectors.

$<4,-2>\cdot<3,-8>=12+(-2)(-8)=12+16=28$

Example 5

What is the angle between $v=<3,5>$ and $u=<2,8>$?

Use the angle between two vectors formula.

$v=<3,5>$ and $u=<2,8>$

$$
\begin{aligned}
\frac{u \cdot v}{|u||v|} &= \cos\theta \\
\frac{<3,5> \cdot <2,8>}{\sqrt{34} \cdot \sqrt{68}} &= \cos\theta \\
\frac{6+35}{\sqrt{34} \cdot \sqrt{68}} &= \cos\theta \\
\cos^{-1}\left(\frac{41}{\sqrt{34} \cdot \sqrt{68}}\right) &= \theta \\
31.49 &\approx \theta
\end{aligned}
$$

Review

Find the dot product for each of the following pairs of vectors.

1. $<2,6> \cdot <-3,5>$
2. $<5,-1> \cdot <4,4>$
3. $<-3,-4> \cdot <2,2>$
4. $<3,1> \cdot <6,3>$
5. $<-1,4> \cdot <2,9>$

Find the angle between each pair of vectors below.

6. $<2,6> \cdot <-3,5>$
7. $<5,-1> \cdot <4,4>$
8. $<-3,-4> \cdot <2,2>$
9. $<3,1> \cdot <6,3>$
10. $<-1,4> \cdot <2,9>$
11. What is $v \cdot v$?
12. How can you use the dot product to find the magnitude of a vector?
13. What is $0 \cdot v$?
14. Show that $(cu) \cdot v = u \cdot (cv)$ where c is a constant.
15. Show that $<2,3>$ is perpendicular to $<1.5,-1>$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.4.

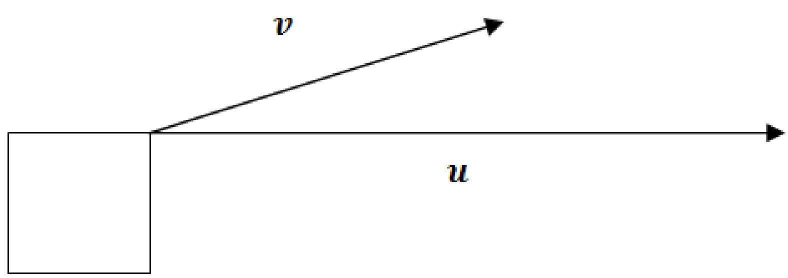
5.5 Vector Projection

Learning Objectives

Here you will project one vector onto another and apply this technique as it relates to force.

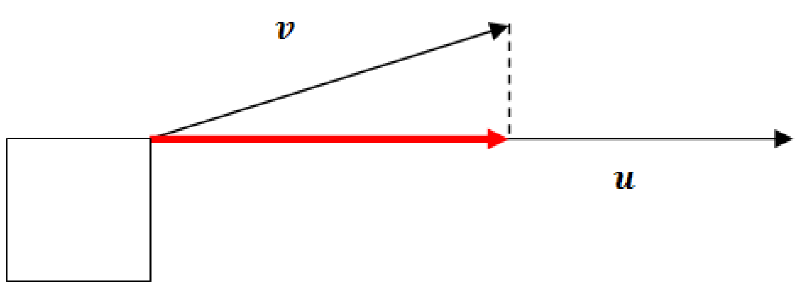
Projecting one vector onto another explicitly answers the question: "how much of one vector goes in the direction of the other vector?" The dot product is useful because it produces a scalar quantity that helps to answer this question. In this concept, you will produce an actual vector not just a scalar.

Why is vector projection useful when considering pulling a box in the direction of v instead of horizontally in the direction of u?



Projections

Consider the question from above.



The definition of **vector projection** for the indicated red vector is the called $proj_u v$. When you read $proj_u v$ you should say "the vector projection of v onto u." This implies that the new vector is going in the direction of u. The vector projection is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector. The parallel vector is the vector projection. Conceptually, this means that if someone was pulling the box at an angle and strength of vector v then some of their energy would be wasted pulling the box up and some of the energy would actually contribute to pulling the box horizontally.

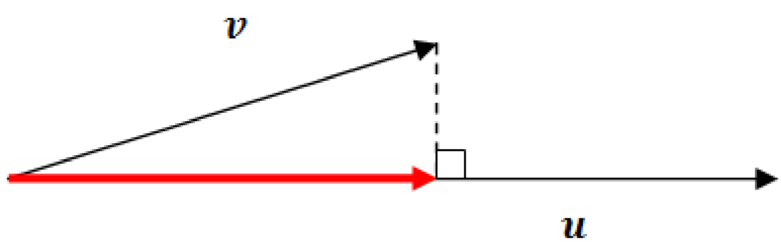
The definition of **scalar projection** is simply the length of the vector projection. When the scalar projection is positive it means that the angle between the two vectors is less than $90°$. When the scalar projection is negative it means that the two vectors are heading in opposite directions.

The vector projection formula can be written two ways. The version on the left is most simplified, but the version on the right makes the most sense conceptually. A.

$proj_u v = \left(\frac{v \cdot u}{|u|^2}\right) u = \left(\frac{v \cdot u}{|u|}\right) \frac{u}{|u|}$

The proof of the **vector projection formula** is as follows:

Given two vectors u, v, what is $proj_u v$?



First note that the projected vector in red will go in the direction of u. This means that it will be a product of the unit vector $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). In order to find the scalar projection, note the right triangle, the unknown angle θ between the two vectors and the cosine ratio.

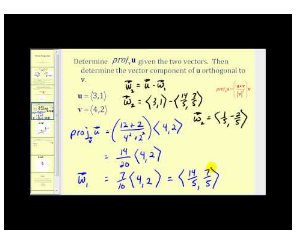
$\cos\theta = \frac{scalar\ projection}{|v|}$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now just substitute and simplify to find the length of the scalar projection.

$$\cos\theta = \frac{scalar\ projection}{|v|}$$
$$\frac{u \cdot v}{|u||v|} = \frac{scalar\ projection}{|v|}$$
$$\frac{u \cdot v}{|u|} = scalar\ projection$$

Now you have the length of the vector projection and the direction you want it to go:

$proj_u v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$



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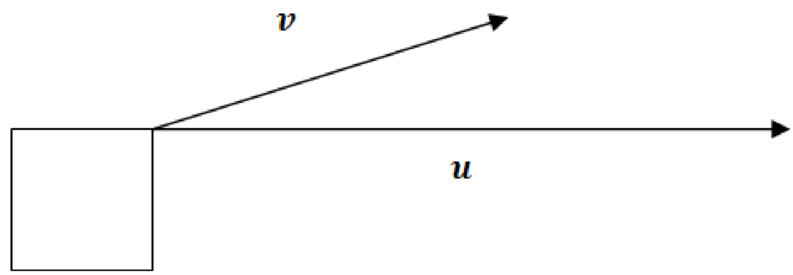
Examples

Example 1

Earlier, you were asked why vector projection useful when considering pulling a box in the direction of instead of horizontally in the direction of u.Vector projection is useful in physics applications involving force and work.



When the box is pulled by vector v some of the force is wasted pulling up against gravity. In real life this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

Example 2

Find the scalar projection of vector $v = < 3, 4 >$ onto vector $u = < 5, -12 >$.

As noted earlier, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u \cdot v}{|u|}\right)$ where u is the vector being projected onto.

$\frac{u \cdot v}{|u|} = \frac{<5,-12> \cdot <3,4>}{13} = \frac{15-48}{13} = -\frac{33}{13}$

Example 3

Find the vector projection of vector $v = < 3, 4 >$ onto vector $u = < 5, -12 >$

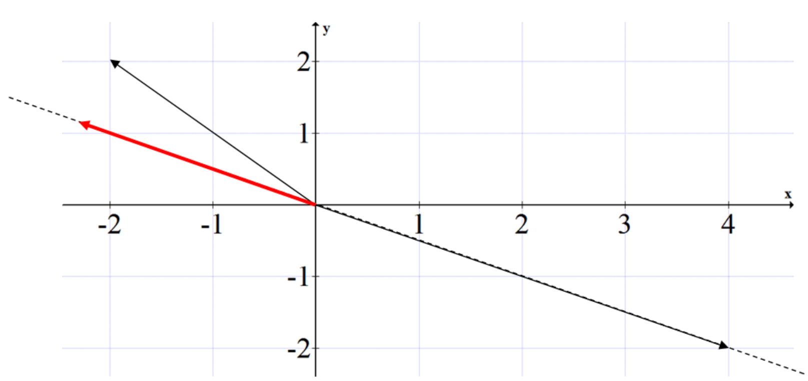
Since the scalar projection has already been found in Example 2, you should multiply the scalar by the "onto" unit vector.

$-\frac{33}{13} < \frac{5}{13}, -\frac{12}{13} > = < -\frac{165}{169}, \frac{396}{169} >$

Example 4

Sketch the vector $< -2, -2 >$ and $< 4, -2 >$. Explain using a sketch why a negative scalar projection of $< -2, -2 >$ onto $< 4, -2 >$ makes sense.

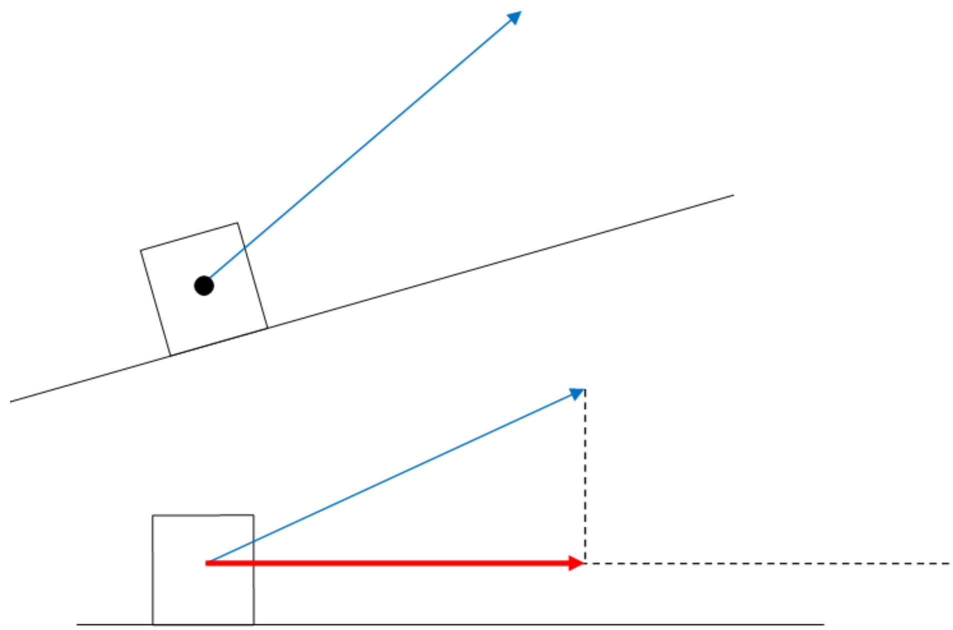
First plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $< -2, 2 >$ goes in the opposite direction of the vector $< 4, -2 >$. This will create a vector projection going in the opposite direction of $< 4, -2 >$.



Example 5

A father is pulling his daughter up a hill. The hill has a 20° incline. The daughter is on a sled that sits on the ground and has a rope that the father pulls with a force of 100 lb as he walks. The rope makes a 39° angle with the slope. What is the effective force that the father exerts moving his daughter and the sled up the hill?

The box represents the girl and the sled. The blue arrow indicates the father's 100 lb force. Notice that the question asks for simply the amount of force which means scalar projection. Since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.



The components of the father's force vector is $100 < \cos 39^\circ, \sin 39^\circ >$ and the "onto" vector is any vector horizontally to the right. Since we are only looking for the length of the horizontal component and you already have the angle between the two vectors, the scalar projection is:

$100 \cdot \cos 39^\circ \approx 77.1\ lb$

Review

1. Sketch vectors $< 2,4 >$ and $< 2,1 >$.

2. What is the vector projection of $< 2,4 >$ onto $< 2,1 >$? Sketch the projection.

3. Sketch vectors $< -2,1 >$ and $< -1,3 >$.

4. What is the vector projection of $< -1,3 >$ onto $< -2,1 >$? Sketch the projection.

5. Sketch vectors $< 6,2 >$ and $< 8,1 >$.

6. What is the vector projection of $< 6,2 >$ onto $< 8,1 >$? Sketch the projection.

7. Sketch vectors $< 1,7 >$ and $< 6,3 >$.

8. What is the vector projection of $< 1,7 >$ onto $< 6,3 >$? Sketch the projection.

9. A box is on the side of a hill inclined at 30°. The weight of the box is 40 pounds. What is the magnitude of the force required to keep the box from sliding down the hill?

10. Sarah is on a sled on the side of a hill inclined at 60°. The weight of Sarah and the sled is 125 pounds. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill.

11. A 1780 pound car is parked on a street that makes an angle of 15° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

12. A 1900 pound car is parked on a street that makes an angle of 10° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

13. A 30 pound force that makes an angle of 32° with an inclined plane is pulling a box up the plane. The inclined plane makes a 20° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

14. A 22 pound force that makes an angle of 12° with an inclined plane is pulling a box up the plane. The inclined plane makes a 25° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

15. Anne pulls a wagon on a horizontal surface with a force of 50 pounds. The handle of the wagon makes an angle of 30° with the ground. What is the magnitude of the effective force pulling the wagon?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.5.

5.6 Polar Coordinates

Introduction

This chapter introduces and explores the polar coordinate system, which is based on a radius and theta. Students will learn how to plot points and basic graphs in this system as well as convert x and y coordinates into polar coordinates and vice versa. We will explore the different graphs that can be generated in the polar system and also use polar coordinates to better understand different aspects of complex numbers.

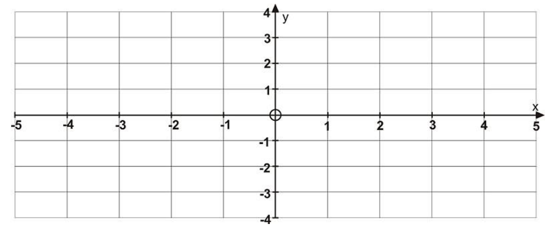
Learning Objectives

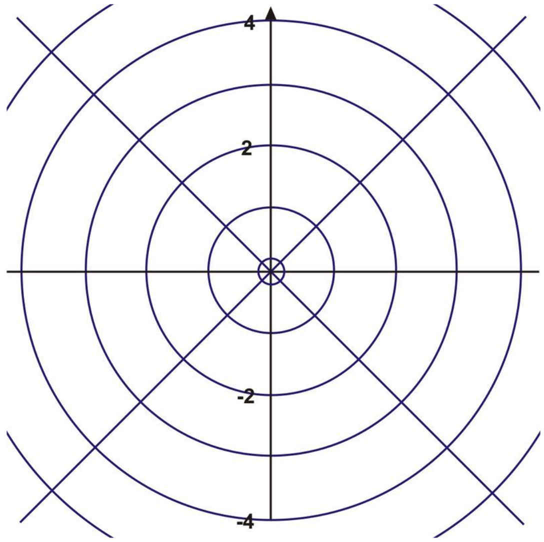
- Distinguish between and understand the difference between a rectangular coordinate system and a polar coordinate system.
- Plot points with polar coordinates on a polar plane.

Plotting Polar Coordinates

The graph paper that you have used for plotting points and sketching graphs has been rectangular grid paper. All points were plotted in a rectangular form (x, y) by referring to a perpendicular $x-$ and $y-$axis. In this section you will discover an alternative to graphing on rectangular grid paper –graphing on circular grid paper.

Look at the two options below:



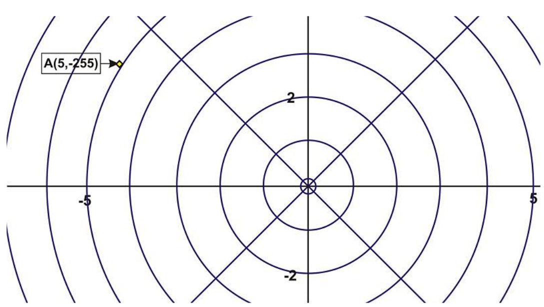


You are all familiar with the rectangular grid paper shown above. However, the circular paper lends itself to new discoveries. The paper consists of a series of concentric circles-circles that share a common center. The common center O, is known as the pole or origin and the polar axis is the horizontal line r that is drawn from the pole in a positive direction. The point P that is plotted is described as a directed distance r from the pole and by the angle that $\overline{OP}$ makes with the polar axis. The coordinates of P are (r,θ).

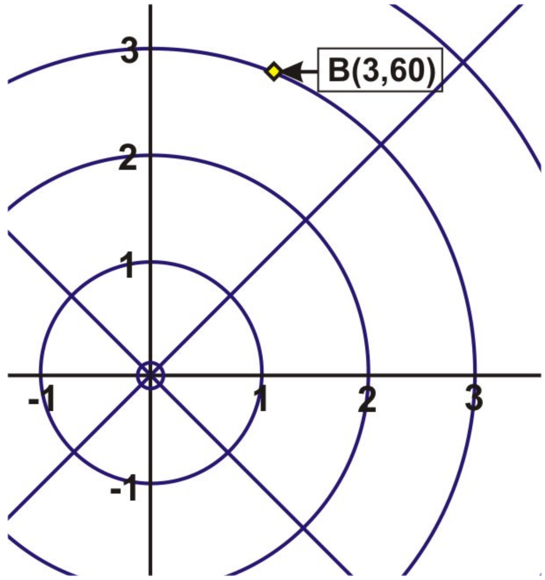
These coordinates are the result of assuming that the angle is rotated counterclockwise. If the angle were rotated clockwise then the coordinates of P would be $(r,-\theta)$. These values for P are called polar coordinates and are of the form $P(r,\theta)$ where r is the absolute value of the distance from the pole to P and θ is the angle formed by the polar axis and the terminal arm $\overline{OP}$.

Example 1: Plot the point $A(5,-255^\circ)$ and the point $B(3,60^\circ)$

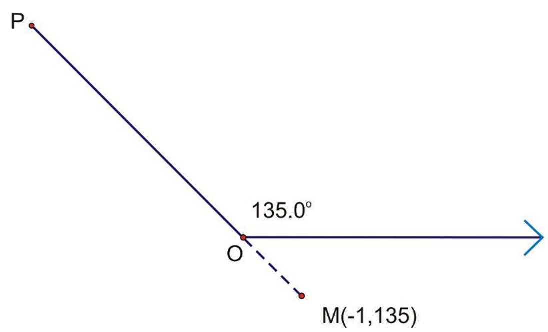
Solution, A: To plot A, move from the pole to the circle that has $r=5$ and then rotate 255° **clockwise** from the polar axis and plot the point on the circle. Label it A.

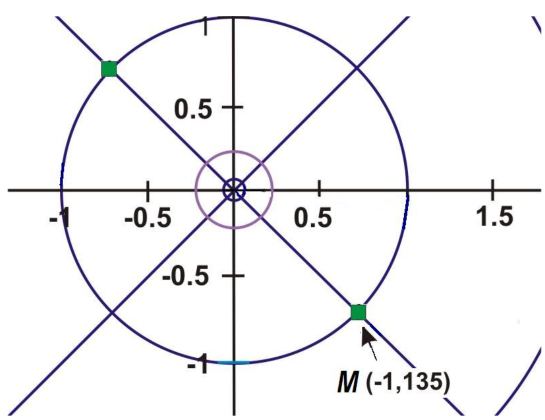


Solution, B: To plot B, move from the pole to the circle that has $r=3$ and then rotate 60° **counter clockwise** from the polar axis and plot the point on the circle. Label it B.



These points that you have plotted have r values that are greater than zero. How would you plot a polar point in which the value of r is less than zero? How could you plot these points if you did not have polar paper? If you were asked to plot the point $(-1, 135^\circ)$ or $\left(-1, \frac{3\pi}{4}\right)$ you would rotate the terminal arm $\overline{OP}$ counterclockwise 135° or $\frac{3\pi}{4}$. (Remember that the angle can be expressed in either degrees or radians). To accommodate $r = -1$, extend the terminal arm $\overline{OP}$ in the opposite direction the number of units equal to $|r|$. Label this point M or whatever letter you choose. The point can be plotted, without polar paper, as a rotation about the pole as shown below.

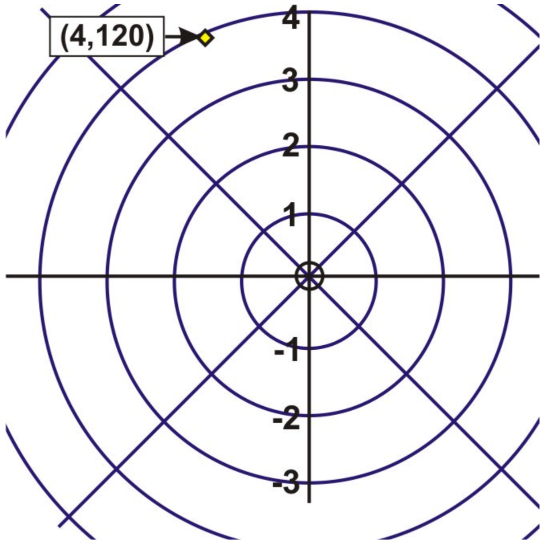




The point is reflected across the pole to point M.

There are multiple representations for the coordinates of a polar point $P(r, \theta)$. If the point P has polar coordinates (r, θ), then P can also be represented by polar coordinates $(r, \theta + 360k^\circ)$ or $(-r, \theta + [2k+1]180^\circ)$ if θ is measured in degrees or by $(r, \theta + 2\pi k)$ or $(-r, \theta + [2k+1]\pi)$ if θ is measured in radians. Remember that k is any integer and represents the number of rotations around the pole. Unless there is a restriction placed upon θ, there will be an infinite number of polar coordinates for $P(r,\theta)$.

Example 2: Determine four pairs of polar coordinates that represent the following point $P(r,\theta)$ such that $-360^\circ \leq \theta \leq 360^\circ$.



Solution: Pair 1 $\rightarrow (4,120^\circ)$. Pair 2 $\rightarrow (4,-240^\circ)$ comes from using $k=-1$ and $(r,\theta+360^\circ k),(4,120^\circ+360(-1))$. Pair 3 $\rightarrow (-4,300^\circ)$ comes from using $k=0$ and $(-r,\theta+[2k+1]180^\circ),(-4,120^\circ+[2(0)+1]180^\circ)$. Pair 4 $\rightarrow (-4,-60^\circ)$ comes from using $k=-1$ and $(-r,\theta+[2k+1]180^\circ),(-4,120^\circ+[2(-1)+1]180^\circ)$.

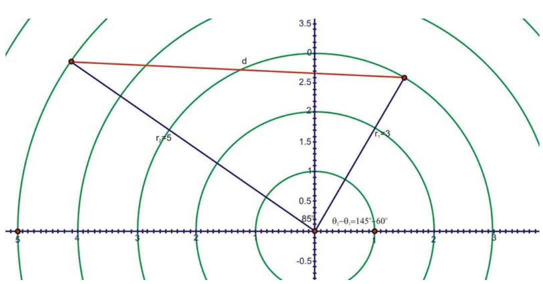
These four pairs of polar coordinates all represent the same point P. You can apply the same procedure to determine polar coordinates of points that have θ measured in radians. This will be an exercise for you to do at the end of the lesson.

The Distance between Two Polar Coordinates

Just like the Distance Formula for x and y coordinates, there is a way to find the distance between two polar coordinates. One way that we know how to find distance, or length, is the Law of Cosines, $a^2 = b^2+c^2-2bc\cos A$ or $a = \sqrt{b^2+c^2-2bc\cos A}$. If we have two points (r_1,θ_1) and (r_2,θ_2), we can easily substitute r_1 for b and r_2 for c. As for A, it needs to be the angle between the two radii, or $(\theta_2-\theta_1)$. Finally, a is now distance and you have $d = \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}$.

Example 3: Find the distance between $(3,60^\circ)$ and $(5,145^\circ)$.

Solution: After graphing these two points, we have a triangle. Using the new Polar Distance Formula, we have $d = \sqrt{3^2+5^2-2(3)(5)\cos 85^\circ} \approx 5.6$.



Example 4: Find the distance between $(9,-45^\circ)$ and $(-4,70^\circ)$.

Solution: This one is a little trickier than the last example because we have negatives. The first point would be plotted in the fourth quadrant and is equivalent to $(9,315^\circ)$. The second point would be $(4,70^\circ)$ reflected across the pole, or $(4,250^\circ)$. Use these two values of θ for the formula. Also, the radii should always be positive when put into the formula. That being said, the distance is $d = \sqrt{9^2+4^2-2(9)(4)\cos(315-250)^\circ} \approx 8.16$.

Points to Consider

- How is the polar coordinate system similar/different from the rectangular coordinate system?
- How do you plot a point on a polar coordinate grid?
- How do you determine the coordinates of a point on a polar grid?
- How do you calculate the distance between two points that have polar coordinates?

Review Questions

1. Graph each point:
   a. $M(2.5,210^\circ)$
   b. $S\left(-3.5,\frac{5\pi}{6}\right)$
   c. $A\left(1,\frac{3\pi}{4}\right)$
   d. $Y\left(5.25,-\frac{\pi}{3}\right)$
2. For the given point $A\left(-4,\frac{\pi}{4}\right)$, list three different pairs of polar coordinates that represent this point such that $-2\pi \le \theta \le 2\pi$.
3. For the given point $B(2,120^\circ)$, list three different pairs of polar coordinates that represent this point such that $-2\pi < \theta < 2\pi$.
4. Given P_1 and P_2, calculate the distance between the points.
   a. $P_1(1,30^\circ)$ and $P_2(6,135^\circ)$
   b. $P_1(2,-65^\circ)$ and $P_2(9,85^\circ)$
   c. $P_1(-3,142^\circ)$ and $P_2(10,-88^\circ)$
   d. $P_1(5,-160^\circ)$ and $P_2(16,-335^\circ)$

5.7 Graphing Basic Polar Equations

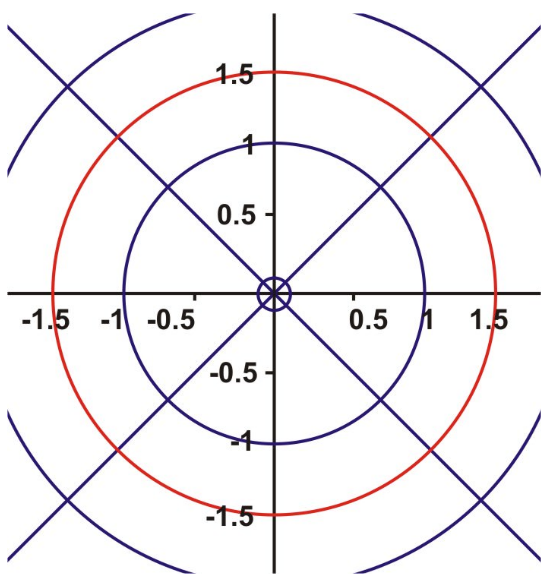
Learning Objectives

- Graph polar equations.
- Graph and recognize limaçons and cardioids.
- Determine the shape of a limaçon from the polar equation.

Just as in graphing on a rectangular grid, you can also graph polar equations on a polar grid. These equations may be simple or complex. To begin, you should try something simple like $r = k$ or $\theta = k$ where k is a constant. The solution for $r = 1.5$ is simply all ordered pairs such that $r = 1.5$ and θ is any real number. The same is true for the solution of $\theta = 30^\circ$. The ordered pairs will be any real number for r and θ will equal 30°. Here are the graphs for each of these polar equations.

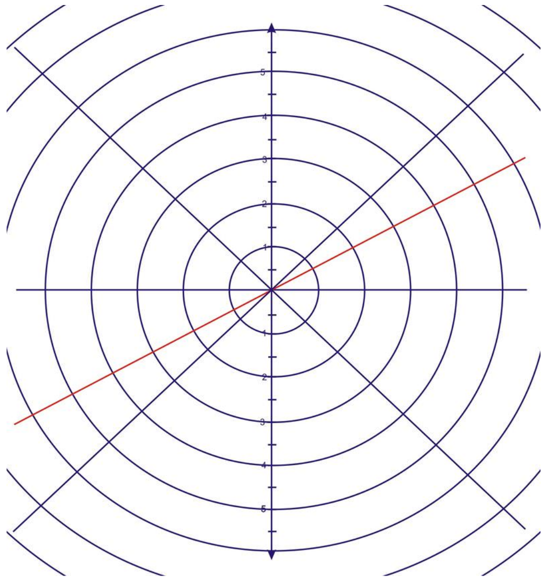
Example 1: On a polar plane, graph the equation $r = 1.5$

Solution: The solution is all ordered pairs of (r, θ) such that r is always 1.5. This means that it doesn't matter what θ is, so the graph is a circle with radius 1.5 and centered at the origin.



Example 2: On a polar plane, graph the equation $\theta = 30^\circ$

Solution: For this example, the r value, or radius, is arbitrary. θ must equal 30°, so the result is a straight line, with an angle of elevation of 30°.



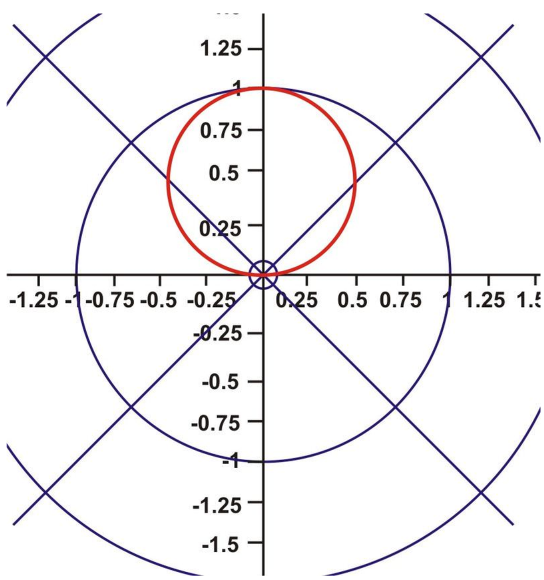
To begin graphing more complicated polar equations, we will make a table of values for $y = \sin\theta$ or in this case $r = \sin\theta$. When the table has been completed, the graph will be drawn on a polar plane by using the coordinates (r, θ).

Example 3: Create a table of values for $r = \sin\theta$ such that $0^\circ \le \theta \le 360^\circ$ and plot the ordered pairs. (Note: Students can be directed to use intervals of 30° or allow them to create their own tables.)

TABLE 5.1:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	**0**	**0.5**	**0.9**	**1**	**0.9**	**0.5**	**0**	**-0.5**	**-0.9**	**-1**	**-0.9**	**-0.5**	**0**

Remember that the values of $\sin\theta$ are the $r-$values.



This is a sinusoid curve of one revolution.

We will now repeat the process for $r = \cos\theta$.

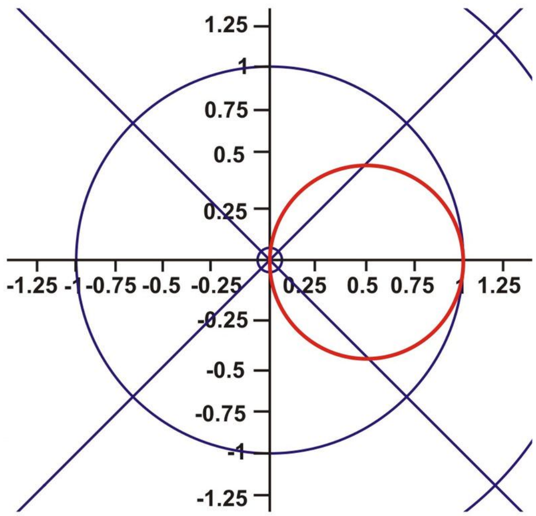
Example 4: Create a table of values for $r = \cos\theta$ such that $0^\circ \le \theta \le 360^\circ$ and plot the ordered pairs. (Note: Students can be directed to use intervals of 30° or allow them to create their own tables.)

TABLE 5.2:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	**1**	**0.9**	**0.5**	**0**	**-0.5**	**-0.9**	**-1**	**-0.9**	**-0.5**	**0**	**0.5**	**0.9**	**1**

Remember that the values of $\cos\theta$ are the $r-$values.

This is also a sinusoid curve of one revolution.



Notice that both graphs are circles that pass through the pole and have a diameter of one unit. These graphs can be altered by adding a number to the function or by multiplying the function by a constant or by doing both. We will explore the results of these alterations by first creating a table of values and then by graphing the resulting coordinates (r, θ).

Example 5: Create a table of values for $r = 2 + 3\sin\theta$ such that $0 \le \theta \le 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 3\sin\theta$ are the $r-$values.

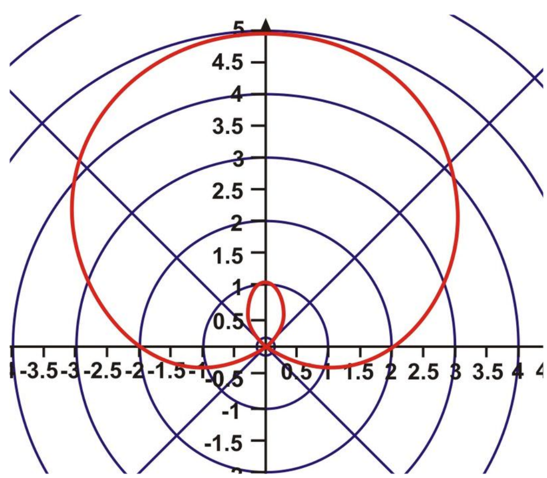


TABLE 5.3:

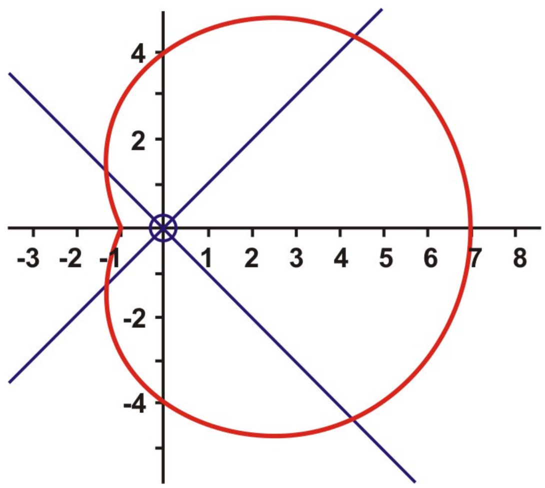
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$2 + 3\sin\theta$	**2.0**	**3.5**	**4.6**	**5.0**	**4.6**	**3.5**	**2.0**	**0.5**	**-0.6**	**-1.0**	**-0.6**	**0.5**	**2.0**

This sinusoid curve is called a limaçon. It has $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ as its polar equation. Not all limaçons have the inner loop as a part of the shape. Some may curve to a point, have a simple indentation (known as a dimple) or curve outward. The shape of the limaçon depends upon the ratio of $\frac{a}{b}$ where a is a constant and b is the coefficient of the trigonometric function. In example 5, the ratio of $\frac{a}{b} = \frac{2}{3}$ which is < 1. All limaçons that meet this criterion will have an inner loop.

Using the same format as was used in the examples above, the following limaçons were graphed. If you like, you may create the table of values for each of these functions.

i) $r = 4 + 3\cos\theta$ such that $0 \leq \theta \leq 2\pi$

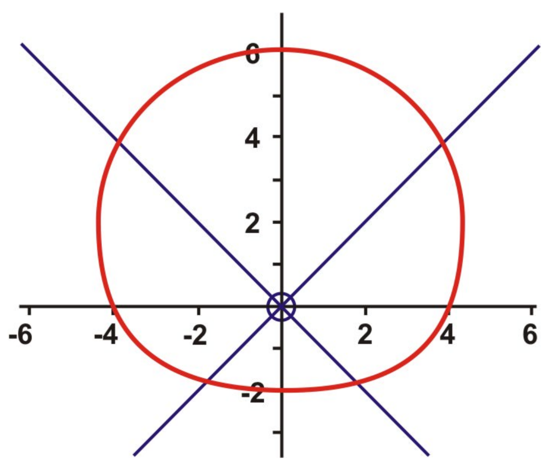
$\frac{a}{b} = \frac{4}{3}$ which is > 1 but < 2



This is an example of a dimpled limaçon.

ii) $r = 4 + 2\sin\theta$ such that $0 \leq \theta \leq 2\pi$

$\frac{a}{b} = \frac{4}{2}$ which is ≥ 2



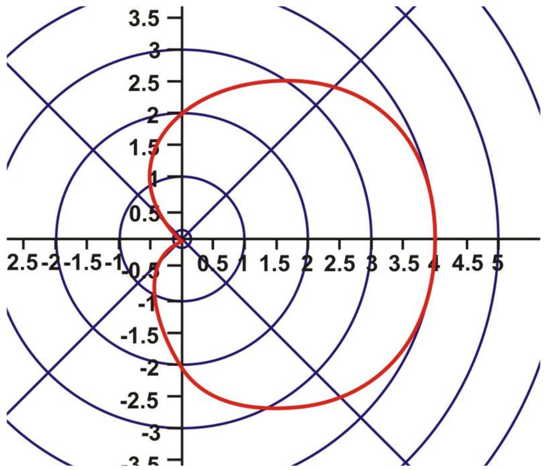
This is an example of a convex limaçon.

Example 6: Create a table of values for $r = 2 + 2\cos\theta$ such that $0 \leq \theta \leq 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 2\cos\theta$ are the $r-$values.

TABLE 5.4:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$2 + 2\cos$	**4.0**	**3.7**	**3.0**	**2.0**	**1.0**	**0.27**	**0**	**.27**	**1.0**	**2.0**	**3.0**	**3.7**	**4.0**

This type of curve is called a cardioid. It is a special type of limaçon that has $r = a + a\cos\theta$ or $r = a + a\sin\theta$ as its polar equation. The ratio of $\frac{a}{b} = \frac{2}{2}$ which is equal to 1.



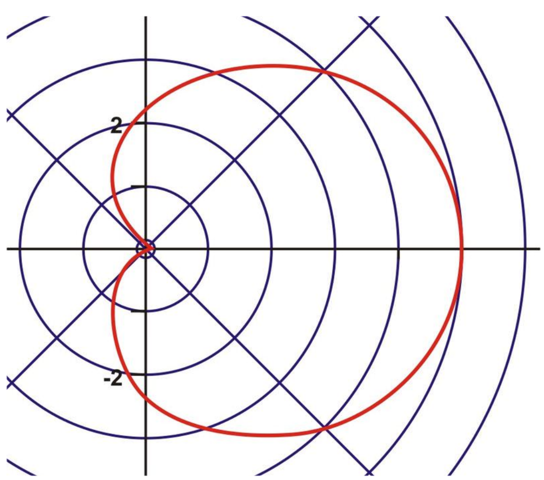
Examples 3 and 4 were shown with θ measured in degrees while examples 5 and 6 were shown with θ measured in radians. The results in the tables and the resulting graphs will be the same in both units.

Now that you are familiar with the limaçon and the cardioid, also called classical curves, it is time to examine the polar pattern of the cardioid microphone. The polar pattern is modeled by the polar equation $r = 2.5 + 2.5\cos\theta$. The values of a and b are equal which means that the ratio $\frac{a}{b} = 1$. Therefore the limaçon will be a cardioid.

Create a table of values for $r = 2.5 + 2.5\cos\theta$ such that $0^\circ \leq \theta \leq 360^\circ$ and graph the results.

TABLE 5.5:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$2.5 + 2.5\cos\theta$	**5.0**	**4.7**	**3.8**	**2.5**	**1.3**	**0.3**	**0**	**0.3**	**1.3**	**2.5**	**3.8**	**4.7**	**5.0**



Transformations of Polar Graphs

Equations of limaçons have two general forms:

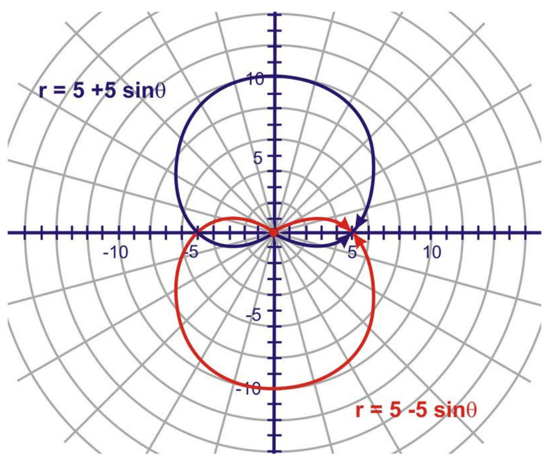
$$r = a \pm b\sin\theta \qquad \text{and} \qquad r = a \pm b\cos\theta$$

The values of "a'' and "b'' will determine the shape of the graph and whether or not it passes through the origin. When the values of "a'' and "b'' are equal, the graph will be a rounded heart-shape called a **cardioid**. The general polar equation of a cardioid can be written as $r = a(1 \pm \sin\theta)$ and $r = a(1 \pm \cos\theta)$. Note: The general polar equation of a cardioid can also be written as $r = a(-1 \pm \sin\theta)$ and $r = a(-1 \pm \cos\theta)$. This will be discussed later in the chapter.

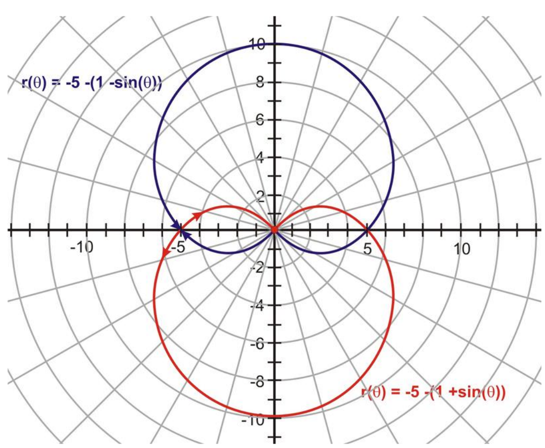
Example 7: Graph the following polar equations on the same polar grid and compare the graphs.

$$\begin{array}{ll} r = 5 + 5\sin\theta & r = 5 - 5\sin\theta \\ r = 5(1 + \sin\theta) & r = 5(1 - \sin\theta) \end{array}$$

Solution:



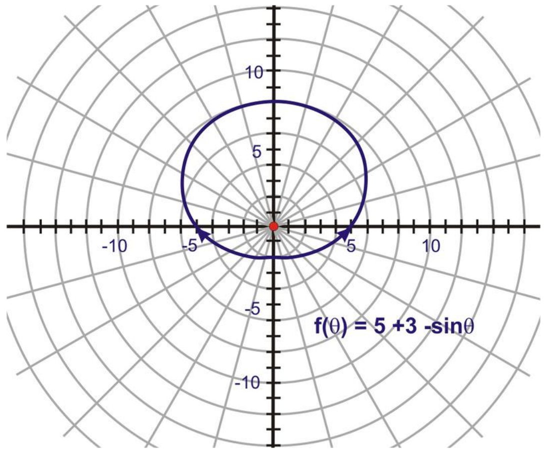
The cardioid is symmetrical about the positive $y-$axis and the point of indentation is at the pole. The result of changing + to - is a reflection in the $x-$axis. The cardioid is symmetrical about the negative $y-$axis and the point of indentation is at the pole.



Changing the value of "a'' to a negative did not change the graph of the cardioid.

Example 8: What effect will changing the values of a and b have on the cardioid if $a > b$? We can discover the answer to this question by plotting the graph of $r = 5 + 3\sin\theta$.

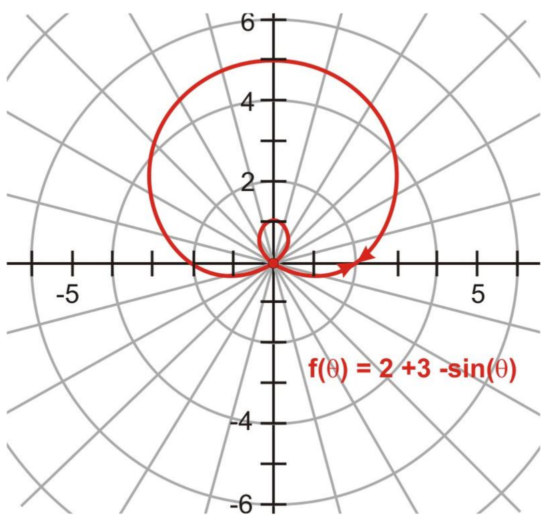
Solution:



The cardioid is symmetrical about the positive $y-$axis and the point of indentation is pulled away from the pole.

Example 9: What effect will changing the values of a and b or changing the function have on the cardioid if $a < b$? We can discover the answer to this question by plotting the graph of $r = 2 + 3\sin\theta$.

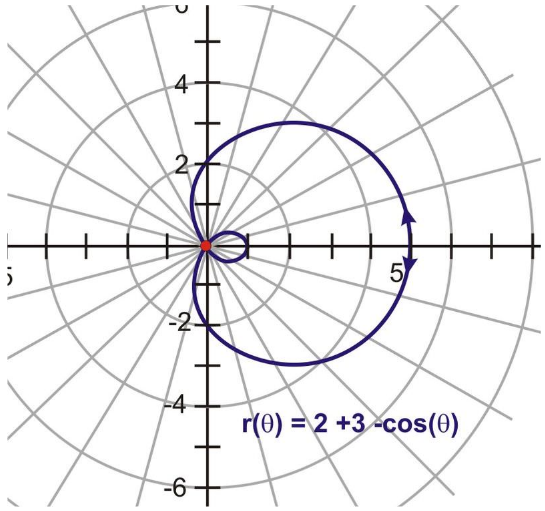
Solution:



The cardioid is now a looped limaçon symmetrical about the positive $y-$axis. The loop crosses the pole.

$$r = 2 + 3\cos\theta$$

The cardioid is now a looped limaçon symmetrical about the positive $x-$axis. The loop crosses the pole. Changing the function to cosine rotated the limaçon 90° clockwise.



As you have seen from all of the graphs, transformations can be performed by making changes in the constants and/or the functions of the polar equations.

Applications

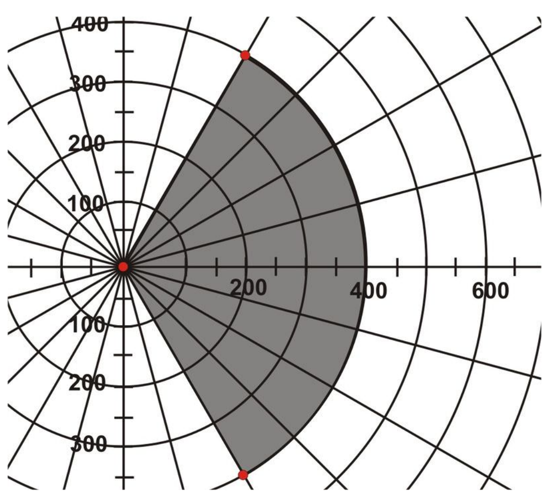
In this subsection we will explore examples of real-world problems that use polar coordinates and polar equations as their solutions.

Example 10: A local charity is sponsoring an outdoor concert to raise money for the children's hospital. To accommodate as many patrons as possible, they are importing bleachers so that all the fans will be seated during the performance. The seats will be placed in an area such that $\frac{-\pi}{3} \leq \theta \leq \frac{\pi}{3}$ and $0 \leq r \leq 4$, where r is measured in hundreds of feet. The stage will be placed at the origin (pole) and the performer will face the audience in the direction of the polar axis (r).

a. Create a polar graph of this area.

b. If all the seats are occupied and each seat takes up 5 square feet of space, how many people will be seated in the bleachers?

Solution: Now that the region has been graphed, the next step is to calculate the area of this sector. To do this, use the formula $A = \frac{1}{2}r^2\theta$.



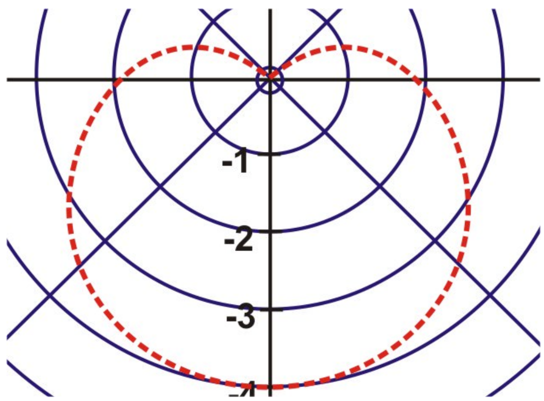
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(400)^2\left(\frac{2\pi}{3}\right)$$
$$A \approx 167552\ ft^2.$$

$$167552\ ft^2. \div 5\ ft^2. \approx 33510$$

The number of people in the bleachers is 33510.

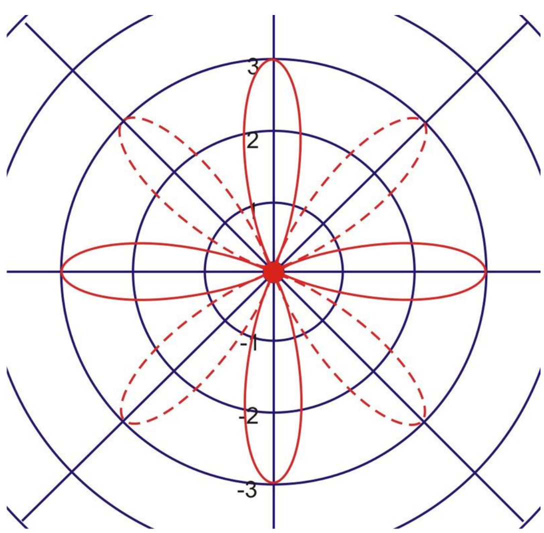
Example 11: When Valentine's Day arrives, hearts can be seen everywhere. As an alternative to purchasing a greeting card, use a computer to create a heart shape. Write an equation that could be used to create this heart and be careful to ensure that it is displayed in the correct position.

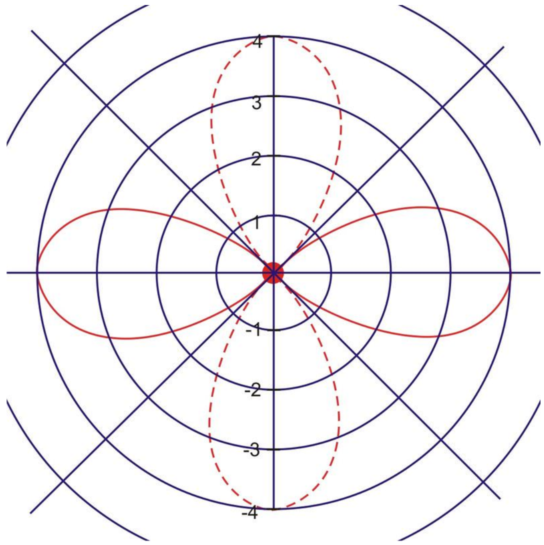
Solution: The classical curve that resembles a heart is a cordioid. You may have to experiment with the equation to create a heart shape that is displayed in the correct direction. One example of an equation that produces a proper heart shape is $r = -2 - 2\sin\theta$.

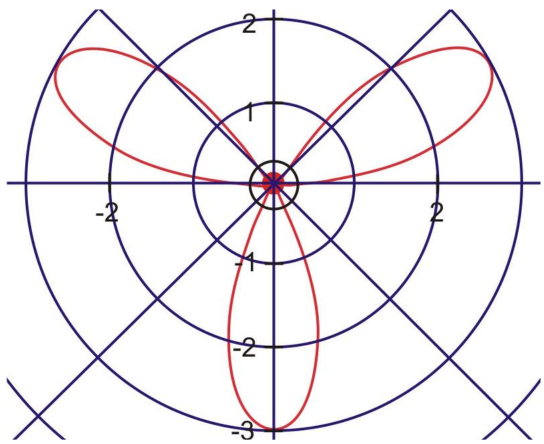


You can create other hearts by replacing the number 2 in the equation. Another equation is $r = -3 - 3\sin\theta$.

Example 12: For centuries, people have been making quilts. These are frequently created by sewing a uniform fabric pattern onto designated locations on the quilt. Using the equation that models a rose curve, create three patterns that could be used for a quilt. Write the equation for each rose and sketch its graph. Explain why the patterns have different numbers of petals. Can you create a sample quilt?

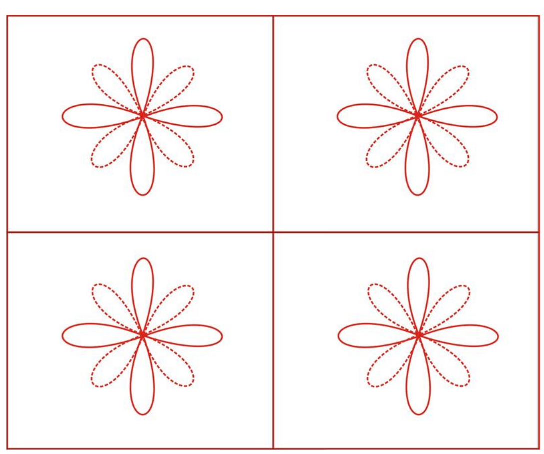






Solution: The rose curve is a graph of the polar equation of the form $r = a \cos n\theta$ or $r = a \sin n\theta$. If n is odd, then the number of petals will be equal to n. If n is even, then the number of petals will be equal to $2n$.

A Sample Quilt:



Graphing Polar Equations on the Calculator

You can use technology, the TI graphing calculator, to create these graphs. However, there are steps that must be followed in order to graph polar equations correctly on the graphing calculator. We will go through the step by step process to plot the polar equation $r = 3\cos\theta$.

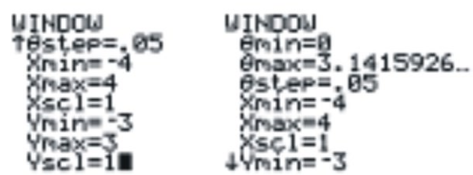
Example 13: Graph $r = 3\cos\theta$ using the TI-83 graphing calculator.

Solution: Press the $\boxed{\text{MODE}}$ button. Scroll down to **Func** and over to highlight **Pol**. Also, while on this screen, make sure that **Radian** is highlighted. Now you must edit the axes for the graph. Press $\boxed{\text{WINDOW}}$ 0 $\boxed{\text{ENTER}}$ $\boxed{2^{\text{nd}}}$ $[\pi]$ $\boxed{\text{ENTER}}$ .05 E

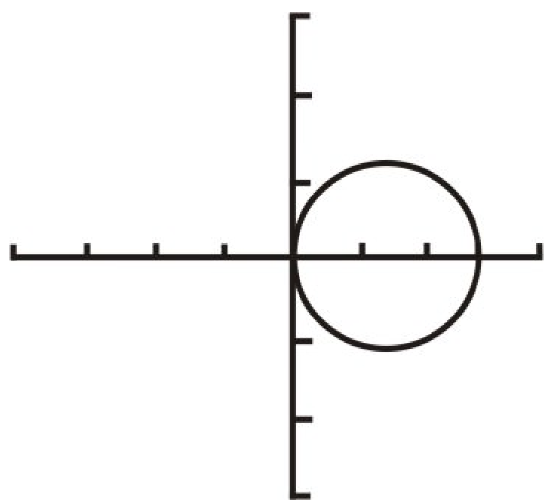
$\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$ $(-)$ 3 $\boxed{\text{ENTER}}$ 3 $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$.

When you have completed these steps, the screen should look like this:

The second **WINDOW** shows part of the first screen since you had to scroll down to access the remaining items.



Enter the equation. Press $\boxed{Y = 3\cos X, T, \theta, n)}$ Press $\boxed{\text{GRAPH}}$.

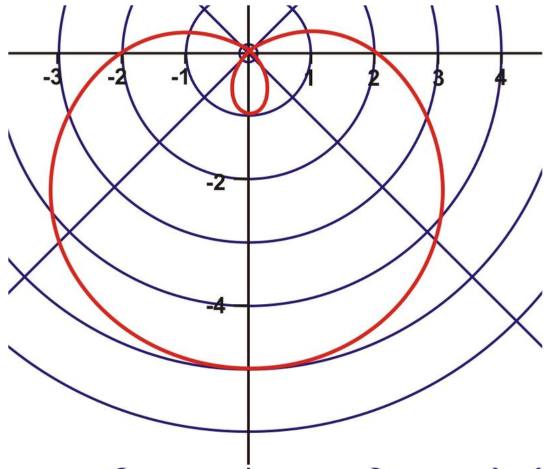


Sometimes the polar equation you graph will look more like an ellipse than a circle. If this happens, press $\boxed{\text{ZOOM}}$ 5 to set a square viewing window. This will make the graph appear like a circle.

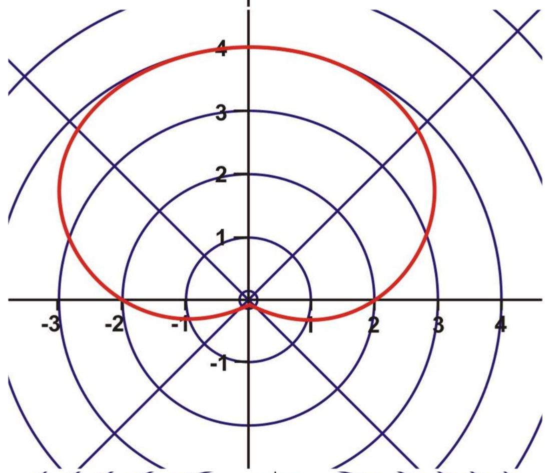
Note: If you want the calculator to graph complete rose petals when n is even, you must set $\theta\text{max} = 2\pi$.

Review Questions

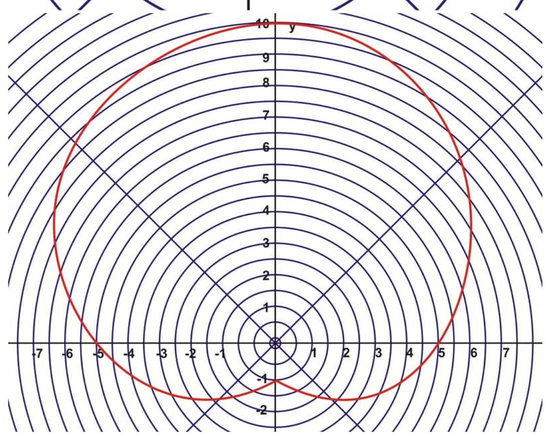
1. Name the classical curve in each of the following diagrams and explain why you feel you're your answer is correct. Also, find the equation of each curve.



a.



b.



c.

2. Graph each curve below. Comparing your answers from part one, determine if you can find a pattern for how to find the equation of a classical curve from its graph.

   a. $r = -3 - 3\cos\theta$
   b. $r = 2 + 4\sin\theta$
   c. $r = 4$
   d. $\theta = \frac{\pi}{2}$
   e. $r = 5 + 3\cos\theta$
   f. $r = -6 - 5\sin\theta$

3. Another classical curve we saw is called a rose and it is modeled by the function $r = a\cos n\theta$ or $r = a\sin n\theta$ where n is any positive integer. Graph $r = 4\cos 2\theta$ and $r = 4\cos 3\theta$. Is there a difference in the curves? Explain.
4. Graph the roses below. Determine if you can find a pattern for how to find the equation of a rose from its graph.

a. $r = 3\sin 4\theta$
b. $r = 2\sin 5\theta$
c. $r = 3\cos 3\theta$
d. $r = -4\sin 2\theta$
e. $r = 5\cos 4\theta$
f. $r = -2\cos 6\theta$

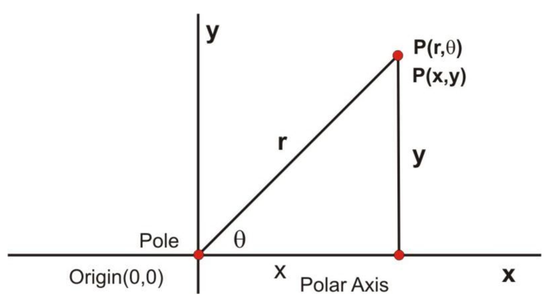
5.8 Converting Between Systems

Learning Objectives

- Convert rectangular coordinates to polar coordinates.
- Convert equations given in rectangular form to equations in polar form and vice versa.

Polar to Rectangular

Just as x and y are usually used to designate the rectangular coordinates of a point, r and θ are usually used to designate the polar coordinates of the point. r is the distance of the point to the origin. θ is the angle that the line from the origin to the point makes with the positive $x-$axis. The diagram below shows both polar and Cartesian coordinates applied to a point P. By applying trigonometry, we can obtain equations that will show the relationship between polar coordinates (r, θ) and the rectangular coordinates (x, y)



The point P has the polar coordinates (r, θ) and the rectangular coordinates (x, y).

Therefore

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

These equations, also known as coordinate conversion equations, will enable you to convert from polar to rectangular form.

Example 1: Given the following polar coordinates, find the corresponding rectangular coordinates of the points: $W(4, -200^\circ), H\left(4, \frac{\pi}{3}\right)$

Solution:

a) For $W(4, -200^\circ), r = 4$ and $\theta = -200^\circ$

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ x &= 4\cos(-200^\circ) & y &= 4\sin(-200^\circ) \\ x &= 4(-.9396) & y &= 4(.3420) \\ x &\approx -3.76 & y &\approx 1.37 \end{aligned}$$

The rectangular coordinates of W are approximately $(-3.76, 1.37)$.

b) For $H\left(4, \frac{\pi}{3}\right), r = 4$ and $\theta = \frac{\pi}{3}$

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ x &= 4\cos\frac{\pi}{3} & y &= 4\sin\frac{\pi}{3} \\ x &= 4\left(\frac{1}{2}\right) & y &= 4\left(\frac{\sqrt{3}}{2}\right) \\ x &= 2 & y &= 2\sqrt{3} \end{aligned}$$

The rectangular coordinates of H are $(2, 2\sqrt{3})$ or approximately $(2, 3.46)$.

In addition to writing polar coordinates in rectangular form, the coordinate conversion equations can also be used to write polar equations in rectangular form.

Example 2: Write the polar equation $r = 4\cos\theta$ in rectangular form.

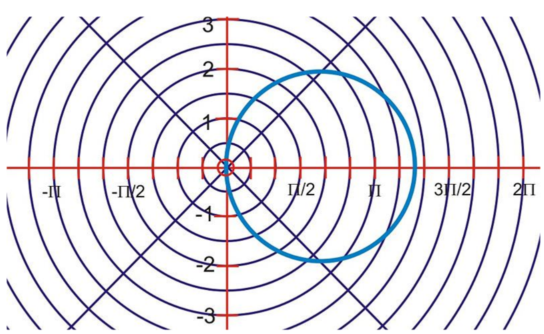
Solution:

$$\begin{aligned} r &= 4\cos\theta & & \\ r^2 &= 4r\cos\theta & & \textit{Multiply both sides by r.} \\ x^2 + y^2 &= 4x & & r^2 = x^2 + y^2 \textit{ and } x = r\cos\theta \end{aligned}$$

The equation is now in rectangular form. The r^2 and θ have been replaced. However, the equation, as it appears, does not model any shape with which we are familiar. Therefore, we must continue with the conversion.

$$\begin{aligned} x^2 - 4x + y^2 &= 0 & & \\ x^2 - 4x + 4 + y^2 &= 4 & & \textit{Complete the square for } x^2 - 4x. \\ (x-2)^2 + y^2 &= 4 & & \textit{Factor } x^2 - 4x + 4. \end{aligned}$$

The rectangular form of the polar equation represents a circle with its centre at (2, 0) and a radius of 2 units.



This is the graph represented by the polar equation $r = 4\cos\theta$ for $0 \le \theta \le 2\pi$ or the rectangular form $(x-2)^2 + y^2 = 4$.

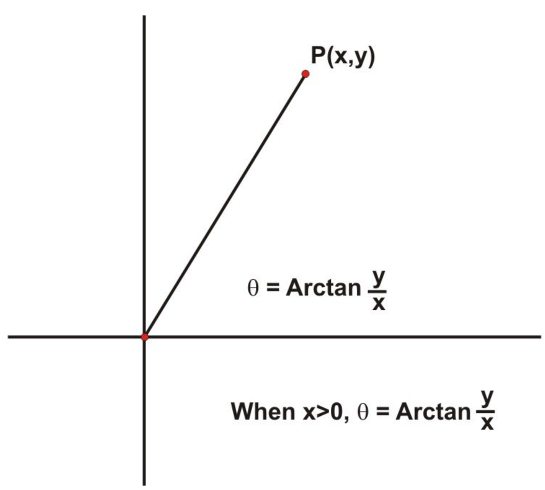
Example 3: Write the polar equation $r = 3\csc\theta$ in rectangular form.

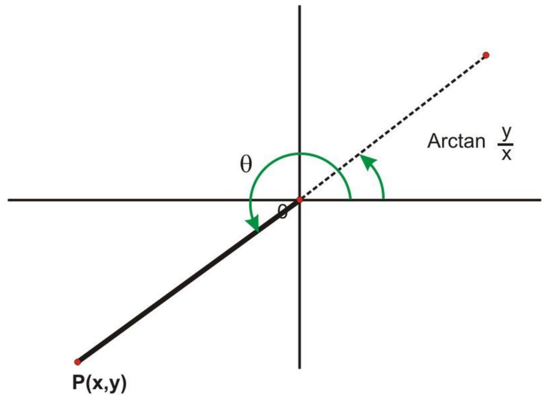
Solution:

$$\begin{aligned}
r &= 3\csc\theta \\
\frac{r}{\csc\theta} &= 3 && \textit{divide by} \csc\theta \\
r \cdot \frac{1}{\csc\theta} &= 3 \\
r\sin\theta &= 3 && \sin\theta = \frac{1}{\csc\theta} \\
y &= 3 && y = r\sin\theta
\end{aligned}$$

Rectangular to Polar

When converting rectangular coordinates to polar coordinates, we must remember that there are many possible polar coordinates. We will agree that when converting from rectangular coordinates to polar coordinates, one set of polar coordinates will be sufficient for each set of rectangular coordinates. Most graphing calculators are programmed to complete the conversions and they too provide one set of coordinates for each conversion. The conversion of rectangular coordinates to polar coordinates is done using the Pythagorean Theorem and the Arctangent function. The Arctangent function only calculates angles in the first and fourth quadrants so π radians must be added to the value of θ for all points with rectangular coordinates in the second and third quadrants.





When x<0, θ = Arctan $\frac{y}{x} + \pi$

In addition to these formulas, $r = \sqrt{x^2+y^2}$ is also used in converting rectangular coordinates to polar form.

Example 4: Convert the following rectangular coordinates to polar form.

$P(3,-5)$ and $Q(-9,-12)$

Solution: For $P(3,-5)$ $x = 3$ and $y = -5$. The point is located in the fourth quadrant and $x > 0$.

$$\begin{aligned} r &= \sqrt{x^2+y^2} & \theta &= Arc\ \tan\frac{y}{x} \\ r &= \sqrt{(3)^2+(-5)^2} & \theta &= \tan^{-1}\left(-\frac{5}{3}\right) \\ r &= \sqrt{34} & \theta &\approx -1.03 \\ r &\approx 5.83 \end{aligned}$$

The polar coordinates of $P(3,-5)$ are $P(5.83,-1.03)$.

For $Q(-9,-12)$ $x = -9$ and $y = -5$. The point is located in the third quadrant and $x < 0$.

$$\begin{aligned} r &= \sqrt{x^2+y^2} & \theta &= Arc\tan\frac{y}{x}+\pi \\ r &= \sqrt{(-9)^2+(-12)^2} & \theta &= \tan^{-1}\left(\frac{-12}{-9}\right)+\pi \\ r &= \sqrt{225} & \theta &\approx 4.07 \\ r &= 15 \end{aligned}$$

The polar coordinates of $Q(-9,-12)$ are $Q(15,4.07)$

Converting Equations

To write a rectangular equation in polar form, the conversion equations of $x = r\cos\theta$ and $y = r\sin\theta$ are used.

Example 5: Write the rectangular equation $x^2+y^2 = 2x$ in polar form.

Solution: Remember $r = \sqrt{x^2+y^2}, r^2 = x^2+y^2$ and $x = r\cos\theta$.

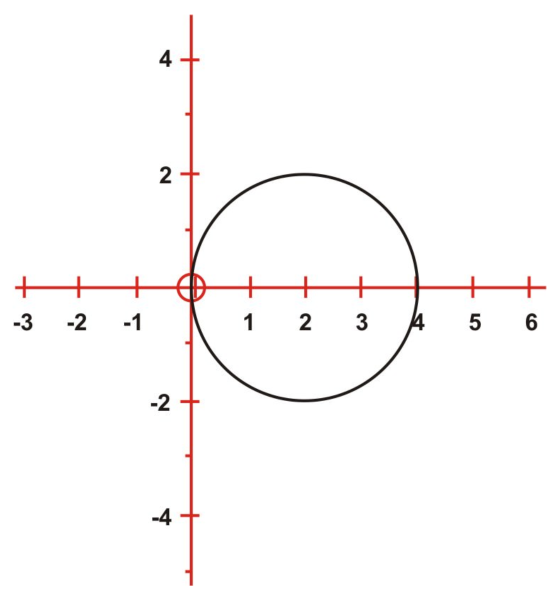
$$\begin{aligned}
x^2+y^2&=2x \\
r^2&=2(r\cos\theta) && \textit{Pythagorean Theorem and } x=r\cos\theta \\
r^2&=2r\cos\theta \\
r&=2\cos\theta && \textit{Divide each side by } r
\end{aligned}$$

Example 6: Write the rectangular equation $(x-2)^2+y^2=4$ in polar form.

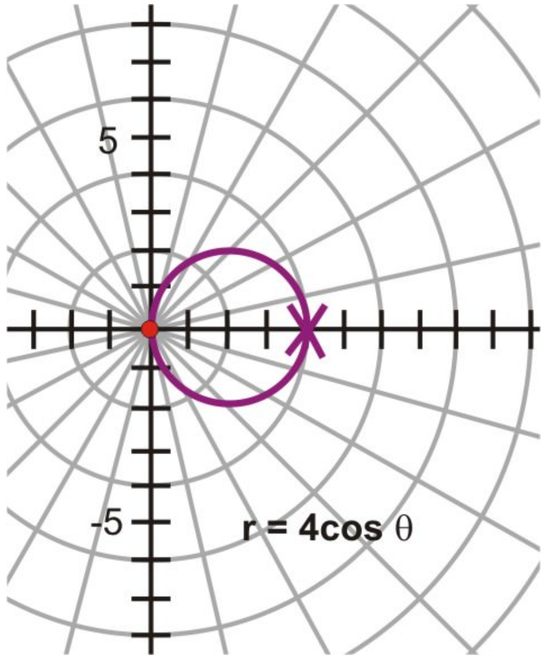
Solution: Remember $x=r\cos\theta$ and $y=r\sin\theta$.

$$\begin{aligned}
&(x-2)^2+y^2=4 \\
&(r\cos\theta-2)^2+(r\sin\theta)^2=4 && x=r\cos\theta \textit{ and } y=r\sin\theta \\
&r^2\cos^2\theta-4r\cos\theta+4+r^2\sin^2\theta=4 && \textit{expand the terms} \\
&r^2\cos^2\theta-4r\cos\theta+r^2\sin^2\theta=0 && \textit{subtract 4 from each side} \\
&r^2\cos^2\theta+r^2\sin^2\theta=4r\cos\theta && \textit{isolate the squared terms} \\
&r^2(\cos^2\theta+\sin^2\theta)=4r\cos\theta && \textit{factor } r^2 - \textit{a common factor} \\
&r^2=4r\cos\theta && \textit{Pythagorean Identity} \\
&r=4\cos\theta && \textit{Divide each side by } r
\end{aligned}$$

If the graph of the polar equation is the same as the graph of the rectangular equation, then the conversion has been determined correctly.



$(x-2)^2+y^2=4$



The rectangular equation $(x-2)^2+y^2=4$ represents a circle with center (2, 0) and a radius of 2 units. The polar equation $r=4\cos\theta$ is a circle with center (2, 0) and a radius of 2 units.

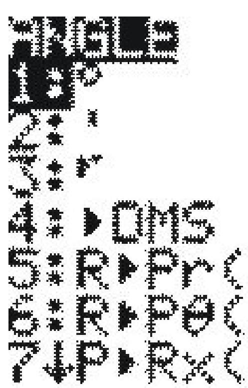
Converting Using the Graphing Calculator

You have learned how to convert back and forth between polar coordinates and rectangular coordinates by using the various formulae presented in this lesson. The TI graphing calculator allows you to use the angle function to convert coordinates quickly from one form to the other. The calculator will provide you with only one pair of polar coordinates for each pair of rectangular coordinates.

Example 7: Express the rectangular coordinates of $A(-3,7)$ as polar coordinates.

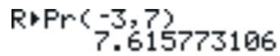
Polar coordinates are expressed in the form (r,θ). An angle can be measured in either degrees or radians, and the calculator will express the result in the form selected in the MODE menu of the calculator.

Press MODE and cursor down to Radian Degree. Highlight **Degree.** Press 2nd mode to return to home screen. To access the angle menu of the calculator press 2nd APPS and this screen will appear:



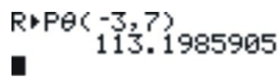
Cursor down to **5** and press ENTER. The following screen will appear

. Press **-3, 7)** ENTER and the value of *r* will appear



. Access the angle menu again by pressing 2nd APPS. When the angle menu screen appears, cursor down to **6** and pres ENTER or press **6** on the calculator. The screen

will appear. Press **-3, 7)** ENTER and the value of θ will appear.



This procedure can be repeated to determine the rectangular coordinates in radians. Before starting, press MODE and cursor down to Radian Degree and highlight **Radian**.

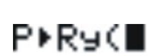
Example 8: Express the polar coordinates of $(300, 70^\circ)$ in rectangular form.

The angle θ is given in degrees so the mode menu of the calculator should also be set in degree. Therefore, press MODE and cursor down to Radian Degree and highlight **degree.** Press 2nd mode to return to home screen. To access the angle menu of the calculator press 2nd APPS and this screen will appear:

Cursor down to 7 and press ENTER or press 7 on the calculator. The following screen will screen will appear:

Press **300, 70)** and the value of *x* will appear

Access the angle menu again by pressing 2nd APPS. When the angle menu screen appears, cursor down to **8** and pres ENTER or press **8** on the calculator. The screen



will appear. Press **300,70)** $\boxed{\text{ENTER}}$ and the value of y will appear

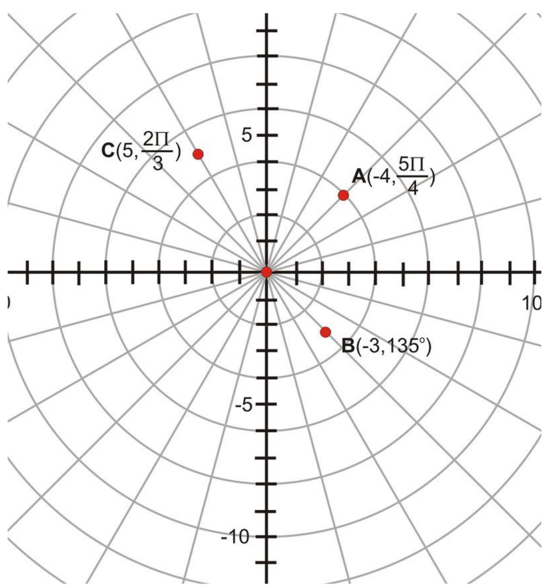
.

Points to Consider

- When we convert coordinates from polar form to rectangular form, the process is very straightforward. However, when converting a coordinate from rectangular form to polar form there are some choices to make. For example the point 0,1 could translate to $(1,2\pi)$ or to $(1,-4\pi)$, and so on.
- Are there any advantages to using polar coordinates instead of rectangular coordinates? List any situations in which this is the case. What types of curves are easier to draw with polar coordinates?
- List situations in which rectangular coordinates are preferable.

Review Questions

1. For the following polar coordinates that are shown on the graph, determine the rectangular coordinates for each point.



2. Write the following polar equations in rectangular form.
   a. $r = 6\cos\theta$
   b. $r\sin\theta = -3$
   c. $r = 2\sin\theta$
   d. $r\sin^2\theta = 3\cos\theta$
3. Write the following rectangular points in polar form.
   a. $A(-2,5)$ using radians
   b. $B(5,-4)$ using radians
   c. $C(1,9)$ using degrees

d. $D(-12,-5)$ using degrees

4. Write the rectangular equations in polar form.

a. $(x-4)^2+(y-3)^2=25$
b. $3x-2y=1$
c. $x^2+y^2-4x+2y=0$
d. $x^3=4y^2$

5.9 More with Polar Curves

Learning Objectives

- Graph polar curves to see the points of intersection of the curves.
- Graph equivalent polar curves.
- Recognize equivalent polar curves from their equations.

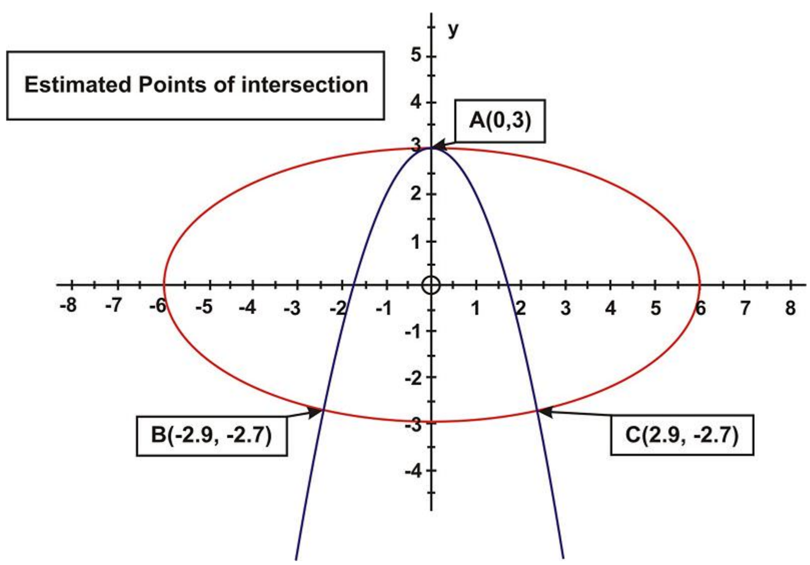
Intersections of Polar Curves

When you worked with a system of linear equations with two unknowns, finding the point of intersection of the equations meant finding the coordinates of the point that satisfied both equations. If the equations are rectangular equations for curves, determining the point(s) of intersection of the curves involves solving the equations algebraically since each point will have one ordered pair of coordinates associated with it.

Example 1: Solve the following system of equations algebraically:

$$\begin{aligned} x^2+4y^2-36&=0\\ x^2+y&=3 \end{aligned}$$

Solution: Before solving the system, graph the equations to determine the number of points of intersection.



The graph of $x^2+4y^2-36=0$ is an ellipse and the graph represented by $x^2+y=3$ is a parabola. There are three points of intersection. To determine the exact values of these points, algebra must be used.

$$x^2+4y^2-36=0 \rightarrow x^2+4y^2=36 \qquad x^2+4y^2+0y=36 \qquad x^2+4y^2+0y=36$$
$$x^2+y=3 \rightarrow x^2+0y^2+y=3 \qquad -1(x^2+0y^2+y=3) \qquad -x^2-0y^2-y=-3$$
$$4y^2-y=33$$

Using the quadratic formula, $a=4$ $b=-1$ $c=-33$

$$y=\frac{-(-1)\pm\sqrt{(-1)^2-4(4)(-33)}}{2(4)}$$
$$y=\frac{1+23}{8}=3 \qquad y=\frac{1-23}{8}=-2.75$$

These values must be substituted into one of the original equations.

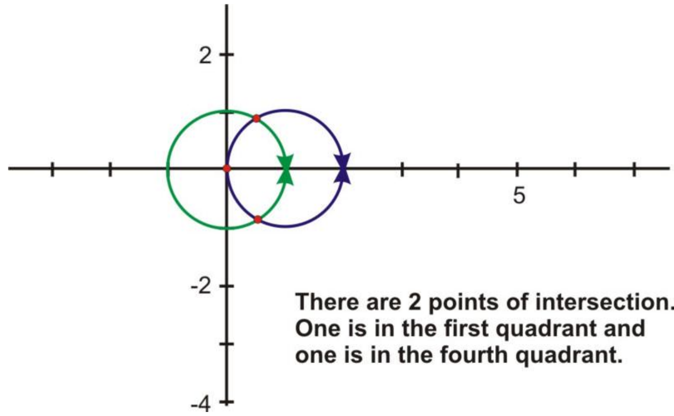
$$\begin{aligned} x^2+y&=3 \\ x^2+3&=3 \\ x^2&=0 \\ x&=0 \end{aligned} \qquad\qquad \begin{aligned} x^2+y&=3 \\ x^2+(-2.75)&=3 \\ x^2&=5.75 \\ x&=\pm\sqrt{5.75}\approx 2.4 \end{aligned}$$

The three points of intersection as determined algebraically in Cartesian representation are $\mathrm{A}(0,3), \mathrm{B}(2.4,-2.75)$ and $\mathrm{C}(2.4,2.75)$.

If we are working with polar equations to determine the polar coordinates of a point of intersection, we must remember that there are many polar coordinates that represent the same point. Remember that switching to polar form changes a great deal more than the notation. Unlike the Cartesian system which has one name for each point, the polar system has an infinite number of names for each point. One option would be to convert the polar coordinates to rectangular form and then to convert the coordinates for the intersection points back to polar form. Perhaps the best option would be to explore some examples. As these examples are presented, be sure to use your graphing calculator to create your own visual representations of the equations presented.

Example 2: Determine the polar coordinates for the intersection point(s) of the following polar equations: $r=1$ and $r=2\cos\theta$.

Solution: Begin with the graph. Using the process described in the technology section in this chapter; create the graph of these polar equations on your graphing calculator. Once the graphs are on the screen, use the **trace** function and the arrow keys to move the cursor around each graph. As the cursor is moved, you will notice that the equation of the curve is shown in the upper left corner and the values of θ, x, y are shown (in decimal form) at the bottom of the screen. The values change as the cursor is moved.



$$r = 1 \qquad\qquad 2\cos\theta = 1$$
$$r = 2\cos\theta \qquad\qquad \cos\theta = \frac{1}{2}$$
$$\cos^{-1}(\cos\theta) = \cos^{-1}\frac{1}{2}$$

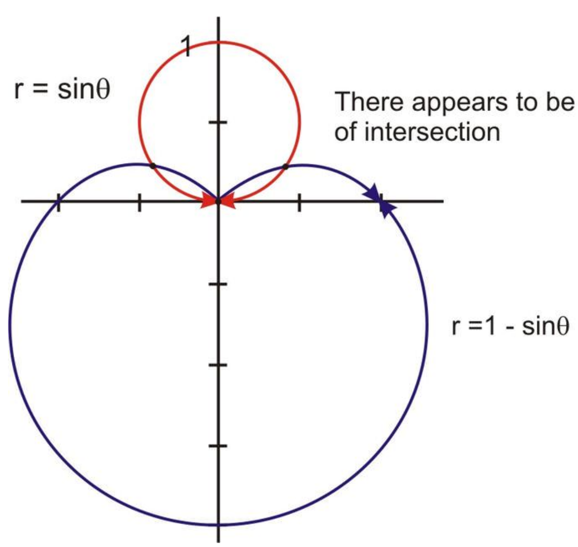
$\theta = \frac{\pi}{3}$ in the first quadrant and $\theta = \frac{5\pi}{3}$ in the fourth quadrant.

The points of intersection are $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right)$. However, these two solutions only cover the possible values $0 \le \theta \le 2\pi$. If you consider that $\cos\theta = \frac{1}{2}$ is true for an infinite number of theta these solutions must be extended to include $\left(1, \frac{\pi}{3}\right)$ and $\left(1, \frac{5\pi}{3}\right) + 2\pi k, k\varepsilon Z$. Now the solutions include all possible rotations.

This example was solved as any system of rectangular equations would be solved. Does this approach work all the time?

Example 3: Find the intersection of the graphs of $r = \sin\theta$ and $r = 1 - \sin\theta$

Solution: Begin with the graph. You can create these graphs using your graphing calculator.



$$\begin{array}{ll}
r = \sin\theta & \sin\theta = 1 - \sin\theta \\
r = 1 - \sin\theta & 2\sin\theta = 1 \\
 & \sin\theta = \frac{1}{2} \\
r = \sin\theta & \theta = \frac{\pi}{6} \text{ in the first quadrant and } \theta = \frac{5\pi}{6} \text{ in the second quadrant.} \\
r = \sin\frac{\pi}{6} & \text{The intersection points are } \left(\frac{1}{2}, \frac{\pi}{6}\right) \text{ and } \left(\frac{1}{2}, \frac{5\pi}{6}\right) \\
r = \frac{1}{2} & \text{Another intersection point seems to be the origin } (0,0).
\end{array}$$

If you consider that $\sin\theta = \frac{1}{2}$ is true for an infinite number of theta as was $\cos\theta = \frac{1}{2}$ in the previous example, the same consideration must be applied to include all possible solutions. To prove if the origin is indeed an intersection point, we must determine whether or not both curves pass through (0, 0).

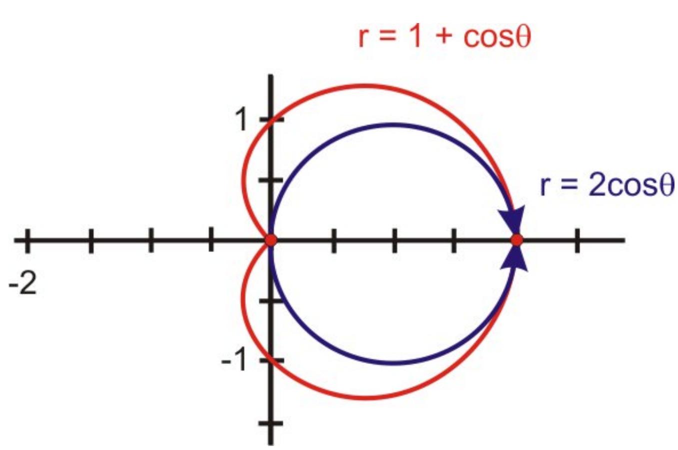
$$\begin{array}{ll}
r = \sin\theta & r = 1 - \sin\theta \\
0 = \sin\theta & 0 = 1 - \sin\theta \\
r = 0 & 1 = \sin\theta \\
 & \frac{\pi}{2} = 0
\end{array}$$

From this investigation, the point (0, 0) was on the curve $r = \sin\theta$ and the point $\left(0, \frac{\pi}{2}\right)$ was on the curve $r = 1 - \sin\theta$. Because the second coordinates are different, it seems that they are two different points. However, the coordinates represent the same point (0,0). The intersection points are $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and (0,0).

Sometimes it is helpful to convert the equations to rectangular form, solve the system and then convert the polar coordinates back to polar form.

Example 4: Find the intersection of the graphs of $r = 2\cos\theta$ and $r = 1 + \cos\theta$

Solution: Begin with the graph:



$r = 2\cos\theta$ expressed in rectangular form

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta && \textit{Multiply by } r \\ x^2+y^2 &= 2x && \textit{Substitution} \end{aligned}$$

$r = 1 + cos\theta$ expressed in rectangular form

$$\begin{aligned} r &= 1+\cos\theta \\ r^2 &= r + r\cos\theta && \textit{Multiply by } r \\ x^2+y^2 &= \sqrt{x^2+y^2}+x && \textit{Substitution} \end{aligned}$$

The equations are now in rectangular form. Solve the system of equations.

$$\begin{aligned} x^2+y^2 &= 2x \\ x^2+y^2 &= \sqrt{x^2+y^2}+x \\ 2x &= \sqrt{2x}+x \\ x &= \sqrt{2x} \\ x^2 &= 2x \\ x^2-2x &= 0 \\ x(x-2) &= 0 \\ x &= 0 && x-2=0 \\ & && x=2 \end{aligned}$$

Substituting these values into the first equation:

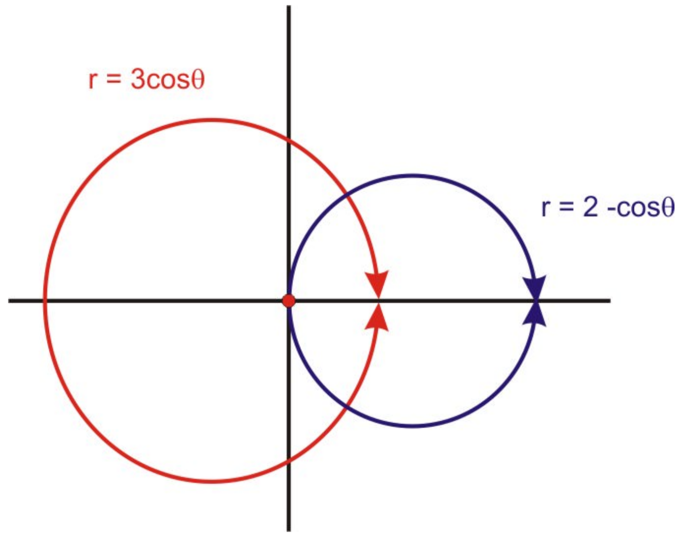
$$\begin{aligned} x^2+y^2 &= 2x & x^2+y^2 &= 2x \\ (0)^2+y^2 &= 2(0) & (2)^2+y^2 &= 2(2) \\ y^2 &= 0 & 4+y^2 &= 4 \\ y &= 0 & y^2 &= 0 \\ & & y &= 0 \end{aligned}$$

The points of intersection are $(0,0)$ and $(2,0)$.

The rectangular coordinates are $(0,0)$ and $(2,0)$. Converting these coordinates to polar coordinates give the same coordinates in polar form. The points can be converted by using the angle menu of the TI calculator.

Example 5: Josie is drawing a mural with polar equations. One mural is represented by the equation $r = 3\cos\theta$ and the other by $r = 2 - \cos\theta$. She wants to see where they will intersect before she transfers her image onto the wall where she is painting.

Solution: To determine where they will intersect, we will begin with a graph.



$$\begin{aligned} r &= 3\cos\theta \\ r &= 2-\cos\theta \\ 3\cos\theta &= 2-\cos\theta \\ 3\cos\theta+\cos\theta &= 2 \\ 4\cos\theta &= 2 \\ \cos\theta &= \frac{2}{4}=\frac{1}{2} \\ \theta &= \frac{\pi}{3} \quad \text{and} \quad \theta=\frac{5\pi}{3} \end{aligned}$$

$$\begin{aligned} r &= 3\cos\theta \\ r &= 3\cos\frac{\pi}{3} \\ r &= 3\cdot\frac{1}{2}=\frac{3}{2} \end{aligned} \qquad \begin{aligned} r &= 3\cos\theta \\ r &= 3\cos\frac{5\pi}{3} \\ r &= 3\cdot\frac{1}{2}=\frac{3}{2} \end{aligned}$$

Josie's murals would intersect and two points $\left(\frac{3}{2},\frac{\pi}{3}\right)$ and $\left(\frac{3}{2},\frac{5\pi}{3}\right)$.

Equivalent Polar Curves

The expression "same only different" comes into play in this lesson. We will graph two distinct polar equations that will produce two equivalent graphs. Use your graphing calculator and create these curves as the equations are presented.

Previously, graphs were generated of a limaçon, a dimpled limaçon, a looped limaçon and a cardioid. All of these were of the form $r=a\pm b\sin\theta$ or $r=a\pm b\cos\theta$. The easiest way to see what polar equations produce equivalent curves is to use either a graphing calculator or a software program to generate the graphs of various polar equations.

Example 6: Plot the following polar equations and compare the graphs.

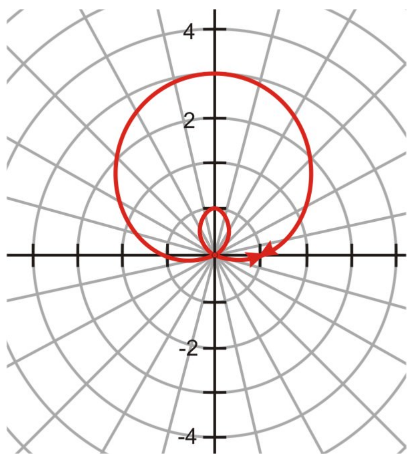
a)

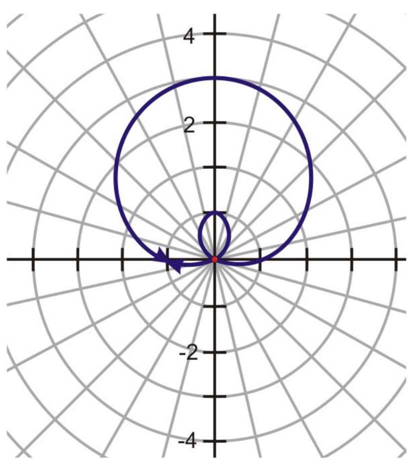
$$r = 1 + 2\sin\theta$$
$$r = -1 + 2\sin\theta$$

b)

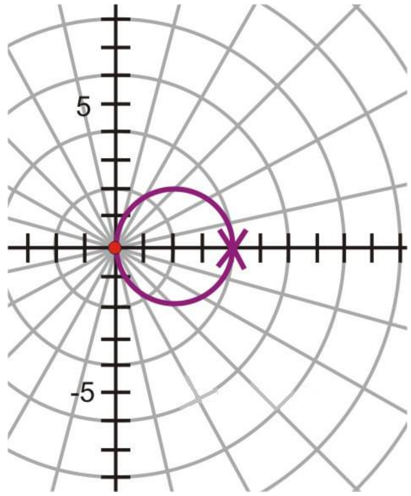
$$r = 4\cos\theta$$
$$r = 4\cos(-\theta)$$

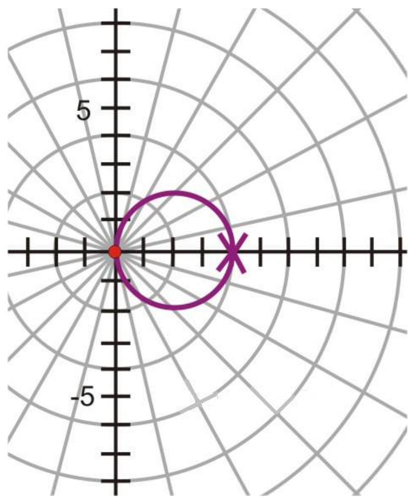
Solution: By looking at the graphs, the result is the same. So, even though a is different in both, they have the same graph. We can assume that the sign of a does not matter.





b) These functions also result in the same graph. Here, θ differed by a negative. So we can assume that the sign of θ does not change the appearance of the graph.

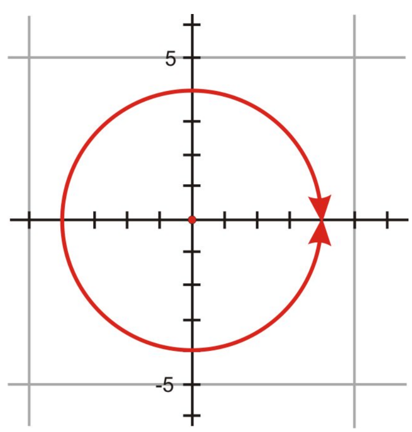


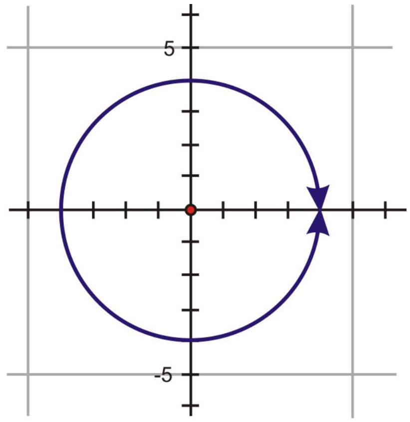


Example 7:

Graph the equations $x^2 + y^2 = 16$ and $r = 4$. Describe the graphs.

Solution:





Both equations, one in rectangular form and one in polar form, are circles with a radius of 4 and center at the origin.

Example 8: Graph the equations $(x-2)^2+(y+2)^2=8$ and $r=4\cos\theta-4\sin\theta$. Describe the graphs.

Solution: There is not a visual representation shown here, but on your calculator you should see that the graphs are circles centered at (2, -2) with a radius $2\sqrt{2}\approx 2.8$.

Points to Consider

- When looking for intersections, which representation is easier to work with? Look over the examples and find some in which doing the algebra in polar coordinates is more direct than finding intersections in Cartesian form.
- Will polar curves always intersect?
- If not, when will intersection not occur?
- If two polar curves have different equations, can they be the same curve?

Review Questions

1. Find the intersection of the graphs of $r=\sin 3\theta$ and $r=3\sin\theta$.
2. Find the intersection of the graphs of $r=2+2\sin\theta$ and $r=2-2\cos\theta$
3. Write the rectangular equation $x^2+y^2=6x$ in polar form and graph both equations. Should they be equivalent?
4. Determine if $r=-2+\sin\theta$ and $r=2-\sin\theta$ are equivalent *without* graphing.
5. Determine if $r=-3+4\cos(-\pi)$ and $r=3+4\cos\pi$ are equivalent *without* graphing.
6. Graph the equations $r=7-3\cos\frac{\pi}{3}$ and $r=7-3\cos\left(-\frac{\pi}{3}\right)$. Are they equivalent?
7. Formulate a theorem about equivalent polar curves resulting from the equations $r=a\pm b\cos\theta$ or $r=a\pm b\sin\theta$. What can be different to yield the same graph? What must be the same? Explain your answer and show graphs to support your conclusions.
8. Determine two polar curves that will never intersect.

5.10 Parameters and Parameter Elimination

Learning Objectives

Here you will represent equations and graphs in a new way with parametric equations.

In a **parametric equation**, the variables x and y are not dependent on one another. Instead, both variables are dependent on a third variable, t. This is the **parameter** or a number that affects the behavior of the equation. Usually t will stand for time. A real world example of the relationship between x, y and t is the height, weight and age of a baby.

Both the height and the weight of a baby depend on time, but there is also clearly a positive relationship between just the height and weight of the baby. By focusing on the relationship between the height and the weight and letting time hide in the background, you create a parametric relationship between the three variables.

What other types of real world situations are modeled with parametric equations?

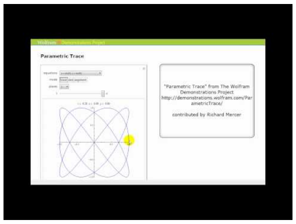
Eliminating the Parameter

In your graphing calculator there is a parametric mode. Once you put your calculator into parametric mode, on the graphing screen you will no longer see $y = \_\_$, instead, you will see:

Notice how for plot one, the calculator is asking for two equations based on variable T:

$$x_{1T} = f(t)$$
$$y_{1T} = g(t)$$

This is called parametric form. **Parametric form** refers to a relationship that includes $x = f(t)$ and $y = g(t)$. In order to transform a parametric equation into a normal one, you need to do a process called "eliminating the parameter." **"Eliminating the parameter"** is a phrase that means to turn a parametric equation that has $x = f(t)$ and $y = g(t)$ into just a relationship between y and x. You are eliminating t. To do this, you must solve the $x = f(t)$ equation for $t = f^{-1}(x)$ and substitute this value of t into the y equation. This will produce a normal function of y based on x.



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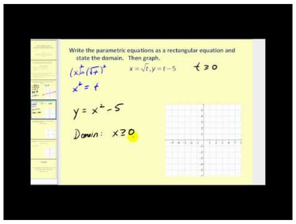

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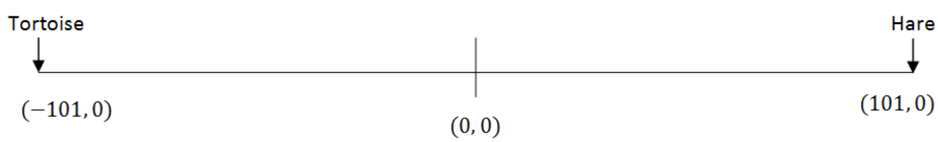
There are two major benefits of graphing in parametric form. First, it is straightforward to graph a portion of a regular function using the T_{min}, T_{max} and T_{step} in the window setting. Second, parametric form enables you to graph projectiles in motion and see the effects of time.

A tortoise and a hare start 202 feet apart and then race to a flag halfway between them. The hare decides to take a nap and give the tortoise a 21 second head start. The hare runs at 9.8 feet per second and the tortoise hustles along at 3.2 feet per second. This situation can be represented by parametric equations and we can use the equations to determine who wins this epic race and by how much.

First draw a picture and then represent each character with a set of parametric equations.



The tortoise's position is (-101, 0) at $t = 0$ and (-97.8, 0) at $t = 1$. You can deduce that the equation modeling the tortoise's position is:

$$\begin{aligned} x_1 &= -101 + 3.2 \cdot t \\ y_1 &= 0 \end{aligned}$$

The hare's position is (101, 0) at $t = 21$ and (91.2, 0) at $t = 22$. Note that it does not make sense to make equations modeling the hare's position before 21 seconds have elapsed because the Hare is napping and not moving. You can set up an equation to solve for the hare's theoretical starting position had he been running the whole time.

$$\begin{aligned} x_2 &= b - 9.8t \\ 101 &= b - 9.8 \cdot 21 \\ 305.8 &= b \end{aligned}$$

The hare's position equation after $t = 21$ can be modeled by:

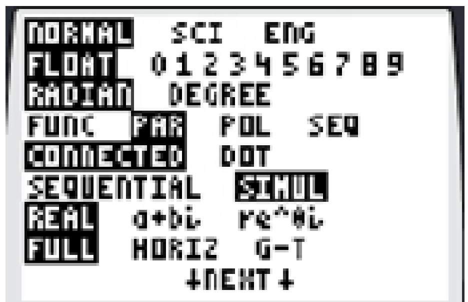
$$x_2 = 305.8 - 9.8 \cdot t$$
$$y_2 = 0$$

The tortoise crosses $x = 0$ when $t \approx 31.5$. The hare crosses $x = 0$ when $t \approx 31.2$. The hare wins by about 1.15 feet.

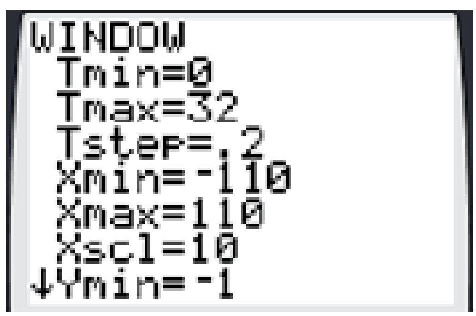
Now, use your calculator to display these parametric equations. T

here are many settings you should know for parametric equations that bring questions like this to life. The TI-84 has features that allow you to see the race happen.

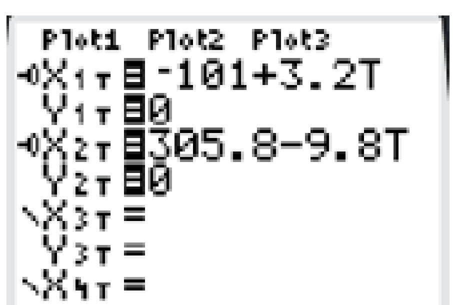
First, set the mode to simultaneous graphing. This will show both the tortoise and hare's position at the same time.



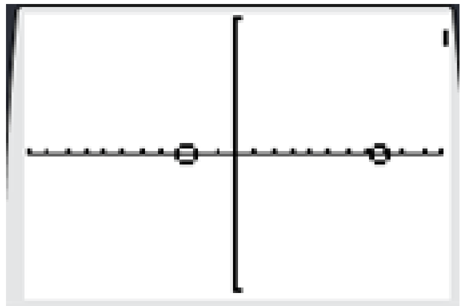
Next, change the graphing window so that t varies between 0 and 32 seconds. The T_{step} determines how often the calculator will calculate points. The larger the T_{step}, the faster and less accurately the graph will plot. Also change the x to vary between -110 and 110 so you can see the positions of both characters.

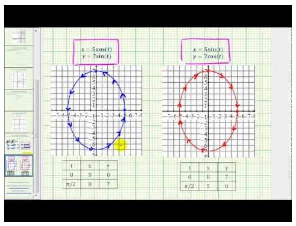


Input the parametric equations. Toggle to the left of the x and change the cursor from a line to a line with a bubble at the end. This shows their position more clearly.



Now when you graph you should watch the race unfold as the two position graphs race towards each other.





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Examples

Example 1

Earlier, you were asked what types of real world situations can be modeled by parametric equations. Parametric equations are often used when only a portion of a graph is useful. By limiting the domain of t, you can graph the precise interval of the function you want. Parametric equations are also useful when two different variables jointly depend on a third variable and you wish to look at the relationship between the two dependent variables. This is very common in statistics where an underlying variable may actually be the cause of a problem and the observer can only examine the relationship between the outcomes that they see. In the physical world, parametric equations are exceptional at graphing position over time because the horizontal and vertical vectors of objects in free motion are each dependent on time, yet independent of one another.

Example 2

Eliminate the parameter in the following equations.

$$x = 6t - 2$$
$$y = 5t^2 - 6t$$

$x = 6t - 2$ So $\frac{x+2}{6} = t$. Now, substitute this value for t into the second equation:

$y = 5\left(\frac{x+2}{6}\right)^2 - 6\left(\frac{x+2}{6}\right)$

Example 3

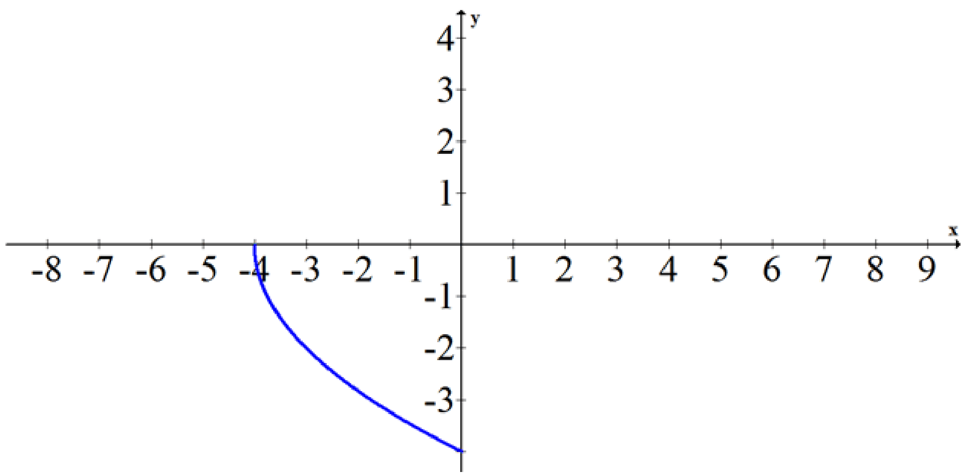
For the given parametric equation, graph over each interval of t.

$x = t^2 - 4$

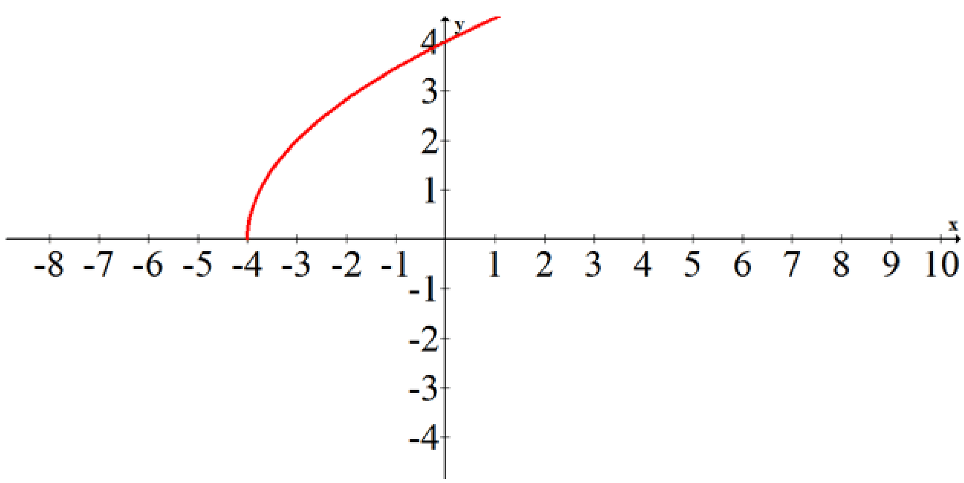
$y = 2t$

1. $-2 \leq t \leq 0$
2. $0 \leq t \leq 5$
3. $-3 \leq t \leq 2$

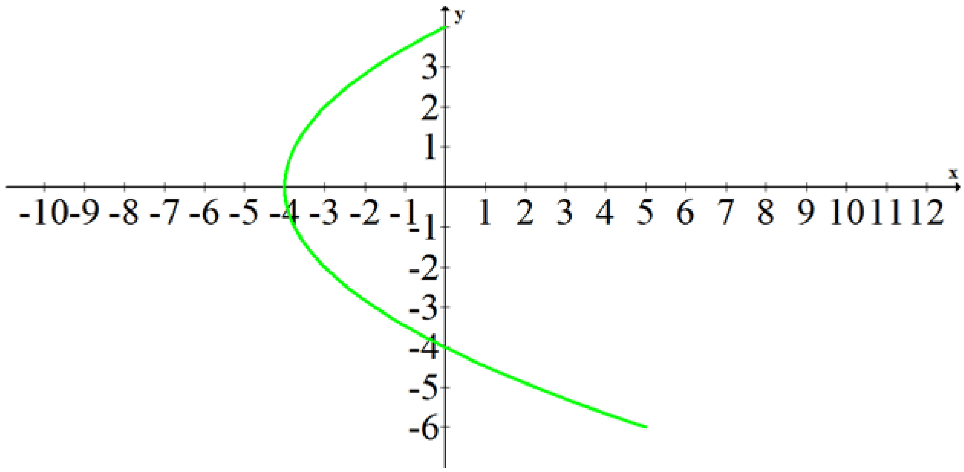
a. A good place to start is to find the coordinates where t indicates the graph will start and end. For $-2 \leq t \leq 0$, $t = -2$ and $t = 0$ indicate that the points (0, -4) and (-4, 0) are the endpoints of the graph.



b. $0 \leq t \leq 5$



c. $-3 \leq t \leq 2$



Example 4

Eliminate the parameter and graph the following parametric curve.

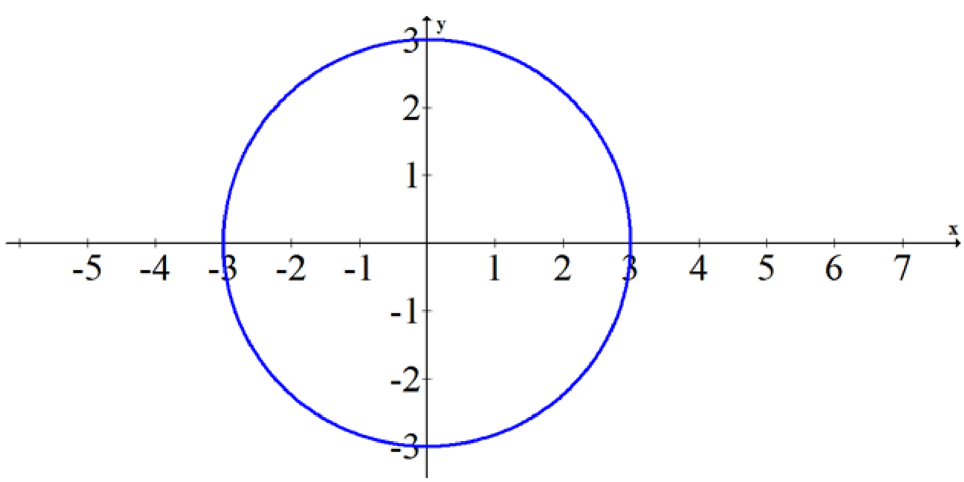
$x = 3 \cdot \sin t$

$y = 3 \cdot \cos t$

When parametric equations involve trigonometric functions you can use the Pythagorean Identity, $\sin^2 t + \cos^2 t = 1$. In this problem, $\sin t = \frac{x}{3}$ (from the first equation) and $\cos t = \frac{y}{3}$ (from the second equation). Substitute these values into the Pythagorean Identity and you have:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$x^2 + y^2 = 9$$

This is a circle centered at the origin with radius 3.



Example 5

Find the parameterization for the line segment connecting the points (1, 3) and (4, 8).

Use the fact that a point plus a vector yields another point. A vector between these points is $< 4-1, 8-3 > = < 3, 5 >$

Thus the point (1, 3) plus t times the vector $< 3,5 >$ will produce the point (4, 8) when $t = 1$ and the point (1, 3) when $t = 0$.

$(x,y) = (1,3) + t \cdot < 3,5 >, \text{ for } 0 \leq t \leq 1$

You than then break up this vector equation into parametric form.

$$x = 1 + 3t$$
$$y = 3 + 5t$$
$$0 \leq t \leq 1$$

Review

Eliminate the parameter in the following sets of parametric equations.

1. $x = 3t - 1;\ y = 4t^2 - 2t$

2. $x = 3t^2 + 6t;\ y = 2t - 1$

3. $x = t + 2;\ y = t^2 + 4t + 4$

4. $x = t - 5;\ y = t^3 + 1$

5. $x = t + 4;\ y = t^2 - 5$

For the parametric equation $x = t, y = t^2 + 1$, graph over each interval of t.

6. $-2 \leq t \leq -1$

7. $-1 \leq t \leq 0$

8. $-1 \leq t \leq 1$

9. $-2 \leq t \leq 2$

10. $-5 \leq t \leq 5$

11. Eliminate the parameter and graph the following parametric curve: $x = \sin t,\ y = -4 + 3\cos t$.

12. Eliminate the parameter and graph the following parametric curve: $x = 1 + 2\cos t,\ y = 1 + 2\sin t$.

13. Using the previous problem as a model, find a parameterization for the circle with center (2, 4) and radius 3.

14. Find the parameterization for the line segment connecting the points (2, 7) and (1, 4).

15. Find a parameterization for the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Use the fact that $\cos^2 t + \sin^2 t = 1$. Check your answer with your calculator.

16. Find a parameterization for the ellipse $\frac{(x-4)^2}{9} + \frac{(y+1)^2}{36} = 1$. Check your answer with your calculator.

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.3.

5.11 Applications of Parametric Equations

Learning Objectives

Here you will use parametric equations to represent the vertical and horizontal motion of objects over time.

A regular function has the ability to graph the height of an object over time. Parametric equations allow you to actually graph the complete position of an object over time. For example, parametric equations allow you to make a graph that represents the position of a point on a Ferris wheel. All the details like height off the ground, direction, and speed of spin can be modeled using the parametric equations.

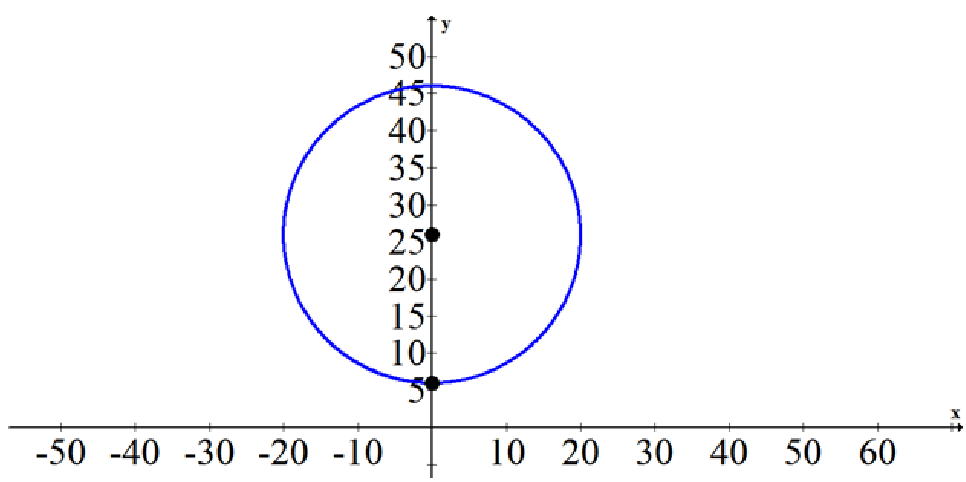
What is the position equation and graph of a point on a Ferris wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds?

Applying Parametric Equations

There are two types of parametric equations that are typical in real life situations. The first is circular motion as was described in the concept problem. The second is projectile motion.

Circular Motion

Parametric equations that describe **circular motion** will have x and y as periodic functions of sine and cosine. Either x will be a sine function and y will be a cosine function or the other way around. The best way to come up with parametric equations is to first draw a picture of the circle you are trying to represent.



Next, it is important to note the starting point, center point and direction. You should already have the graphs of sine and cosine memorized so that when you see a pattern in words or as a graph, you can identify what you see as $+\sin, -\sin, +\cos, -\cos$.

Take the example given above with the Ferris Wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds. The the vertical component starts at a low point of 6, travels to a middle point of 26 and then a height of 46 and back down. This is a $-\cos$ pattern. The amplitude of the $-\cos$ is 20 and the vertical shift is 26. Lastly, the period is 60. You can use the period to help you find b.

$$60 = \frac{2\pi}{b}$$
$$b = \frac{\pi}{30}$$

Thus the vertical parameterization is:

$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$

The horizontal parameterization is found by noticing that the x values start at 0, go up to 20, go back to 0, then down to -20, and finally back to 0. This is a $+\sin$ pattern with amplitude 20. The period is the same as with the vertical component.

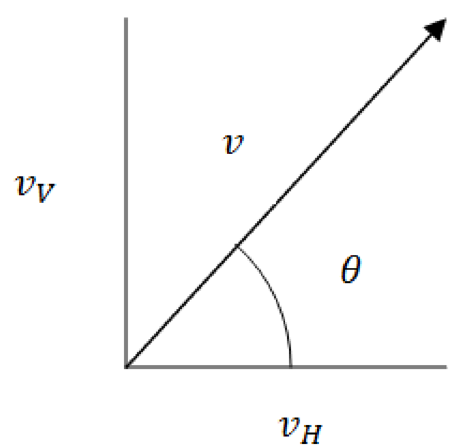
Thus parametric equations for the point on the wheel are:

$$x = 20\sin\left(\frac{\pi}{30}t\right)$$
$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$$

Note that **horizontal and vertical components** of parametric equations are the $x =$ and $y =$ functions respectively or the horizontal and vertical parameterization.

Projectile Motion

Projectile motion has a vertical component that is quadratic and a horizontal component that is linear. This is because there are 3 parameters that influence the position of an object in flight: starting height, initial velocity, and the force of gravity. The horizontal component is independent of the vertical component. This means that the starting horizontal velocity will remain the horizontal velocity for the entire flight of the object.



Note that gravity, g, has a force of about $-32\ ft/s^2$ or $-9.81\ m/s^2$. The examples and practice questions in this concept will use feet.

If an object is launched from the origin at a velocity of v then it has horizontal and vertical components that can be found using basic trigonometry.

$$\sin\theta = \frac{v_V}{v} \rightarrow v \cdot \sin\theta = v_V$$
$$\cos\theta = \frac{v_H}{v} \rightarrow v \cdot \cos\theta = v_H$$

The horizontal component is basically finished. The only adjustments that would have to be made are if the starting location is not at the origin, wind is added or if the projectile travels to the left instead of the right. See Example A.

$x = t \cdot v \cdot \cos\theta$

The vertical component also needs to include gravity and the starting height. The general equation for the vertical component is:

$y = \frac{1}{2} \cdot g \cdot t^2 + t \cdot v \cdot \sin\theta + k$

The constant g represents gravity, t represents time, v represents initial velocity and k represents starting height. You will explore this equation further in calculus and physics. Note that in this concept, most answers will be found and confirmed using technology such as your graphing calculator.

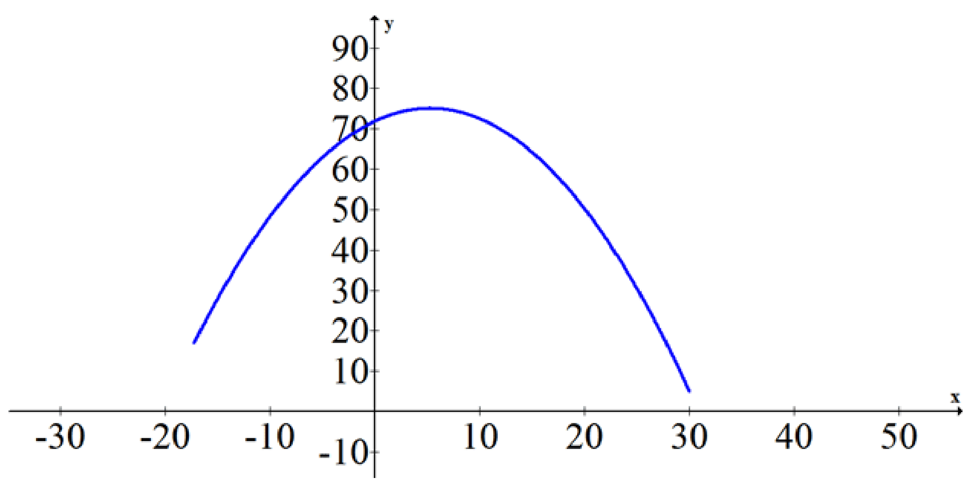
Examples

Example 1

A ball is thrown from the point (30, 5) at an angle of $\frac{4\pi}{9}$ to the left at an initial velocity of 68 ft/s. Model the position of the ball over time using parametric equations. Use your graphing calculator to graph your equations for the first four seconds while the ball is in the air.

The horizontal component is $x = -t \cdot 68 \cdot \cos(\frac{4\pi}{9}) + 30$. Note the negative sign because the object is traveling to the left and the +30 because the object starts at (30, 5).

The vertical component is $y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 68 \cdot \sin(\frac{4\pi}{9}) + 5$. Note that $g = -32$ because gravity has a force of -32 ft/s^2 and the $+5$ because the object starts at (30, 5).



Example 2

When does the ball from Example 1 reach its maximum and when does the ball hit the ground? How far did the person throw the ball?

To find when the function reaches its maximum, you can find the vertex of the parabola. Analytically this is messy because of the decimal coefficients in the quadratic. Use your calculator to approximate the maximum after you have graphed it. Depending on how small you make your T_{step} should find the maximum height to be about 75 feet.

To find out when the ball hits the ground, you can set the vertical component equal to zero and solve the quadratic equation. You can also use the table feature on your calculator to determine when the graph goes from having a positive vertical value to a negative vertical value. The benefit for using the table is that it simultaneously tells you the x value of the zero.

After about 4.2588 seconds the ball hits the ground at (-20.29, 0). This means the person threw the ball from (30, 5) to (-20.29, 0), a horizontal distance of just over 50 feet.

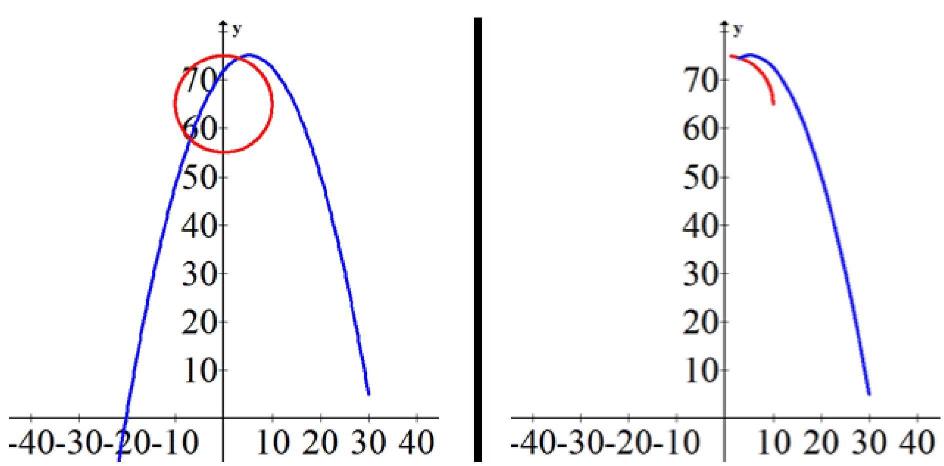
Example 3

Kieran is on a Ferris wheel and his position is modeled by the parametric equations:

$$x_K = 10 \cdot \cos\left(\frac{\pi}{5}t\right)$$
$$y_K = 10 \cdot \sin\left(\frac{\pi}{5}t\right) + 65$$

Jason throws the ball modeled by the equation in Example 1 towards Kieran who can catch the ball if it gets within three feet. Does Kieran catch the ball?

This question is designed to demonstrate the power of your calculator. If you simply model the two equations simultaneously and ignore time you will see several points of intersection. This graph is shown below on the left. These intersection points are not interesting because they represent where Kieran and the ball are at the same place but at different moments in time.



When the T_{max} is adjusted to 2.3 so that each graph represents the time from 0 to 2.3, you get a better sense that at about 2.3 seconds the two points are close. This graph is shown above on the right.

You can now use your calculator to help you determine if the distance between Kieran and the ball actually does go below 3 feet. Start by plotting the ball's position in your calculator as x_1 and y_1 and Kieran's position as x_2 and y_2. Then, plot a new parametric equation that compares the distance between these two points over time. You can put this under x_3 and y_3.A calculator can reference **internal variables** like x_1, y_1 that have already been set in the calculator's memory to form new variables like x_3, y_3. Note that you can find the x_1, x_2, y_1, y_2 entries in the vars and parametric menu.

$$x_3 = t$$
$$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Now when you graph, you should change your window settings and let t vary between 0 and 4, the x window show between 0 and 4 and the y window show between 0 and 5. This way it should be clear if the distance truly does get below 3 feet.

Depending on how accurate your T_{step} is, you should find that the distance is below 3 feet. Kieran does indeed catch the ball.

Example 4

At what velocity does a football need to be thrown at a 45° angle in order to make it all the way across a football field?

A football field is 100 yards or 300 feet. The parametric equations for a football thrown from (300, 0) back to the origin at speed v are:

$$x = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot sin\left(\frac{\pi}{4}\right)$$

Substituting the point (0, 0) in for (x, y) produces a system of two equations with two variables v, t.

$$0 = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$0 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot \sin\left(\frac{\pi}{4}\right)$$

You can solve this system many different ways.

$t = \frac{5\sqrt{3}}{2} \approx 4.3\ seconds, v = 40\sqrt{6} \approx 97.98\ ft/s$

In order for someone to throw a football at a 45° angle all the way across a football field, they would need to throw at about 98 ft/s which is about 66.8 mph.

$\frac{98\ feet}{1\ sec} \cdot \frac{3600\ sec}{1\ hour} \cdot \frac{1\ mile}{5280\ feet} \approx \frac{66.8\ miles}{1\ hour}$

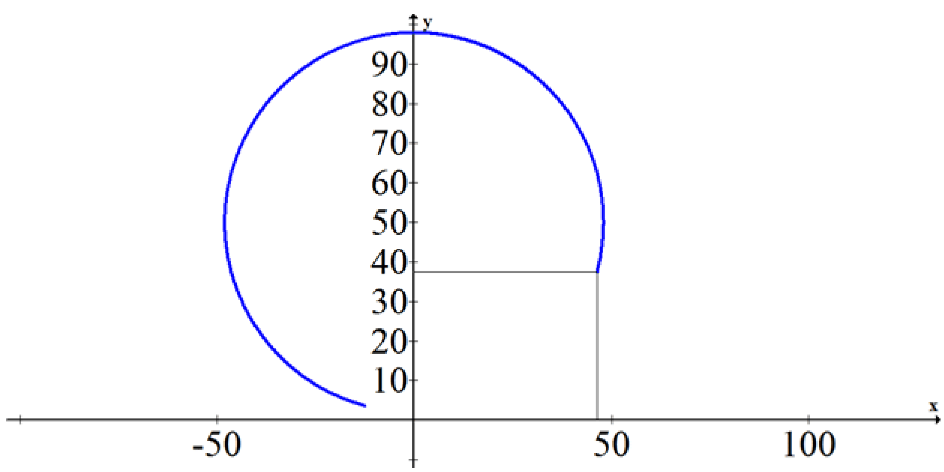
Example 5

Nikki got on a Ferris wheel ten seconds ago. She started 2 feet off the ground at the lowest point of the wheel and will make a complete cycle in four minutes. The ride reaches a maximum height of 98 feet and spins clockwise. Write parametric equations that model Nikki's position over time. Where will Nikki be three minutes from now?

Don't let the 10 second difference confuse you. In order to deal with the time difference, use $\left(t+\frac{1}{6}\right)$ instead of t in each equation. When $t = 0$, ten seconds $\left(\frac{1}{6} \text{ of a minute}\right)$ have already elapsed.

$$x = -48 \cdot \sin\left(\frac{\pi}{2}\left(t+\frac{1}{6}\right)\right)$$
$$y = -48 \cdot \cos\left(\frac{\pi}{2}\left(t+\frac{1}{6}\right)\right) + 50$$

At $t = 3, x \approx 46.36$ and $y \approx 37.58$



Review

Candice gets on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 103 feet, making a complete cycle in 5 minutes.

1. Write a set of parametric equations to model Candice's position.
2. Where will Candice be in two minutes?
3. Where will Candice be in four minutes?

One minute ago Guillermo got on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 83 feet, making a complete cycle in 6 minutes.

4. Write a set of parametric equations to model Guillermo's position.

5. Where will Guillermo be in two minutes?

6. Where will Guillermo be in four minutes?

Kim throws a ball from (0, 5) to the right at 50 mph at a 45° angle.

7. Write a set of parametric equations to model the position of the ball.

8. Where will the ball be in 2 seconds?

9. How far does the ball get before it lands?

David throws a ball from (0, 7) to the right at 70 mph at a 60° angle. There is a 6 mph wind in David's favor.

10. Write a set of parametric equations to model the position of the ball.

11. Where will the ball be in 2 seconds?

12. How far does the ball get before it lands?

Suppose Riley stands at the point (250, 0) and launches a football at 72 mph at an angle of 60° towards Kristy who is at the origin. Suppose Kristy also throws a football towards Riley at 65 mph at an angle of 45° at the exact same moment. There is a 6 mph breeze in Kristy's favor.

13. Write a set of parametric equations to model the position of Riley's ball.

14. Write a set of parametric equations to model the position of Kristy's ball.

15. Graph both functions and explain how you know that the footballs don't collide even though the two graphs intersect.

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.5.

5.12 The Ambiguous Case

Learning Objectives

- Find possible triangles given two sides and an angle (SSA).
- Use the Law of Cosines and Sines in various ambiguous cases.

In previous sections, we learned about the Law of Cosines and the Law of Sines. We learned that we can use the Law of Cosines when:

1. we know all three sides of a triangle (SSS) and
2. we know two sides and the included angle (SAS).

We learned that we can use the Law of Sines when:

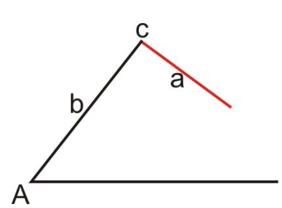
1. we know two angles and a non-included side (AAS) and
2. we know two angles and the included side (ASA).

However, we have not explored how to approach a triangle when we know two sides and a *non*-included angle (SSA). In this section, we will look at why the SSA case is called the ambiguous case, the possible triangles formed by the SSA case, and how to apply the Law of Sines and the Law of Cosines when we encounter the SSA case.

Possible Triangles with SSA

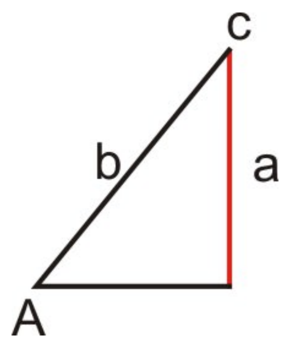
In Geometry, you learned that two sides and a non-included angle do not necessarily define a unique triangle. Consider the following cases given a, b, and $\angle A$:

Case 1: No triangle exists $(a < b)$



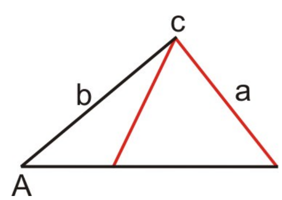
In this case $a < b$ and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2: One triangle exists $(a < b)$



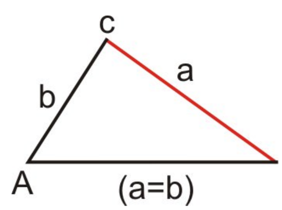
In this case, $a < b$ and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3: Two triangles exist $(a < b)$



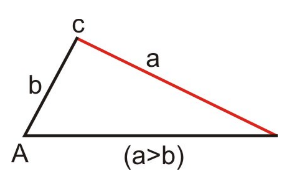
In this case, $a < b$ and side a meets the base at exactly two points. Since two triangles exist, there are two solutions.

Case 4: One triangle exists $(a = b)$



In this case $a = b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5: One triangle exists $(a > b)$



In this case, $a > b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous Case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b. If you are faced with the first situation, where $a < b$, we can still tell how many solutions there will be by using a and $b \sin A$.

TABLE 5.6:

	If:	Then:
a.	$a < b$	No solution, one solution, two solutions
i.	$a < b\sin A$	No solution
ii.	$a = b\sin A$	One solution
iii.	$a > b\sin A$	Two solutions
b.	$a = b$	One solution
c.	$a > b$	One solution

Example 1: Determine if the sides and angle given determine no, one or two triangles. All sets contain an angle, its opposite side and the side between them.

a. $a = 5, b = 8, A = 62.19^\circ$

b. $c = 14, b = 10, B = 15.45^\circ$

c. $d = 16, g = 11, D = 44.94^\circ$

d. $a = 9, b = 7, B = 51.06^\circ$

Solution: Even though a, b and $\angle A$ are not used in every example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the angle.

a. $5 < 8, 8\sin 62.19^\circ = 7.076$. So $5 < 7.076$, which means there is no solution.

b. $10 < 14, 14\sin 15.45^\circ = 3.73$. So $10 > 3.73$, which means there are two solutions.

c. $16 > 11$, there is one solution.

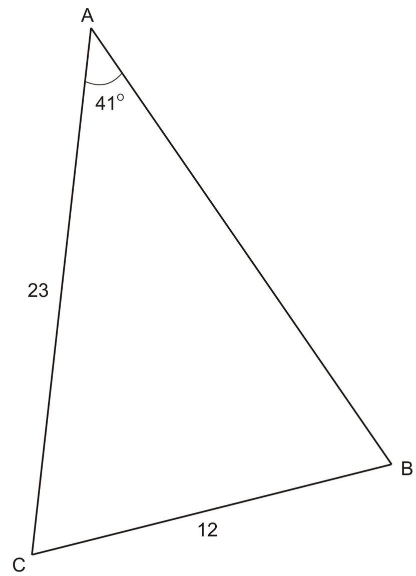
d. $7 < 9, 9\sin 51.06^\circ = 7.00$. So $7 = 7$, which means there is one solution.

In the next two sections we will look at how to use the Law of Cosines and the Law of Sines when faced with the various cases above.

Using the Law of Sines

In triangle ABC below, we know two sides and a non-included angle. Remember that the Law of Sines states: $\frac{\sin A}{a} = \frac{\sin B}{b}$. Since we know a, b, and $\angle A$, we can use the Law of Sines to find $\angle B$. However, since this is the SSA case, we have to watch out for the Ambiguous case. Since $a < b$, we could be faced with either Case 1, Case 2, or Case 3 above.

Example 2: Find $\angle B$.



Solution: Use the Law of Sines to determine the angle.

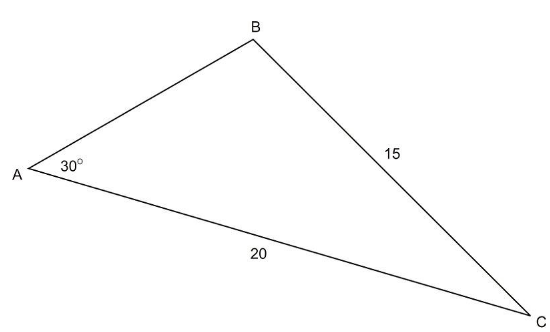
$$\begin{aligned}
\frac{\sin 41}{12} &= \frac{\sin B}{23} \\
23 \sin 41 &= 12 \sin B \\
\frac{23 \sin 41}{12} &= \sin B \\
1.257446472 &= \sin B
\end{aligned}$$

Since no angle exists with a sine greater than 1, there is no solution to this problem.

We also could have compared a and $b \sin A$ beforehand to see how many solutions there were to this triangle.

$a = 12, b \sin A = 15.1$: since $12 < 15.1, a < b \sin A$ which tells us there are no solutions.

Example 3: In triangle $ABC, a = 15, b = 20$, and $\angle A = 30°$. Find $\angle B$.



Solution: Again in this case, $a < b$ and we know two sides and a non-included angle. By comparing a and $b \sin A$, we find that $a = 15, b \sin A = 10$. Since $15 > 10$ we know that there will be two solutions to this problem.

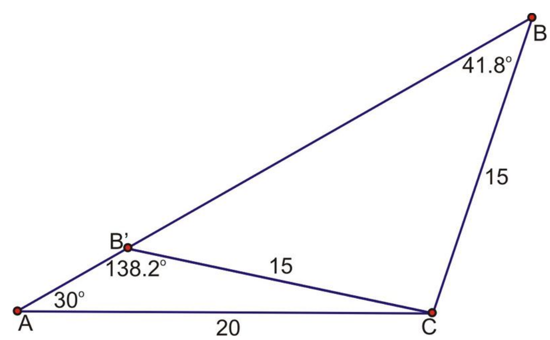
$$\begin{aligned}
\frac{\sin 30}{15} &= \frac{\sin B}{20} \\
20 \sin 30 &= 15 \sin B \\
\frac{20 \sin 30}{15} &= \sin B \\
0.6666667 &= \sin B \\
\angle B &= 41.8^\circ
\end{aligned}$$

There are two angles less than 180° with a sine of 0.6666667, however. We found the first one, 41.8°, by using the inverse sine function. To find the second one, we will subtract 41.8° from 180°, $\angle B = 180^\circ - 41.8^\circ = 138.2^\circ$.

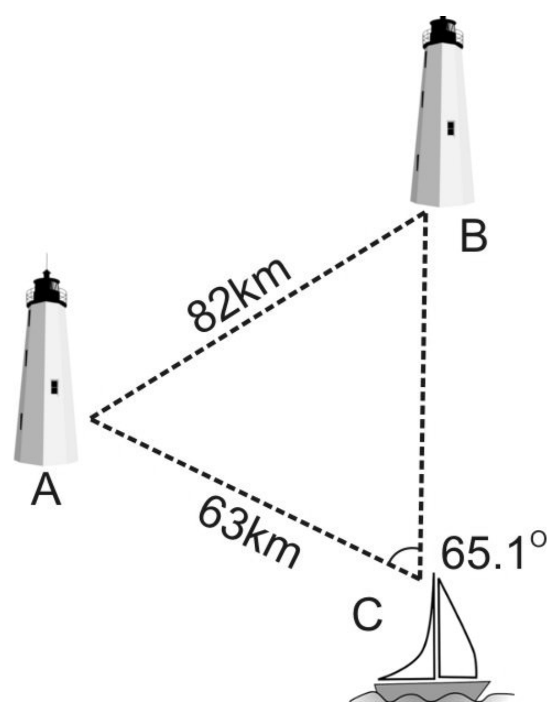
To check to make sure 138.2° is a solution, we will use the Triangle Sum Theorem to find the third angle. Remember that all three angles must add up to 180°.

$$180^\circ - (30^\circ + 41.8^\circ) = 108.2^\circ \qquad \text{or} \qquad 180^\circ - (30^\circ + 138.2^\circ) = 11.8^\circ$$

This problem yields two solutions. Either $\angle B = 41.8^\circ$ or 138.2°.



Example 4: A boat leaves lighthouse A and travels 63km. It is spotted from lighthouse B, which is 82km away from lighthouse A. The boat forms an angle of 65.1° with both lighthouses. How far is the boat from lighthouse B?



Solution: In this problem, we again have the SSA angle case. In order to find the distance from the boat to the lighthouse (a) we will first need to find the measure of angle A. In order to find angle A, we must first use the Law of Sines to find angle B. Since $c > b$, this situation will yield exactly one answer for the measure of angle B.

$$\begin{aligned}\frac{\sin 65.1^\circ}{82} &= \frac{\sin B}{63}\\ \frac{63\sin 65.1^\circ}{82} &= \sin B\\ 0.6969 &\approx \sin B\\ \angle B &= 44.2^\circ\end{aligned}$$

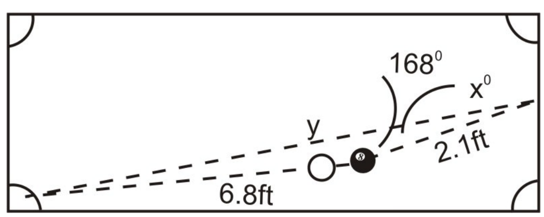
Now that we know the measure of angle B, we can find the measure of angle A, $\angle A = 180^\circ - 65.1^\circ - 44.2^\circ = 70.7^\circ$. Finally, we can use $\angle A$ to find side a.

$$\begin{aligned}\frac{\sin 65.1^\circ}{82} &= \frac{\sin 70.7^\circ}{a}\\ \frac{82\sin 70.7^\circ}{\sin 65.1^\circ} &= a\\ a &= 85.3\end{aligned}$$

The boat is approximately 85.3 km away from lighthouse B.

Using the Law of Cosines

Example 5: In a game of pool, a player must put the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a 168° angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?



Note: This is actually a trick shot performed by spinning the eight ball, and the eight ball will not actually travel in straight-line trajectories. However, to simplify the problem, assume that it travels in straight lines.

Solution: In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X).

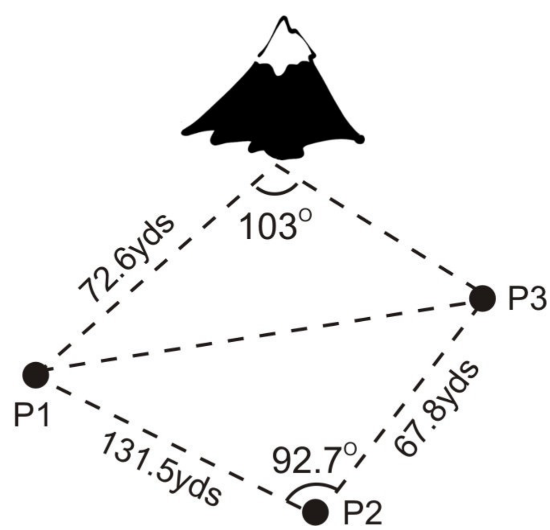
$$\begin{aligned}y^2 &= 6.8^2 + 2.1^2 - 2(6.8)(2.1)\cos 168^\circ\\ y^2 &= 78.59\\ y &= 8.86\ feet\end{aligned}$$

The distance from the spot on the bumper to the pocket is 8.86 feet. We can now use this distance and the Law of Sines to find angle X. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since we know all three sides this problem will yield only one solution.

$$\begin{aligned} \frac{\sin 168^\circ}{8.86} &= \frac{\sin X}{6.8} \\ \frac{6.8 \sin 168^\circ}{8.86} &= \sin X \\ 0.1596 &\approx \sin B \\ \angle B &= 8.77^\circ \end{aligned}$$

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios. We will also look at another real-world application involving the SSA case.

Example 6: Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain. The mountains forms a 103° angle with Person 1 and Person 3, while Person 2 forms a 92.7° angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.



Solution: In the triangle formed by the three people, we know two sides and the included angle (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

To find x:

$$\begin{aligned} x^2 &= 131.5^2 + 67.8^2 - 2(131.5)(67.8)\cos 92.7 \\ x^2 &= 22729.06397 \\ x &= 150.8 \; yds \end{aligned}$$

Now that we know $x = 150.8$, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since $150.8 > 72.6$, we know that we will have only one solution to this problem.

$$\frac{\sin 103}{150.8} = \frac{\sin Y}{72.6}$$
$$\frac{72.6 \sin 103}{150.8} = \sin Y$$
$$0.4690932805 = \sin Y$$
$$28.0 \approx \angle Y$$

Points to Consider

- Why is there only one possible solution to the SSA case if $a > b$?
- Explain why $b > a > b \sin A$ yields two possible solutions to a triangle.
- If we have a SSA angle case with two possible solutions, how can we check both solutions to make sure they are correct?

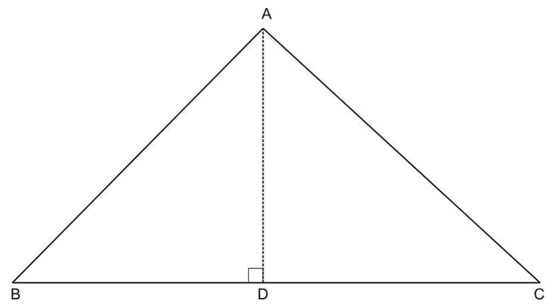
Review Questions

1. Using the table below, determine how many solutions there would be to each problem based on the given information and by calculating $b \sin A$ and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram."

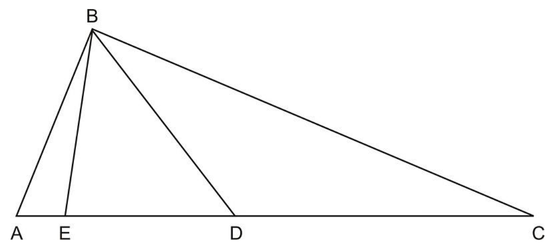
TABLE 5.7:

Given	$a >, =,$ or $< b \sin A$	Diagram	Number of solutions
a. $A = 32.5°, a = 26, b = 37$			
b. $A = 42.3°, a = 16, b = 26$			
c. $A = 47.8°, a = 13.48, b = 18.2$			
d. $A = 51.5°, a = 3.4, b = 4.2$			

2. Using the information in the table above, find all possible measures of angle B if any exist.
3. Prove using the Law of Sines: $\frac{a-c}{c} = \frac{\sin A - \sin C}{\sin C}$
4. Give the measure of a non-included angle and the lengths of two sides so that two triangles exist. Explain why two triangles exist for the measures you came up with.
5. If $a = 22$ and $b = 31$, find the values of A so that:
   a. There is no solution
   b. There is one solution
   c. There are two solutions
6. In the figure below, $AB = 13.7, AD = 9.8$, and $\angle C = 42.6°$. Find $\angle A, \angle B, BC$, and AC.

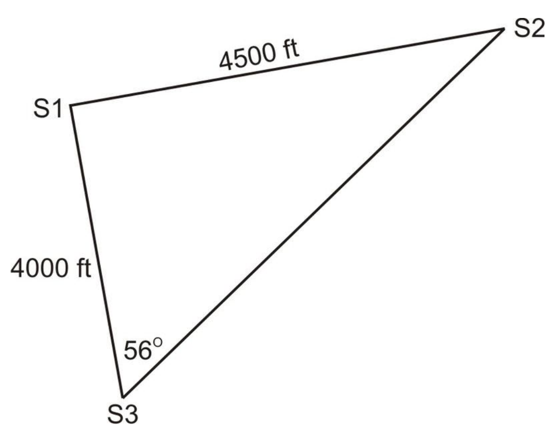


7. In the figure below, $\angle C = 21.8^\circ, BE = 9.9, BD = 10.2, ED = 7.6$, and $\angle ABC = 109.6^\circ$. Find the following:

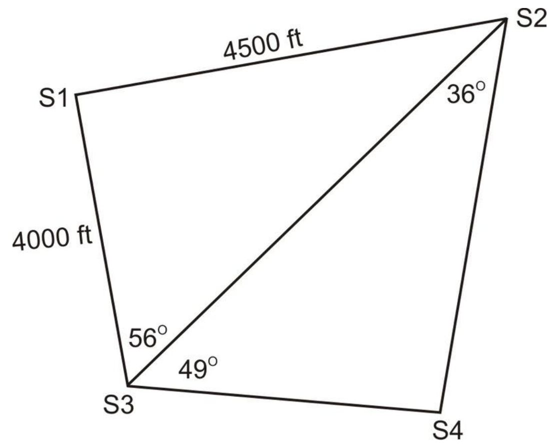


a. $\angle EBD$
b. $\angle BDE$
c. $\angle DEB$
d. $\angle BDC$
e. $\angle BEA$
f. $\angle DBC$
g. $\angle ABE$
h. $\angle BAE$
i. BC
j. AB
k. AE
l. DC
m. AC

8. Radio detection sensors for tracking animals have been placed at three different points in a wildlife preserve. The distance between Sensor 1 and Sensor 2 is 4500ft. The distance between Sensor 1 and Sensor 3 is 4000ft. The angle formed by Sensor 3 with Sensors 1 and 2 is 56°. If the range of Sensor 3 is 6000ft, will it be able to detect all movement from its location to the location of Sensor 2?



9. In problem 8 above, a fourth sensor is placed in the wildlife preserve. Sensor 2 forms a 36 angle with Sensors 3 and 4, and Sensor 3 forms a 49 angle with Sensors 2 and 4. How far away is Sensor 4 from Sensors 2 and 3?


S2
4500 ft
S1
36°
4000 ft
56°
49°
S3
S4

5.13 General Solutions of Triangles

Learning Objectives

- Use the Pythagorean Theorem, trigonometry functions, the Law of Sines, and the Law of Cosines to solve various triangles.
- Understand when it is appropriate to use each method.
- Apply the methods above in real-world and applied problems.

In the previous sections we have discussed a number of methods for finding a missing side or angle in a triangle. Previously, we only knew how to do this in right triangles, but now we know how to find missing sides and angles in oblique triangles as well. By combining all of the methods we've learned up until this point, it is possible for us to find all missing sides and angles in any triangle we are given.

Summary of Triangle Techniques

Below is a chart summarizing the triangle techniques that we have learned up to this point. This chart describes the type of triangle (either right or oblique), the given information, the appropriate technique to use, and what we can find using each technique.

TABLE 5.8:

Type of Triangle:	**Given Information:**	**Technique:**	**What we can find:**
Right	Two sides	Pythagorean Theorem	Third side
Right	One angle and one side	Trigonometric ratios	Either of the other two sides
Right	Two sides	Trigonometric ratios	Either of the other two angles
Oblique	2 angles and a non-included side (AAS)	Law of Sines	The other non-included side
Oblique	2 angles and the included side (ASA)	Law of Sines	Either of the non-included sides
Oblique	2 sides and the angle opposite one of those sides (SSA) –Ambiguous case	Law of Sines	The angle opposite the other side (can yield no, one, or two solutions)
Oblique	2 sides and the included angle (SAS)	Law of Cosines	The third side
Oblique	3 sides	Law of Cosines	Any of the three angles

Using the Law of Cosines

It is possible for us to completely solve a triangle using the Law of Cosines. In order to do this, we will need to apply the Law of Cosines multiple times to find all of the sides and/or angles we are missing.

Example 1: In triangle $ABC, a = 12, b = 13, c = 8$. Solve the triangle.

Solution: Since we are given all three sides in the triangle, we can use the Law of Cosines. Before we can solve the triangle, it is important to know what information we are missing. In this case, we do not know any of the angles, so we are solving for $\angle A, \angle B$, and $\angle C$. We will begin by finding $\angle A$.

$$\begin{aligned}
12^2 &= 8^2 + 13^2 - 2(8)(13)\cos A \\
144 &= 233 - 208\cos A \\
-89 &= -208\cos A \\
0.4278846154 &= \cos A \\
64.7 &\approx \angle A
\end{aligned}$$

Now, we will find $\angle B$ by using the Law of Cosines. Keep in mind that you can now also use the Law of Sines to find $\angle B$. Use whatever method you feel more comfortable with.

$$\begin{aligned}
13^2 &= 8^2 + 12^2 - 2(8)(12)\cos B \\
169 &= 208 - 192\cos B \\
-39 &= -192\cos B \\
0.2031 &= \cos B \\
78.3^\circ &\approx \angle B
\end{aligned}$$

We can now quickly find $\angle C$ by using the Triangle Sum Theorem, $180^\circ - 64.7^\circ - 78.3^\circ = 37^\circ$

Example 2: In triangle $DEF, d = 43, e = 37$, and $\angle F = 124^\circ$. Solve the triangle.

Solution: In this triangle, we have the SAS case because we know two sides and the included angle. This means that we can use the Law of Cosines to solve the triangle. In order to solve this triangle, we need to find side $f, \angle D$, and $\angle E$. First, we will need to find side f using the Law of Cosines.

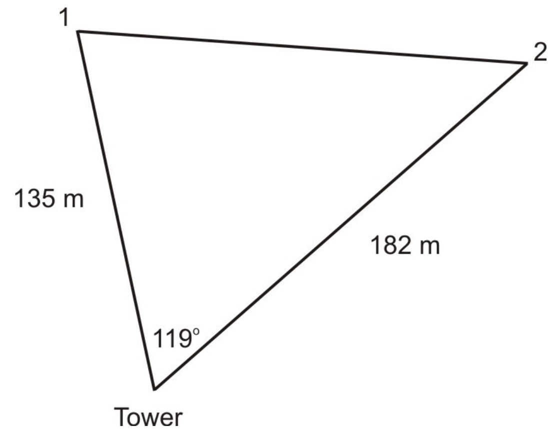
$$\begin{aligned}
f^2 &= 43^2 + 37^2 - 2(43)(37)\cos 124 \\
f^2 &= 4997.351819 \\
f &\approx 70.7
\end{aligned}$$

Now that we know f, we know all three sides of the triangle. This means that we can use the Law of Cosines to find either angle D or angle E. We will find angle D first.

$$\begin{aligned}
43^2 &= 70.7^2 + 37^2 - 2(70.7)(37)\cos D \\
1849 &= 6367.49 - 5231.8\cos D \\
-4518.49 &= -5231.8\cos D \\
0.863658779 &= \cos D \\
30.3^\circ &\approx \angle D
\end{aligned}$$

To find angle E, we need only to use the Triangle Sum Theorem, $\angle E = 180 - (124 + 30.3) = 25.7°$.

Example 3: A control tower is receiving signals from two microchips implanted in wild tigers. Microchip 1 is 135 miles from the control tower and microchip 2 is 182 miles from the control tower. If the control tower forms a 119° angle with both microchips, how far apart are the two tigers?



Solution: To find the distance between the two tigers, we need to find the distance between the two microchips. We will call this distance x. Since we know two sides and the included angle, we can use the Law of Cosines to find x.

$$\begin{aligned} x^2 &= 135^2 + 182^2 - 2(135)(182)\cos 119 \\ x^2 &= 75172.54474 \\ x &= 274.2 \ miles \end{aligned}$$

The two tigers are 274.2 miles apart.

Using the Law of Sines

It is also possible for us to completely solve a triangle using the Law of Sines if we begin with the ASA case, the AAS case, or the SSA case. We must remember that when given the SSA case, it is possible that we may encounter the Ambiguous Case.

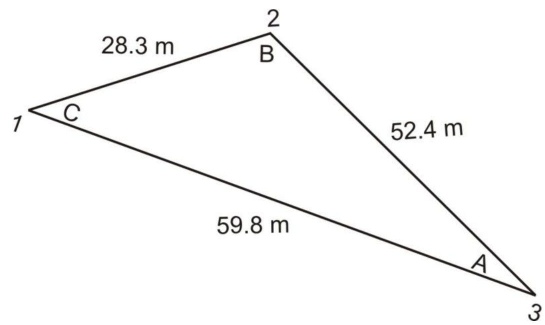
Example 4: In triangle $ABC, A = 43°, B = 82°$, and $c = 10.3$. Solve the triangle.

Solution: This is an example of the ASA case, which means that we can use the Law of Sines to solve the triangle. In order to use the Law of Sines, we must first know angle C, which we can find using the Triangle Sum Theorem, $\angle C = 180° - (43° + 82°) = 55°$.

Now that we know $\angle C$, we can use the Law of Sines to find either side a or side b.

$$\begin{aligned} \frac{\sin 55}{10.3} &= \frac{\sin 43}{a} \\ a &= \frac{10.3 \sin 43}{\sin 55} \\ a &= 8.6 \end{aligned} \qquad \begin{aligned} \frac{\sin 55}{10.3} &= \frac{\sin 82}{b} \\ b &= \frac{10.3 \sin 82}{\sin 55} \\ b &= 12.5 \end{aligned}$$

Example 5: A cruise ship is based at Island 1, but makes trips to Island 2 and Island 3 during the day. Island 3 lies directly east of Island 1. If the distance from Island 1 to Island 2 is 28.3 miles, from Island 2 to 3 is 52.4 miles, and Island 3 to 1 is 59.8 miles, what heading (angle) must the captain:



a. Leave Island 1

b. Leave Island 2

c. Leave Island 3

Solution: In order to find all three angles in the triangle, we must use the Law of Cosines because we are dealing with the SSS case. Once we find one angle using the Law of Cosines, we can use the Law of Sines to find a second angle. Then, we can use the Triangle Sum Theorem to find the third angle.

We will begin by finding $\angle B$ because it is the largest angle.

$$\begin{aligned}
59.8^2 &= 52.4^2 + 28.3^2 - 2(52.4)(28.3)\cos B \\
3576.04 &= 3546.65 - 2965.84\cos B \\
29.39 &= -2965.84\cos B \\
-0.0099095029 &= \cos B \\
B &= 90.6^\circ
\end{aligned}$$

Now that we know $\angle B$, we can find either $\angle A$ or $\angle C$. We will find $\angle C$ first since it is the second largest angle.

$$\begin{aligned}
\frac{\sin 90.6}{59.8} &= \frac{\sin C}{52.4} \\
\frac{52.4\sin 90.6}{59.8} &= \sin C \\
0.876203135 &= \sin C \\
\angle C &= 61.2^\circ
\end{aligned}$$

Now that we know $\angle B$ and $\angle C$, we can use the Triangle Sum Theorem to find $\angle A = 180^\circ - (61.2^\circ + 90.6^\circ) = 28.2^\circ$.

Now, we must convert our angles into headings.

When going from Island 1 to Island 2, 61.2° would be a heading of $N28.8^\circ E$. Also, since $\angle A = 28.2^\circ$, if we were to travel from Island 3 to Island 2, our heading would be $W28.2^\circ N$. This means that when going from Island 2 to Island 3, the heading would be in the exact opposite direction, or $E28.2^\circ S$. When going from Island 3 back to Island 1, since we know that Island 3 is directly east of Island 1, the captain must now sail in the direction opposite of east, or directly west, which is $N90^\circ W$.

Points to Consider

- Is there ever a situation where you would need to use the Law of Sines *before* using the Law of Cosines?
- In what situation might you consider using the Law of Cosines instead of Law of Sines if both were applicable?
- Why do we only have to use the Law of Cosines one time before we can switch to using the Law of Sines?

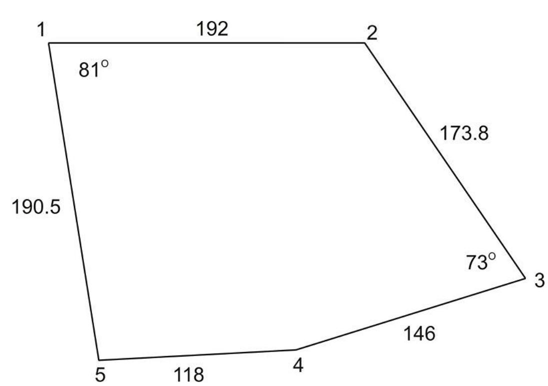
Review Questions

1. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of each triangle and label the given information. Also, state how many solutions (if any) each triangle would have. If a triangle has no solution or two solutions, explain why.

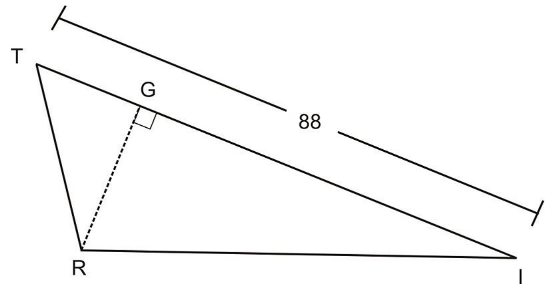
TABLE 5.9:

Given	Drawing	Case	Law	Number of Solutions
a. $A = 69^\circ, B = 12^\circ, a = 22.3$, find b.				
b. $a = 1.4, b = 2.3, C = 58^\circ$, find c.				
c. $a = 3.3, b = 6.1, c = 4.8$, find A.				
d. $a = 15, b = 25, A = 58^\circ$, find B.				
e. $a = 45, b = 60, A = 47^\circ$, find B.				

2. Using the information in the chart above, solve for the requested side or angle.
3. Using the information in the chart in question 1 and your answers from question 2, determine what information you are still missing from each triangle. Then, solve for each piece, solving each triangle.
4. The side of a rhombus is 12 cm and the longer diagonal is 21.5cm. Find the area of the rhombus and the measures of the angles in the rhombus.
5. Find the area of the pentagon below.



6. In the figure drawn below, angle T is 56.8° and $RT = 38$. Using the figure below, find the length of the altitude drawn to the longest side, the area of the two triangles formed by this altitude, RI and angle I.



7. Refer back to Example 5, the island hopping problem. Suppose the cruise ship were based on Island 3 and traveled to Island 2 and then to Island 1, before returning to Island 3.
   a. What would be its heading when going from Island 3 to Island 2?
   b. What would be its heading when going from Island 2 to Island 1?
   c. What would be its heading when going from Island 1 back to Island 3?
8. A golfer is standing on the tee of a golf hole that has a 115° bend to the left. The distance from the tee to the bend is 218 yards. The distance from the bend to the green is 187 yards.
   a. How far would the golfer need to hit the ball if he wanted to make it to the green in one shot?
   b. At what angle would he need to hit the ball?
9. A golfer is standing on the tee, which is 320 yards from the cup on the green. After he hits his first shot, which is sliced to the right, his ball forms a 162.2° angle with the tee and the cup, and the cup forms a 14.2° angle with his ball and the tee.
   a. What is the degree of his slice?
   b. How far was his first shot?
   c. How far away from the cup is he?

5.14 Solving Right Triangles

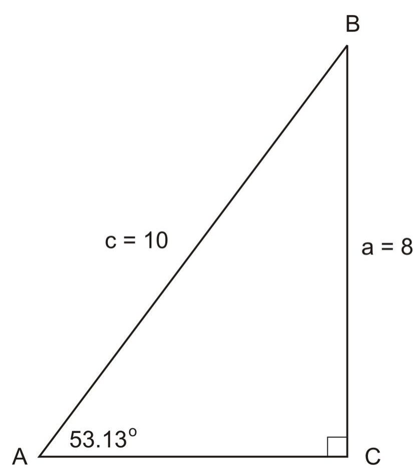
Learning Objectives

- Solve right triangles.
- Find the area of any triangle using trigonometry.
- Solve real-world problems that require you to solve a right triangle.
- Find angle measures using inverse trigonometric functions.

Solving Right Triangles

You can use your knowledge of the Pythagorean Theorem and the six trigonometric functions to solve a right triangle. Because a right triangle is a triangle with a 90 degree angle, solving a right triangle requires that you find the measures of one or both of the other angles. How you solve will depend on how much information is given. The following examples show two situations: a triangle missing one side, and a triangle missing two sides.

Example 1: Solve the triangle shown below.



Solution:

We need to find the lengths of all sides and the measures of all angles. In this triangle, two of the three sides are given. We can find the length of the third side using the Pythagorean Theorem:

$$\begin{aligned}
8^2 + b^2 &= 10^2 \\
64 + b^2 &= 100 \\
b^2 &= 36 \\
b &= \pm 6 \Rightarrow b = 6
\end{aligned}$$

(You may have also recognized the "Pythagorean Triple," 6, 8, 10, instead of carrying out the Pythagorean Theorem.)

You can also find the third side using a trigonometric ratio. Notice that the missing side, b, is adjacent to $\angle A$, and the hypotenuse is given. Therefore we can use the cosine function to find the length of b:

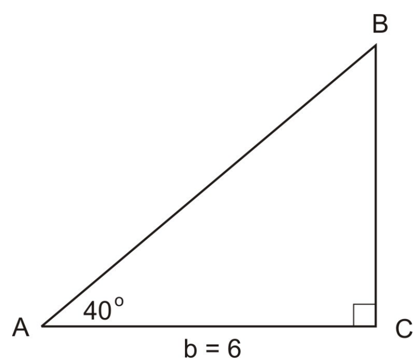
$$\cos 53.13^\circ = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{10}$$
$$0.6 = \frac{b}{10}$$
$$b = 0.6(10) = 6$$

We could also use the tangent function, as the opposite side was given. It may seem confusing that you can find the missing side in more than one way. The point is, however, not to create confusion, but to show that you must look at what information is missing, and choose a strategy. Overall, when you need to identify one side of the triangle, you can either use the Pythagorean Theorem, or you can use a trig ratio.

To solve the above triangle, we also have to identify the measures of all three angles. Two angles are given: 90 degrees and 53.13 degrees. We can find the third angle using the Triangle Sum Theorem, $180 - 90 - 53.13 = 36.87^\circ$.

Now let's consider a triangle that has two missing sides.

Example 2: Solve the triangle shown below.



Solution:

In this triangle, we need to find the lengths of two sides. We can find the length of one side using a trig ratio. Then we can find the length of the third side by using a trig ratio with the same given information, not the side we solved for. This is because the side we found is an *approximation* and would not yield the most accurate answer for the other missing side. *Only use the given information when solving right triangles.*

We are given the measure of angle A, and the length of the side adjacent to angle A. If we want to find the length of the hypotenuse, c, we can use the cosine ratio:

$$\cos 40^\circ = \frac{adjacent}{hypotenuse} = \frac{6}{c}$$
$$\cos 40^\circ = \frac{6}{c}$$
$$c\cos 40^\circ = 6$$
$$c = \frac{6}{\cos 40^\circ} \approx 7.83$$

If we want to find the length of the other leg of the triangle, we can use the tangent ratio. This will give us the most accurate answer because we are not using approximations.

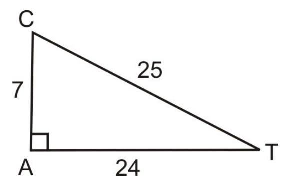
$$\tan 40^\circ = \frac{opposite}{adjacent} = \frac{a}{6}$$
$$a = 6\tan 40^\circ \approx 5.03$$

Now we know the lengths of all three sides of this triangle. In the review questions, you will verify the values of c and a using the Pythagorean Theorem. Here, to finish solving the triangle, we only need to find the measure of $\angle B : 180 - 90 - 40 = 50^\circ$

Notice that in both examples, one of the two non-right angles was given. If neither of the two non-right angles is given, you will need a new strategy to find the angles.

Inverse Trigonometric Functions

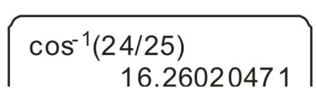
Consider the right triangle below.



From this triangle, we know how to determine all six trigonometric functions for both $\angle C$ and $\angle T$. From any of these functions we can also find the value of the angle, using our graphing calculators. If you look back at #7 from 1.3, we saw that $\sin 30^\circ = \frac{1}{2}$. If you type 30 into your graphing calculator and then hit the $\boxed{\text{SIN}}$ button, the calculator yields 0.5. (Make sure your calculator's mode is set to degrees.)

Conversely, with the triangle above, we know the trig ratios, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. These functions are located above the SIN, COS, and TAN buttons on the calculator. To access this function, press 2^{nd} and the appropriate button and the measure of the angle appears on the screen.

$\cos T = \frac{24}{25} \rightarrow \cos^{-1}\frac{24}{25} = T$ from the calculator we get



Example 3: Find the angle measure for the trig functions below.

a. $\sin x = 0.687$

b. $\tan x = \frac{4}{3}$

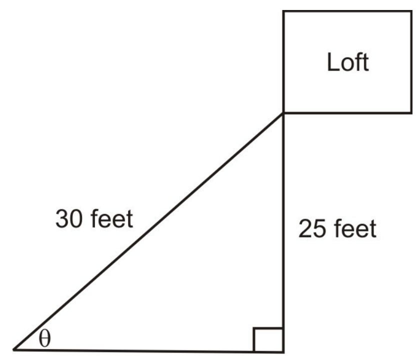
Solution: Plug into calculator.

a. $\sin^{-1} 0.687 = 43.4^\circ$

b. $\tan^{-1}\frac{4}{3} = 53.13^\circ$

Example 4: You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy and to move it manually down a ladder would take too much time and effort. You decide to devise a make shift conveyor belt made of bed sheets that you will attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?

Solution:



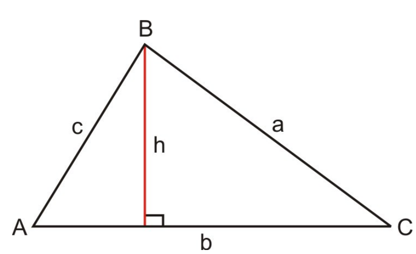
From the picture, we need to use the inverse sine function.

$$\begin{aligned}\sin\theta &= \frac{25\ feet}{30\ feet}\\ \sin\theta &= 0.8333\\ \sin^{-1}(\sin\theta) &= \sin^{-1}0.8333\\ \theta &= 56.4^\circ\end{aligned}$$

The sheets should be anchored at an angle of 56.4°.

Finding the Area of a Triangle

In Geometry, you learned that the area of a triangle is $A=\frac{1}{2}bh$, where b is the base and h is the height, or altitude. Now that you know the trig ratios, this formula can be changed around, using sine.



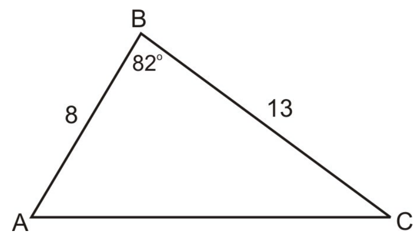
Looking at the triangle above, you can use sine to determine $h, \sin C=\frac{h}{a}$. So, solving this equation for h, we have $a\sin C=h$. Substituting this for h, we now have a new formula for area.

$$A=\frac{1}{2}ab\sin C$$

What this means is you do not need the height to find the area anymore. All you now need is two sides and the angle between the two sides, called the included angle.

Example 5: Find the area of the triangle.

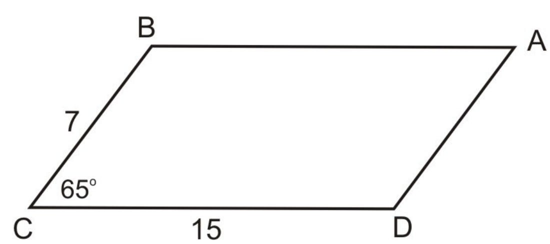
a.



Solution: Using the formula, $A = \frac{1}{2}\, ab \sin C$, we have

$$\begin{aligned} A &= \frac{1}{2} \cdot 8 \cdot 13 \cdot \sin 82^\circ \\ &= 4 \cdot 13 \cdot 0.990 \\ &= 51.494 \end{aligned}$$

Example 6: Find the area of the parallelogram.

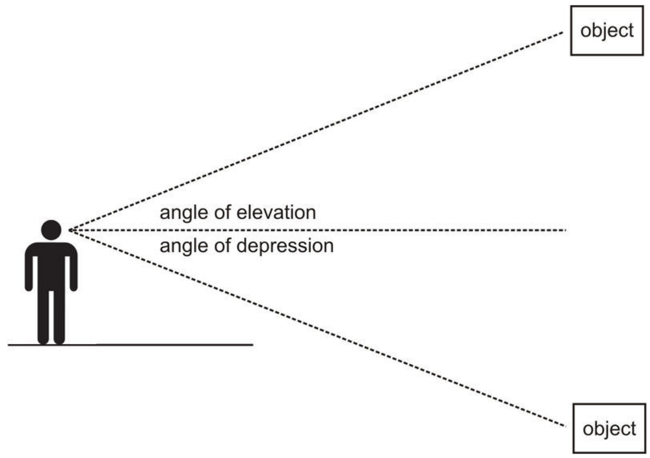


Solution: Recall that a parallelogram can be split into two triangles. So the formula for a parallelogram, using the new formula, would be: $A = 2 \cdot \frac{1}{2}\, ab \sin C$ or $A = ab \sin C$.

$$\begin{aligned} A &= 7 \cdot 15 \cdot \sin 65^\circ \\ &= 95.162 \end{aligned}$$

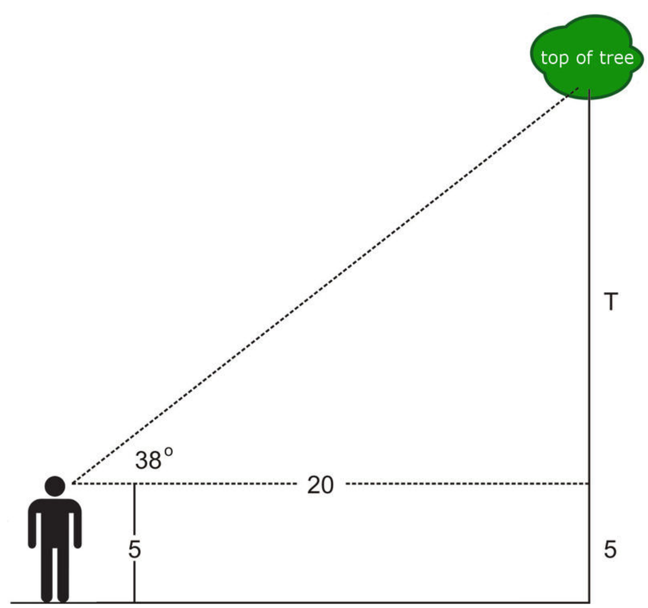
Angles of Elevation and Depression

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.



The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight *down to* an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Example 7: You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?



Solution:

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall.

The figure shows us that once we find the value of T, we have to add 5 feet to this value to find the total height of the triangle. To find T, we should use the tangent value:

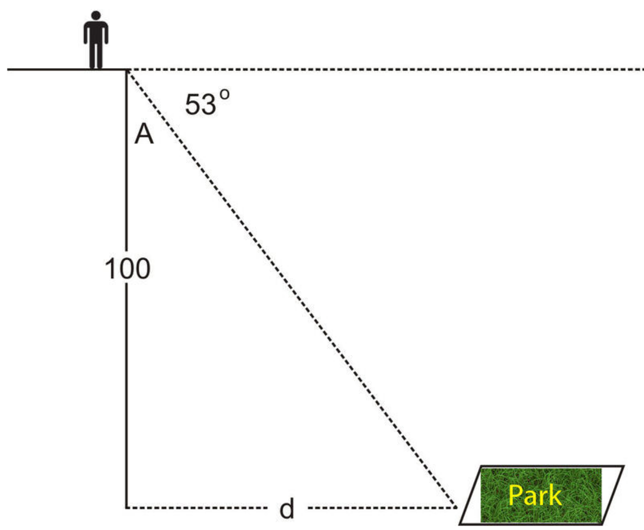
$$\tan 38^\circ = \frac{opposite}{adjacent} = \frac{T}{20}$$
$$\tan 38^\circ = \frac{T}{20}$$
$$T = 20 \tan 38^\circ \approx 15.63$$
$$\text{Height of tree} \approx 20.63\ ft$$

The next example shows an angle of depression.

Example 8: You are standing on top of a building, looking at a park in the distance. The angle of depression is 53°. If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Solution:

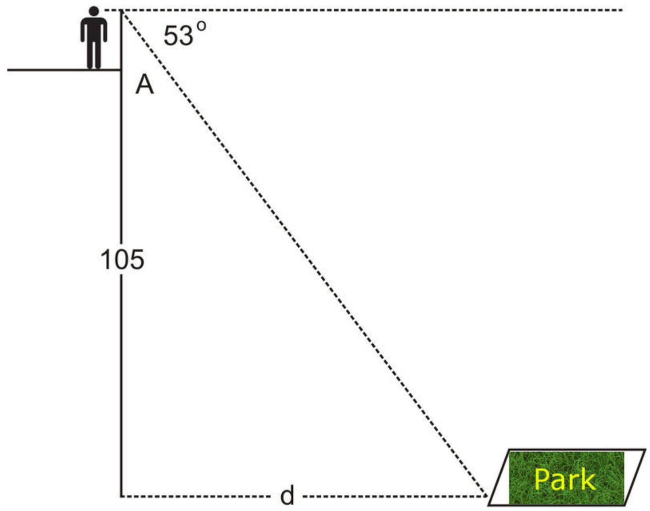
If we ignore the height of the person, we solve the following triangle:



Given the angle of depression is 53°, $\angle A$ in the figure above is 37°. We can use the tangent function to find the distance from the building to the park:

$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{100}$$
$$\tan 37^\circ = \frac{d}{100}$$
$$d = 100 \tan 37^\circ \approx 75.36\ ft.$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:

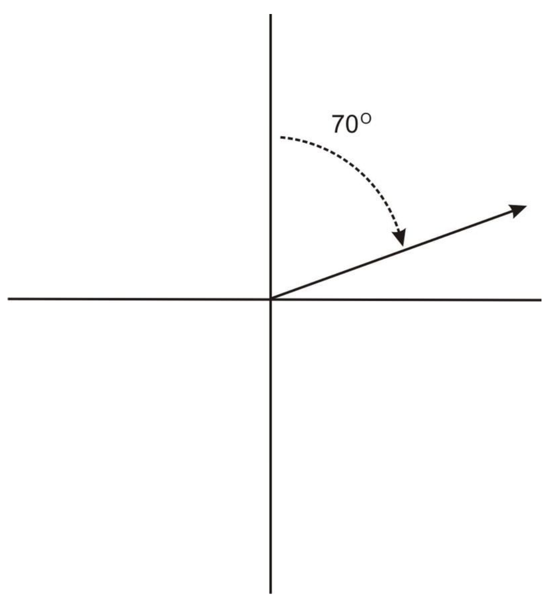


$$\tan 37^\circ = \frac{opposite}{adjacent} = \frac{d}{105}$$
$$\tan 37^\circ = \frac{d}{105}$$
$$d = 105 \tan 37^\circ \approx 79.12 \ ft.$$

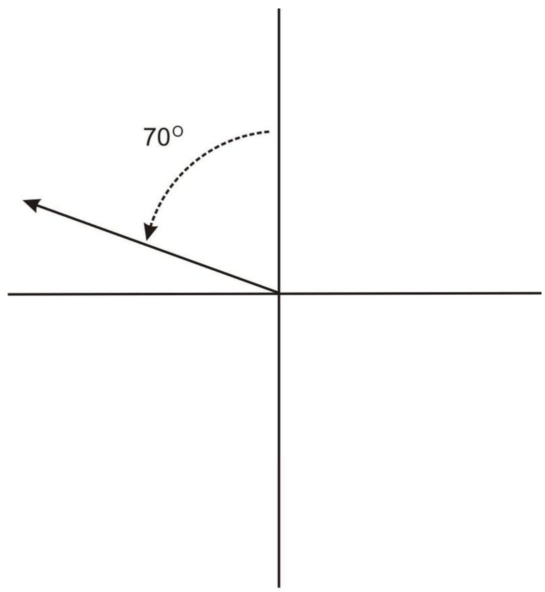
If you are only looking to estimate a distance, then you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Right Triangles and Bearings

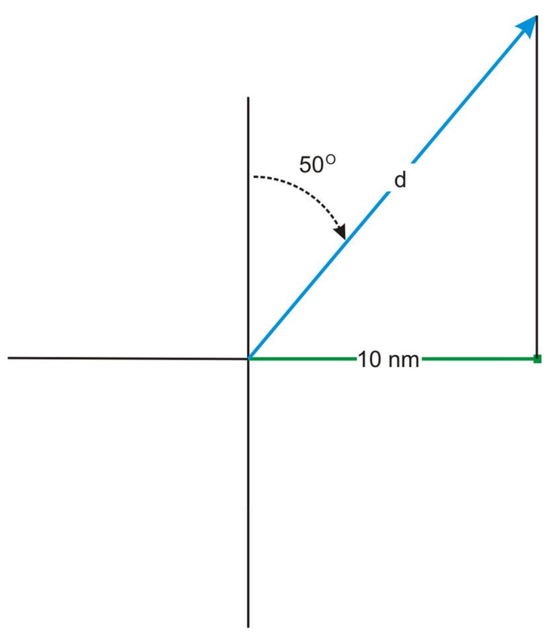
We can also use right triangles to find distances using angles given as bearings. In navigation, a bearing is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north. The graph below shows an angle of 70 degrees:



It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are measured in trigonometry. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, $N70^\circ E$ refers to an angle from the north, towards the east, while $N70^\circ W$ refers to an angle from the north, towards the west. $N70^\circ E$ is the same as the angle shown in the graph above. $N70^\circ W$ would result in an angle in the second quadrant.



Example 9: A ship travels on a $N50^\circ E$ course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?



Solution: The angle between d and 10 nm is the complement of 50°, which is 40°. Therefore we can find d using the cosine function:

$$\cos 40^\circ = \frac{adjacent}{hypotenuse} = \frac{10}{d}$$
$$\cos 40^\circ = \frac{10}{d}$$
$$d \cos 40^\circ = 10$$
$$d = \frac{10}{\cos 40^\circ} \approx 13.05\ nm$$

Other Applications of Right Triangles

In general, you can use trigonometry to solve any problem that involves right triangles. The next few examples show different situations in which a right triangle can be used to find a length or a distance.

Example 10: The wheelchair ramp

In lesson 3 we introduced the following situation: You are building a ramp so that people in wheelchairs can access a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

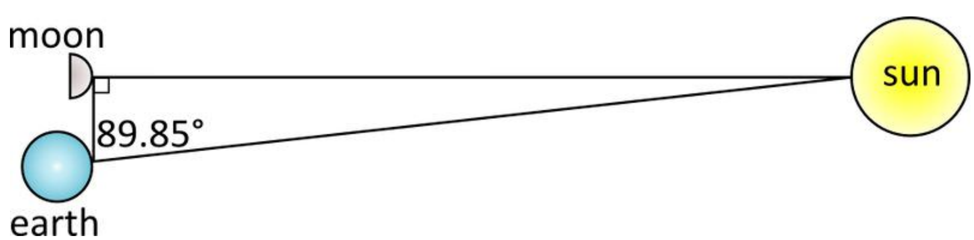
Given that we know the angle of the ramp and the length of the side opposite the angle, we can use the sine ratio to find the length of the ramp, which is the hypotenuse of the triangle:

$$\sin 5^\circ = \frac{8}{L}$$
$$L \sin 5^\circ = 8$$
$$L = \frac{8}{\sin 5^\circ} \approx 91.8\ ft$$

This may seem like a long ramp, but in fact a 5° ramp angle is what is required by the Americans with Disabilities Act (ADA). This explains why many ramps are comprised of several sections, or have turns. The additional distance is needed to make up for the small slope.

Right triangle trigonometry is also used for measuring distances that could not actually be measured. The next example shows a calculation of the distance between the moon and the sun. This calculation requires that we know the distance from the earth to the moon. In chapter 5 you will learn the Law of Sines, an equation that is necessary for the calculation of the distance from the earth to the moon. In the following example, we assume this distance, and use a right triangle to find the distance between the moon and the sun.

Example 11: The earth, moon, and sun create a right triangle during the first quarter moon. The distance from the earth to the moon is about 240,002.5 miles. What is the distance between the sun and the moon?



Solution:

Let $d =$ the distance between the sun and the moon. We can use the tangent function to find the value of d:

$$\tan 89.85^\circ = \frac{d}{240,002.5}$$
$$d = 240,002.5 \tan 89.85^\circ = 91,673,992.71 \ miles$$

Therefore the distance between the sun and the moon is much larger than the distance between the earth and the moon.

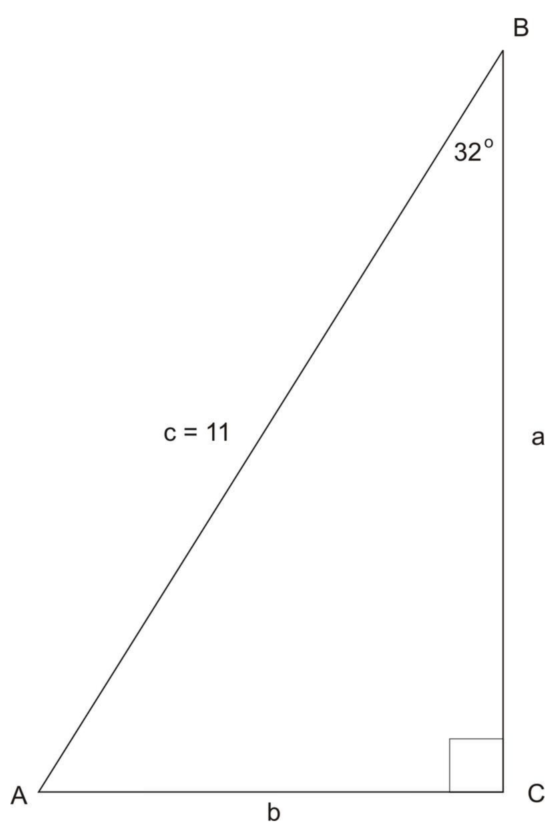
(Source: www.scribd.com, Trigonometry from the Earth to the Stars.)

Points to Consider

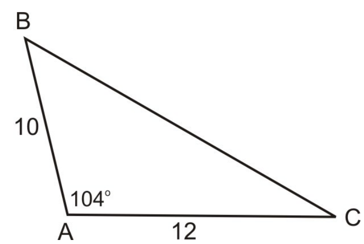
- In what kinds of situations do right triangles naturally arise?
- Are there right triangles that cannot be solved?
- Trigonometry can solve problems at an astronomical scale as well as problems at a molecular or atomic scale. Why is this true?

Review Questions

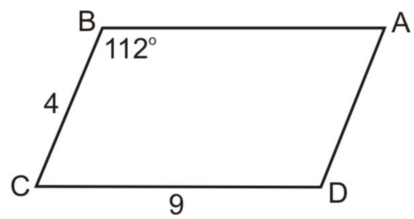
1. Solve the triangle.



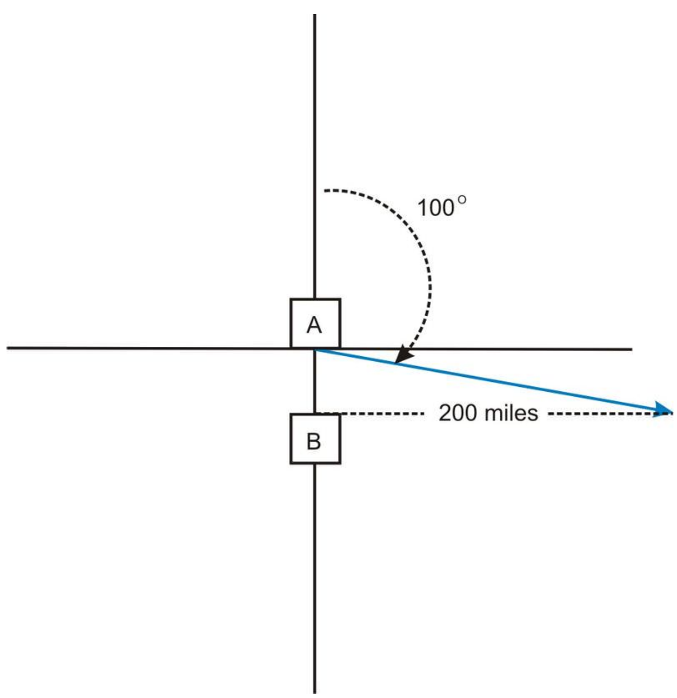
2. Two friends are writing practice problems to study for a trigonometry test. Sam writes the following problem for his friend Anna to solve: In right triangle ABC, the measure of angle C is 90 degrees, and the length of side c is 8 inches. Solve the triangle. Anna tells Sam that the triangle cannot be solved. Sam says that she is wrong. Who is right? Explain your thinking.
3. Use the Pythagorean Theorem to verify the sides of the triangle in example 2.
4. Estimate the measure of angle B in the triangle below using the fact that $\sin B = \frac{3}{5}$ and $\sin 30^\circ = \frac{1}{2}$. Use a calculator to find sine values. Estimate B to the nearest degree.
5. Find the area of the triangle.



6. Find the area of the parallelogram below.

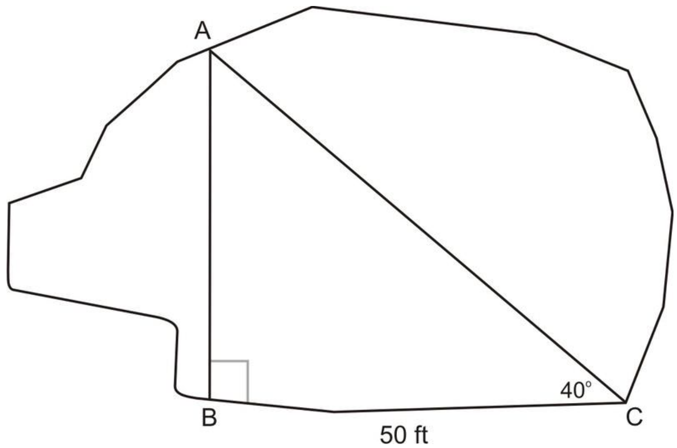


7. The angle of elevation from the ground to the top of a flagpole is measured to be 53°. If the measurement was taken from 15 feet away, how tall is the flagpole?
8. From the top of a hill, the angle of depression to a house is measured to be 14°. If the hill is 30 feet tall, how far away is the house?
9. An airplane departs city A and travels at a bearing of 100°. City B is directly south of city A. When the plane is 200 miles east of city B, how far has the plan traveled? How far apart are City A and City B?

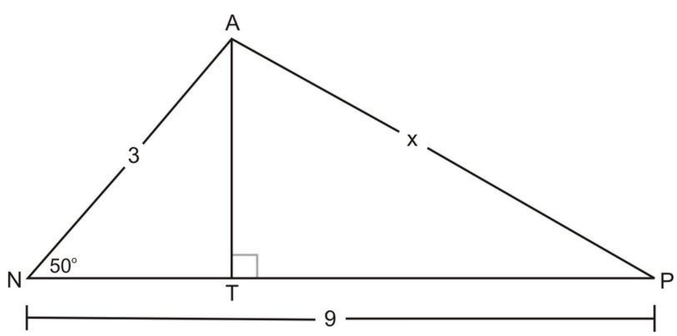


What is the length of the slanted outer wall, w? What is the length of the main floor, f?

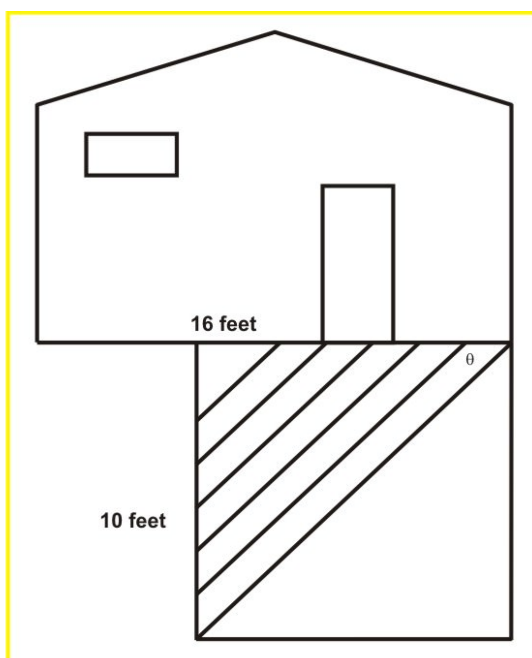
10. A surveyor is measuring the width of a pond. She chooses a landmark on the opposite side of the pond, and measures the angle to this landmark from a point 50 feet away from the original point. How wide is the pond?



11. Find the length of side x:



12. A deck measuring 10 feet by 16 feet will require laying boards with one board running along the diagonal and the remaining boards running parallel to that board. The boards meeting the side of the house must be cut prior to being nailed down. At what angle should the boards be cut?



5.15 Law of Sines

Learning Objectives

Here you will further explore solving non-right triangles in cases where a corresponding side and angle are given using the Law of Sines.

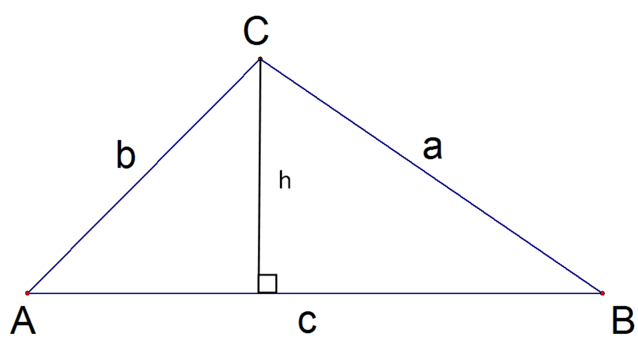
When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but SSA is not one of them. Why not?

The Law of Sines

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The **Law of Sines** states that in every triangle the ratio of each side to the sine of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here is a proof of the Law of Sines:



Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

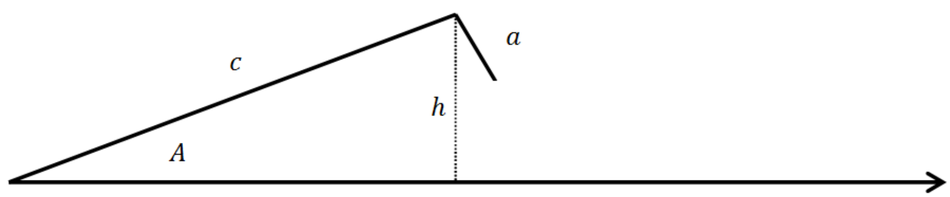
$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.



In this image side a is deliberately too short, but in most problems you will not know this. You will need to compare a to the height.

$$\sin A = \frac{h}{c}$$
$$h = c \ \sin A$$

Case 1: $a < h$

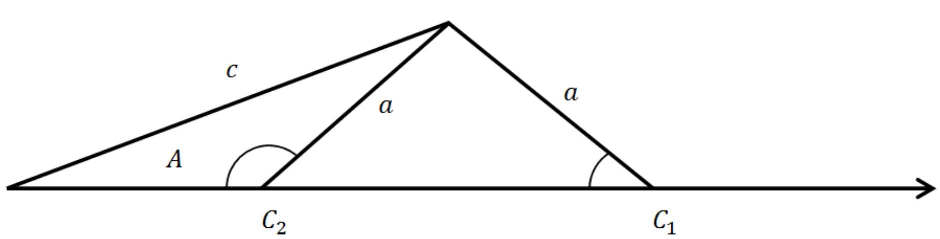
Simply put, side a is not long enough to reach the opposite side and the triangle is impossible.

Case 2: $a = h$

Side a just barely reaches the opposite side forming a 90° angle.

Case 3: $h < a < c$

In this case side a can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of Sines to solve for angle C_1 first and then use the supplement to determine C_2.

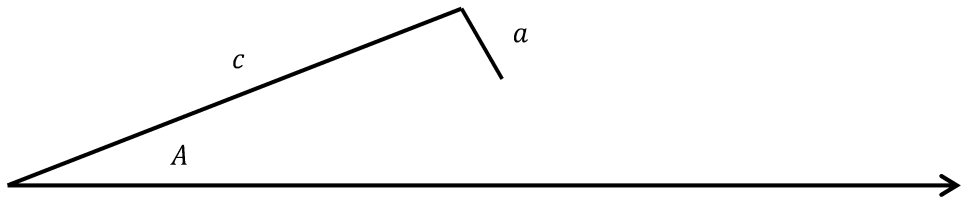


Case 4: $c \leq a$

In this case, side a can only swing towards the exterior of the triangle, only producing C_1.

For the case of SSA, you should always check how many triangles there are before starting to find measures. Take the following triangle:

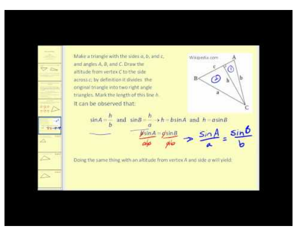
$\angle A = 40^\circ, c = 13$, and $a = 2$.



Before trying find $\angle C$, you need to check that a triangle is possible and if there is more than one solution. Use the equation from above,

$$\sin 40^\circ = \frac{h}{13}$$
$$h = 13 \sin 40^\circ \approx 8.356$$

Because $a < h$ $(2 < 8.356)$, this information does not form a proper triangle.



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58146

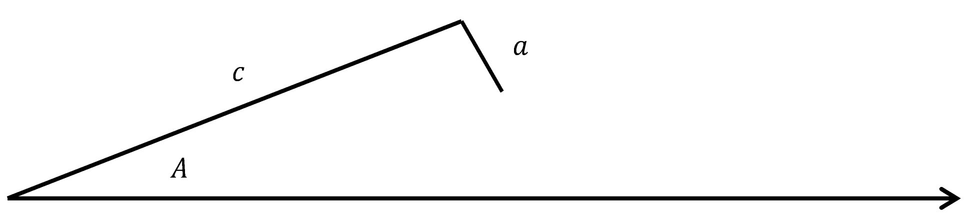
Examples

Example 1

Earlier, you were asked why SSA is not a method to show that two triangles are congruent. SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle. Sometimes there is no triangle, one triangle, or two triangles.

Example 2

$\angle A = 17^\circ, c = 14$, and $a = 4.0932\ldots$ If possible, find $\angle C$.



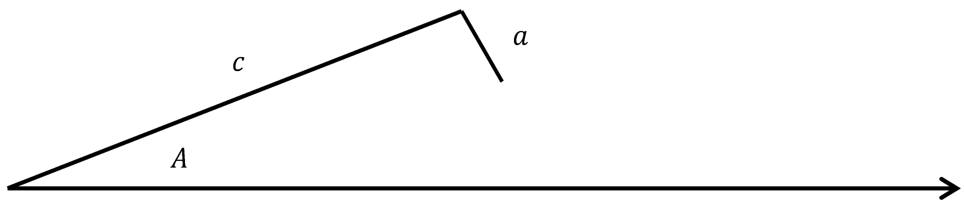
Check that a triangle is possible:

$$\begin{aligned}
\sin 17^\circ &= \frac{h}{14} \\
h &= 14\ \sin 17^\circ \approx 4.0932\ldots
\end{aligned}$$

Since $a = h$, this information forms exactly one triangle and angle C must be 90°.

Example 3

$\angle A = 22^\circ, c = 11$ and $a = 9$. If possible, find $\angle C$.



Check that a triangle is possible:

$$\begin{aligned}
\sin 22^\circ &= \frac{h}{11} \\
h &= 11\ \sin 22^\circ \approx 4.12\ldots
\end{aligned}$$

Since $h < a < c$, there must be two possible angles for angle C.

Apply the Law of Sines:

$$\begin{aligned}
\frac{9}{\sin 22^\circ} &= \frac{11}{\sin C_1} \\
9 \sin C_1 &= 11 \sin 22^\circ \\
\sin C_1 &= \frac{11 \sin 22^\circ}{9} \\
C_1 &= \sin^{-1}\left(\frac{11 \sin 22^\circ}{9}\right) \approx 27.24\ldots^\circ \\
C_2 &= 180 - C_1 = 152.75\ldots^\circ
\end{aligned}$$

Example 4

Given ΔABC where $A = 12^\circ, B = 50^\circ, a = 14$ find b.

$\frac{14}{\sin 12^\circ} = \frac{b}{\sin 50^\circ}$

$b = \frac{14 \sin 50^\circ}{\sin 12^\circ} \approx 51.58\ldots$

Example 5

Given ΔABC where $A = 70^\circ, b = 8, a = 3$, find $\angle B$ if possible.

$\sin 70^\circ = \frac{h}{8}$

$h = 8\sin 70^\circ \approx 7.51\ldots$

Because $a < h$, this triangle is impossible.

Review

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30^\circ, a = 13, b = 15$
2. $A = 22^\circ, a = 21, b = 12$
3. $A = 42^\circ, a = 36, b = 37$

For 4-7, find all possible measures of $\angle B$ (if any exist) for each of the following triangle values.

4. $A = 86^\circ, a = 15, b = 11$
5. $A = 30^\circ, a = 24, b = 43$
6. $A = 48^\circ, a = 34, b = 39$
7. $A = 80^\circ, a = 22, b = 20$

For 8-12, find the length of b for each of the following triangle values.

8. $A = 94^\circ, a = 31, B = 34^\circ$
9. $A = 112^\circ, a = 12, B = 15^\circ$
10. $A = 78^\circ, a = 20, B = 16^\circ$
11. $A = 54^\circ, a = 15, B = 112^\circ$
12. $A = 39^\circ, a = 9, B = 98^\circ$
13. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce two triangles?
14. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce no triangles?
15. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for a that would produce one triangle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.6.

5.16 Law of Cosines

Learning Objectives

Here you will solve non-right triangles with the Law of Cosines.

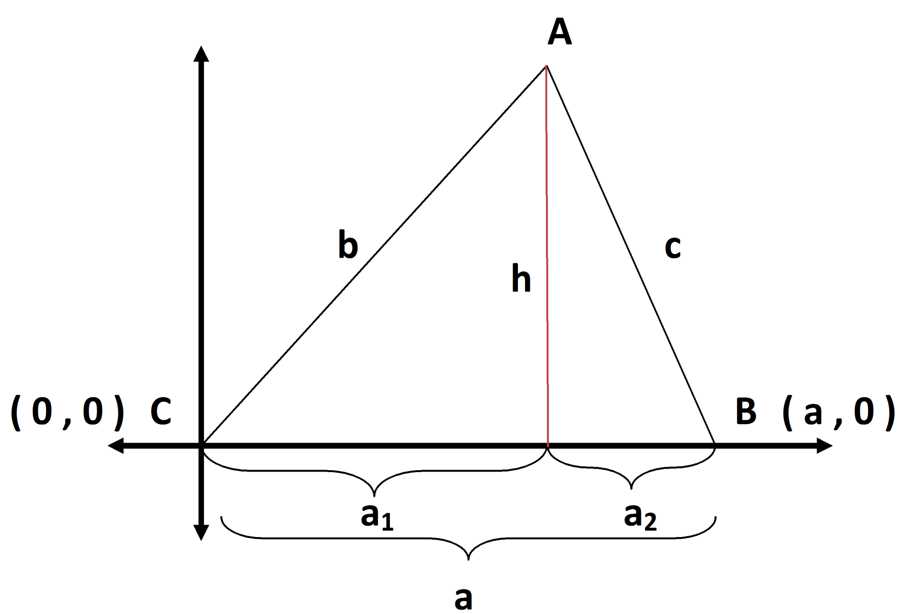
The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

The Law of Cosines

The **Law of Cosines** is:

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

It is important to understand the proof:



You know four facts from the picture:

$a = a_1 + a_2 \qquad (1)$

$b^2 = a_1^2 + h^2 \qquad (2)$

$c^2 = a_2^2 + h^2 \qquad (3)$

$\cos C = \frac{a_1}{b} \qquad (4)$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$a_2 = a - a_1 \qquad (5, \text{ from } 1)$

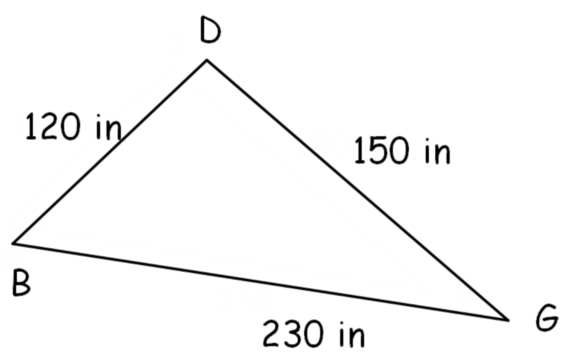
$a_1 = b \cdot \cos C \qquad (6,\ \text{from } 4)$

Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

$$
\begin{aligned}
c^2 &= a_2^2 + h^2 && \text{(3 again)}\\
c^2 &= (a - a_1)^2 + h^2 && \text{(substitute using 5)}\\
c^2 &= a^2 - 2a \cdot a_1 + a_1^2 + h^2 && \text{(FOIL)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + a_1^2 + h^2 && \text{(substitute using 6)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + b^2 && \text{(substitute using 2)}\\
c^2 &= a^2 + b^2 - 2ab \cdot \cos C && \text{(rearrange terms)}
\end{aligned}
$$

There are only two types of problems in which it is appropriate to use the Law of Cosines. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called SSS like in geometry. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called SAS.

Take the following triangle.



The measure of angle D is missing and can be found using the Law of Cosines. It is necessary to set up the Law of Cosines equation very carefully with D corresponding to the opposite side of 230. The letters are not ABC like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C\\
230^2 &= 120^2 + 150^2 - 2 \cdot 120 \cdot 150 \cdot \cos D\\
230^2 - 120^2 - 150^2 &= -2 \cdot 120 \cdot 150 \cdot \cos D\\
\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150} &= \cos D\\
D &= \cos^{-1}\left(\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150}\right) \approx 116.4^\circ \approx 2.03 \textit{ radians}
\end{aligned}
$$

Examples

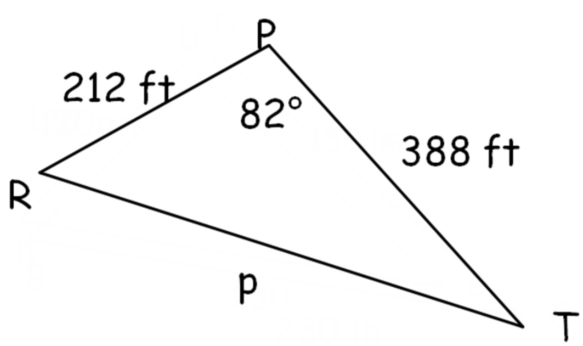
Example 1

Earlier, you were given a triangle with sides 11, 12, and 13 and asked what the measure of the angle opposite 11 is. A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\ 11^2 &= 12^2 + (13)^2 - 2 \cdot 12 \cdot 13 \cdot \cos C \\ C &= \cos^{-1}\left(\frac{11^2 - 12^2 - 13^2}{-2 \cdot 12 \cdot 13}\right) \approx 52.02\ldots^\circ \end{aligned}$$

Example 2

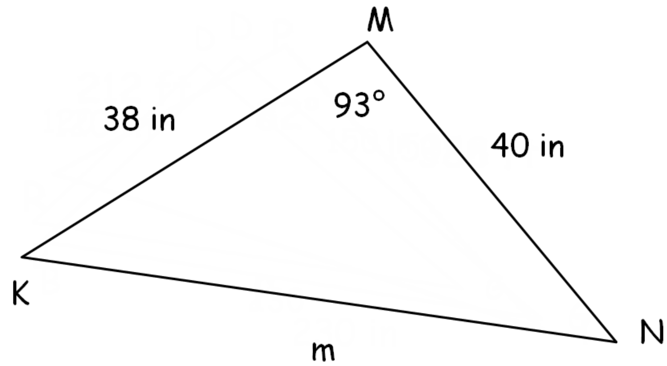
Determine the length of side p.



$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\ p^2 &= 212^2 + 388^2 - 2 \cdot 212 \cdot 388 \cdot \cos 82^\circ \\ p^2 &\approx 194192.02\ldots \\ p &\approx 440.7 \end{aligned}$$

Example 3

Determine the degree measure of angle N.



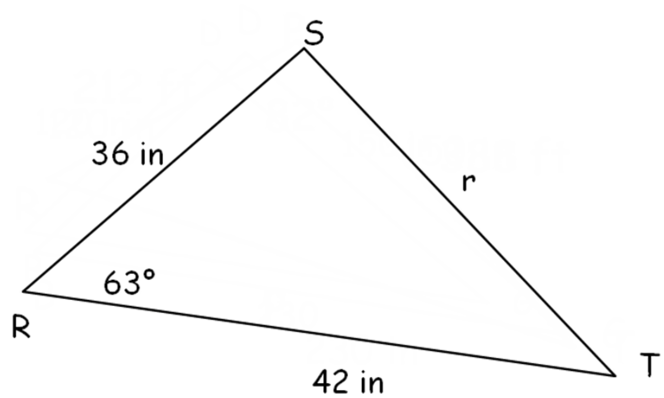
This problem must be done in two parts. First apply the Law of Cosines to determine the length of side m. This is a SAS situation like Example B. Once you have all three sides you will be in the SSS situation like in Example A and can apply the Law of Cosines again to find the unknown angle N.

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cdot \cos C\\ m^2 &= 38^2 + 40^2 - 2 \cdot 38 \cdot 40 \cdot \cos 93^\circ\\ m^2 &\approx 3203.1\ldots\\ m &\approx 56.59\ldots\end{aligned}$$

Now that you have all three sides you can apply the Law of Cosines again to find the unknown angle N. Remember to match angle N with the corresponding side length of 38 inches. It is also best to store m into your calculator and use the unrounded number in your future calculations.

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cdot \cos C\\ 38^2 &= 40^2 + (56.59\ldots)^2 - 2 \cdot 40 \cdot (56.59\ldots) \cdot \cos N\\ 38^2 - 40^2 - (56.59\ldots)^2 &= -2 \cdot 40 \cdot (56.59\ldots) \cdot \cos N\\ \frac{38^2 - 40^2 - (56.59\ldots)^2}{-2 \cdot 40 \cdot (56.59\ldots)} &= \cos N\\ N &= \cos^{-1}\left(\frac{38^2 - 40^2 - (56.59\ldots)^2}{-2 \cdot 40 \cdot (56.59\ldots)}\right) \approx 42.1^\circ\end{aligned}$$

For the next two examples, use the triangle below.



Example 4

Determine the length of side r.

$r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r = 41.07\ldots$

Example 5

Determine the measure of angle T in degrees.

$36^2 = (41.07\ldots)^2 + 42^2 - 2 \cdot (41.07\ldots) \cdot 42 \cdot \cos T$

$T \approx 51.34\ldots^\circ$

Review

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$.

1. Find the measure of angle A.
2. Find the measure of angle B.
3. Find the measure of angle C.
4. Find the measure of angle C in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$.

5. Find the measure of angle D.
6. Find the measure of angle E.
7. Find the measure of angle F.

In $\Delta GHI, g = 19, \angle H = 55^\circ$, and $i = 12$.

8. Find the length of h.
9. Find the measure of angle G.
10. Find the measure of angle I.
11. Explain why the Law of Cosines is connected to the Pythagorean Theorem.
12. What are the two types of problems where you might use the Law of Cosines?

Determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$
14. $a = 1, b = 5, c = 4$
15. $a = 5, b = 6, c = 10$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.5.

5.17 Area of a Triangle

Learning Objectives

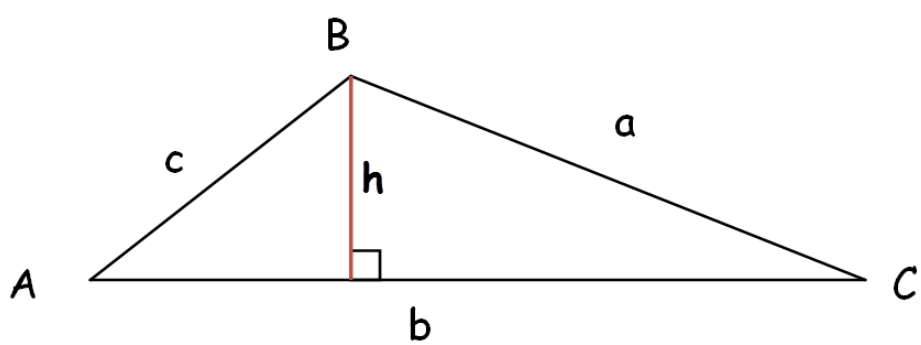
Here you'll use the sine ratio to find the area of non-right triangles in which two sides and the included angle measure are known.

From geometry you already know that the area of a triangle is $\frac{1}{2} \cdot b \cdot h$.

What if you are given the sides of a triangle are 5 and 6 and the angle between the sides is $\frac{\pi}{3}$? You are not directly given the height, but can you still figure out the area of the triangle?

Finding the Area of Triangles

The sine function allows you to find the height of any triangle and substitute that value into the familiar triangle area formula.



Using the sine function, you can isolate h for height:

$$\sin C = \frac{h}{a}$$
$$a \sin C = h$$

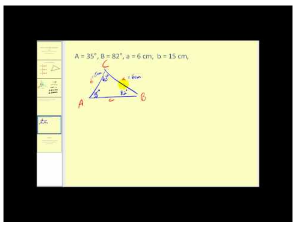
Substituting into the area formula:

$$Area = \frac{1}{2} b \cdot h$$
$$Area = \frac{1}{2} b \cdot a \cdot \sin C$$
$$Area = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

Note that the letters do not have to match exactly because the triangle or formula can just be relabeled. If you were given ΔABC with $A = 22°, b = 6, c = 7$ and asked to find the area, you would use the formula:

$$Area = \frac{1}{2} bc \sin A$$
$$Area = \frac{1}{2} \cdot 6 \cdot 7 \cdot \sin 22° \approx 7.86 \ldots units^2$$

The important part is that neither given side corresponds to the given angle.



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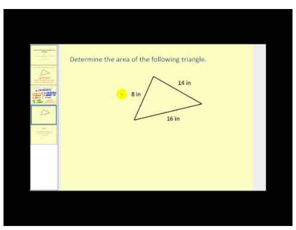
Bonus Video: There is another way of finding the area of a triangle, Heron's Formula. This will be discussed in Example 4.



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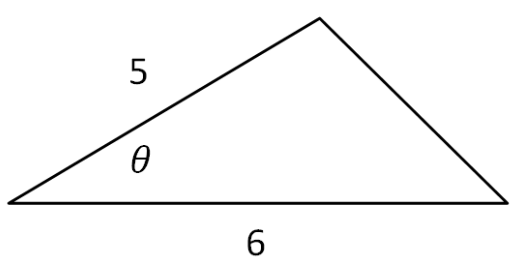
Examples

Example 1

Earlier, you were given the sides of a triangle are 5 and 6 and the angle between the sides is $\theta = \frac{\pi}{3}$ and asked to find the area.



$Area = \frac{1}{2} \cdot 5 \cdot 6 \cdot \sin \frac{\pi}{3} \approx 12.99\ un^2$

Example 2

Given ΔXYZ has area 28 square inches, what is the angle included between side length 8 and 9?

$$\begin{aligned}
Area &= \frac{1}{2} \cdot a \cdot b \cdot \sin C \\
28 &= \frac{1}{2} \cdot 8 \cdot 9 \cdot \sin C \\
\sin C &= \frac{28 \cdot 2}{8 \cdot 9} \\
C &= \sin^{-1} \left(\frac{28 \cdot 2}{8 \cdot 9} \right) \approx 51.05 \ldots^\circ
\end{aligned}$$

Example 3

Given triangle ABC with $A = 12^\circ, b = 4$ and $Area = 1.7\ un^2$, what is the length of side c?

$$\begin{aligned}
Area &= \frac{1}{2} \cdot c \cdot b \cdot \sin A \\
1.7 &= \frac{1}{2} \cdot c \cdot 4 \cdot \sin 12^\circ \\
c &= \frac{1.7 \cdot 2}{4 \cdot \sin 12^\circ} \approx 4.08 \ldots
\end{aligned}$$

Example 4

What is the area of ΔXYZ with $x = 11, y = 12, z = 13$?

Because none of the angles are given, there are two possible solution paths. You could use the Law of Cosines to find one angle. The angle opposite the side of length 11 is $52.02 \ldots^\circ$ therefore the area is:

$Area = \frac{1}{2} \cdot 12 \cdot 13 \cdot \sin 52.02 \ldots \approx 61.5\ un^2$

Another way to find the area is through the use of Heron's Formula which is:

$Area = \sqrt{s(s-a)(s-b)(s-c)}$

Where s is the semi perimeter:

$s = \frac{a+b+c}{2}$

Example 5

The area of a triangle is 3 square units. Two sides of the triangle are 4 units and 5 units. What is the measure of their included angle?

$3 = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin \theta$

$\theta = \sin^{-1} \left(\frac{3 \cdot 2}{4 \cdot 5} \right) \approx 17.45 \ldots^\circ$

Review

For 1-11, find the area of each triangle.

1. ΔABC if $a = 13, b = 15$, and $\angle C = 70^\circ$.
2. ΔABC if $b = 8, c = 4$, and $\angle A = 58^\circ$.
3. ΔABC if $b = 34, c = 29$, and $\angle A = 125^\circ$.
4. ΔABC if $a = 3, b = 7$, and $\angle C = 81^\circ$.
5. ΔABC if $a = 4.8, c = 3.7$, and $\angle B = 54^\circ$.
6. ΔABC if $a = 12, b = 5$, and $\angle C = 22^\circ$.
7. ΔABC if $a = 3, b = 10$, and $\angle C = 65^\circ$.
8. ΔABC if $a = 5, b = 9$, and $\angle C = 11^\circ$.
9. ΔABC if $a = 5, b = 7$, and $c = 8$.
10. ΔABC if $a = 7, b = 8$, and $c = 14$.
11. ΔABC if $a = 12, b = 14$, and $c = 13$.
12. The area of a triangle is 12 square units. Two sides of the triangle are 8 units and 4 units. What is the measure of their included angle?
13. The area of a triangle is 23 square units. Two sides of the triangle are 14 units and 5 units. What is the measure of their included angle?
14. Given ΔDEF has area 32 square inches, what is the angle included between side length 9 and 10?
15. Given ΔGHI has area 15 square inches, what is the angle included between side length 7 and 11?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.7.

5.18 Applications of Basic Triangle Trigonometry

Learning Objectives

Here you will apply your knowledge of trigonometry and problem solving in context.

Deciding when to use SOH, CAH, TOA, Law of Cosines or the Law of Sines is not always obvious. Sometimes more than one approach will work and sometimes correct computations can still lead to incorrect results. This is because correct interpretation is still essential.

If you use both the Law of Cosines and the Law of Sines on a triangle with sides 4, 7, 10 you end up with conflicting answers. Why?

Trigonometry Applications

When applying trigonometry, it is important to have a clear toolbox of mathematical techniques to use. Some of the techniques may be review like the fact that all three angles in a triangle sum to be 180°, other techniques may be new like the Law of Cosines. There also may be some properties that are true and make sense but have never been formally taught. Take a look at all the tools you have in your toolbox to solve applications with trigonometry.

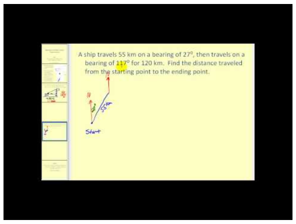
Toolbox

- The three angles in a triangle sum to be 180°.
- There are 360° in a circle and this can help us interpret negative angles as positive angles.
- The Pythagorean Theorem states that for legs a, b and hypotenuse c in a right triangle, $a^2 + b^2 = c^2$.
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the third side.
- The Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- The Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$ (Be careful for the ambiguous case)
- SOH CAH TOA is a mnemonic device to help you remember the three original trig functions:

$$\sin\theta = \frac{opp}{hyp} \quad \cos\theta = \frac{adj}{hyp} \quad \tan\theta = \frac{opp}{adj}$$

- 30-60-90 right triangles have side ratios $x, x\sqrt{3}, 2x$
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$
- Pythagorean number triples are exceedingly common and should always be recognized in right triangle problems. Examples of triples are 3, 4, 5 and 5, 12, 13.

A few definitions will also be necessary to solve applications. **Angle of elevation** is the angle at which you view and object above the horizon. **Angle of depression** is the angle at which you view and object below the horizon. This can be thought of as negative angles of elevation. Bearing is how direction is measured at sea. North is 0º, East is 90º , South is 180º and West is 270º.



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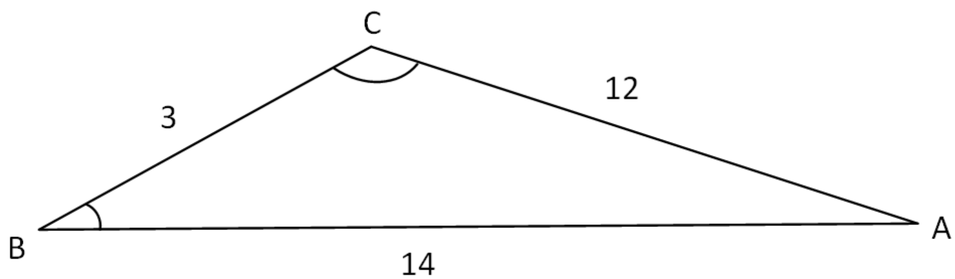
Examples

Example 1

Earlier, you were asked why you may end up with conflicting answers if you use both the Law of Sines and the Law of Cosines. Sometimes when using the Law of Sines you can get answers that do not match the Law of Cosines. Both answers can be correct computationally, but the Law of Sines may involve interpretation when the triangle is obtuse. The Law of Cosines does not require this interpretation step.



First, use Law of Cosines to find $\angle B$:

$$12^2 = 3^2 + 14^2 - 2 \cdot 3 \cdot 14 \cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2 - 3^2 - 14^2}{-2 \cdot 3 \cdot 14}\right) \approx 43.43\ldots^\circ$$

Then, use Law of Sines to find $\angle C$. Use the unrounded value for B even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14 \sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14 \sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of Cosines to double check $\angle C$.

$$14^2 = 3^2 + 12^2 - 2 \cdot 3 \cdot 12 \cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2 \cdot 3 \cdot 12}\right) \approx 126.7^\circ$$

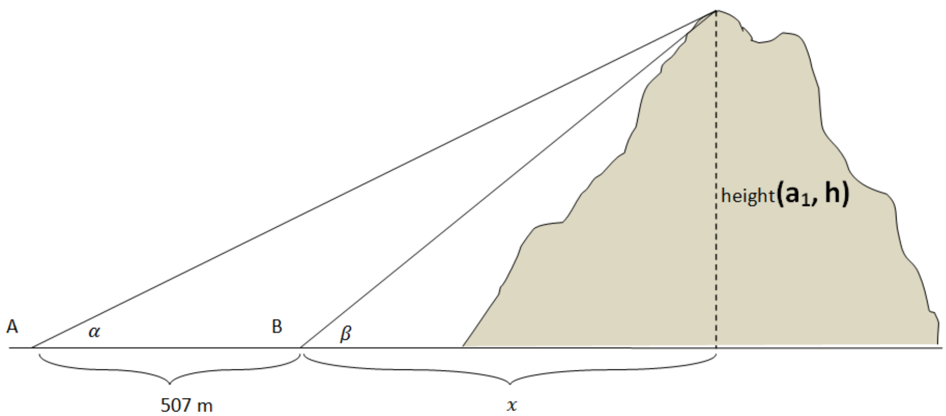
Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles.

Example 2

A surveying crew is given the job of verifying the height of a cliff. From point A, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

Note that α is just the Greek letter alpha and in this case it stands for the number 21.567°. β is the Greek letter beta and it stands for the number 25.683°.

First, sketch the image and label what you know.



Next, because the height is measured at a right angle with the ground, set up two equations. Remember that α and β are just numbers, not variables.

$$\tan\alpha = \frac{h}{507+x}$$
$$\tan\beta = \frac{h}{x}$$

Both of these equations can be solved for h and then set equal to each other to find x.

$$\begin{aligned}
h = \tan\alpha(507+x) &= x\tan\beta \\
507\tan\alpha + x\tan\alpha &= x\tan\beta \\
507\tan\alpha &= x\tan\beta - x\tan\alpha \\
507\tan\alpha &= x(\tan\beta - \tan\alpha) \\
x = \frac{507\tan\alpha}{\tan\beta - \tan\alpha} &= \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 228.7\ meters
\end{aligned}$$

Since the problem asked for the height, you need to substitute x back and solve for h.

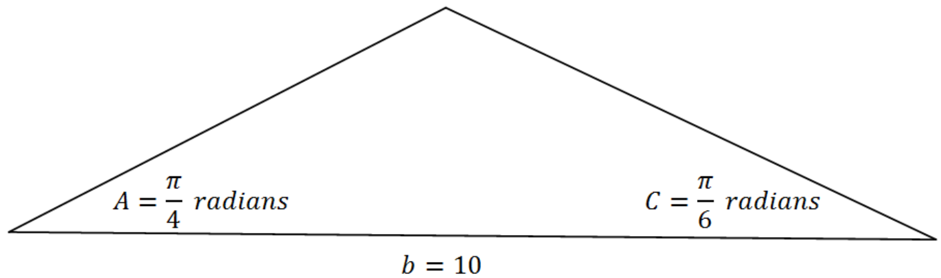
$h = x\tan\beta = 228.7\tan 25.683^\circ \approx 109.99\ meters$

Example 3

Given a triangle with SSS or SAS you know to use the Law of Cosines. In triangles where there are corresponding angles and sides like AAS or SSA it makes sense to use the Law of Sines. What about ASA?

Given ΔABC with $A = \frac{\pi}{4}$ *radians*, $C = \frac{\pi}{6}$ *radians* and $b = 10$ *in* what is a?

First,draw a picture.



The sum of the angles in a triangle is 180°. Since this problem is in radians you either need to convert this rule to radians, or convert the picture to degrees.

$$A = \frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = 45^\circ$$
$$C = \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ$$

The missing angle must be $\angle B = 105^\circ$. Now you can use the Law of Sines to solve for a.

$$\frac{\sin 105^\circ}{10} = \frac{\sin 45^\circ}{a}$$
$$a = \frac{10 \sin 45^\circ}{\sin 105^\circ} \approx 7.32 \ in$$

Example 4

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{10}$. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

When you draw a picture, you see that the given angle $\frac{\pi}{10}$ is not directly inside the triangle between the lighthouse, the boat and the base of the lighthouse. It is complementary to the angle you need.

$$\frac{\pi}{10} + \theta = \frac{\pi}{2}$$
$$\theta = \frac{2\pi}{5}$$

Now that you have the angle, use tangent to solve for x.

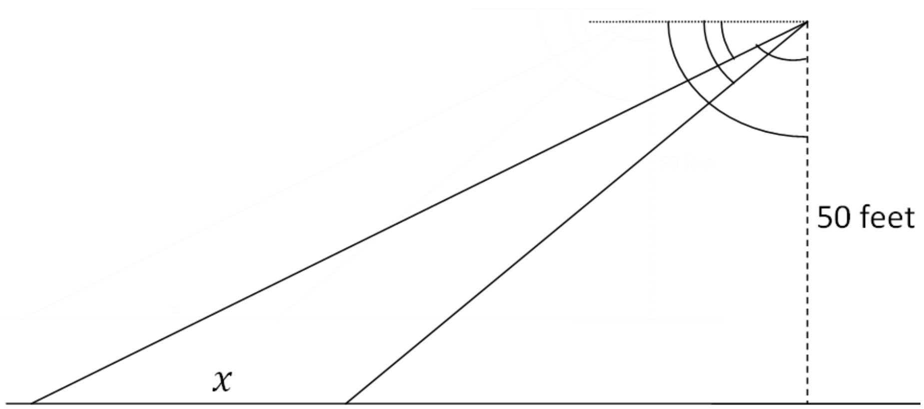
$$\tan \frac{2\pi}{5} = \frac{x}{200}$$
$$x = 200 \tan \frac{2\pi}{5} \approx 615.5 \ldots ft$$

Alternatively, you could have noticed that $\frac{\pi}{10}$ is alternate interior angles with the angle of elevation of the lighthouse from the boat's perspective. This would yield the same distance for x.

Example 5

From the third story of a building (50 feet) David observes a car moving towards the building driving on the streets below. If the angle of depression of the car changes from 21° to 45° while he watches, how far did the car travel?

Draw a very careful picture:



In the upper right corner of the picture there are four important angles that are marked with angles. The measures of these angles from the outside in are $90^\circ, 45^\circ, 21^\circ, 69^\circ$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore you can set up and solve an equation for x.

$$\tan 69^\circ = \frac{x+50}{50}$$
$$x = 50\tan 69^\circ - 50 \approx 80.25\ldots ft$$

The hardest part of this problem is drawing a picture and working with the angles.

Review

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. You want to find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.

2. What methods or techniques will you use?

3. Solve the problem.

From the third story of a building (60 feet) Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from 34° to 62° while he watches. You want to know how far the car traveled.

4. Draw a picture of this situation.

5. What methods or techniques will you use?

6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.

8. What methods or techniques will you use?

9. Solve the problem.

You want to figure out the height of a building. From point A, you measure an angle of elevation to the top of the building to be $\alpha = 10^\circ$. You move 50 feet closer to the building to point B and find that the angle to the top of the building is now $\beta = 60^\circ$.

10. Draw a picture of this situation.

11. What methods or techniques will you use?

12. Solve the problem.

13. Given ΔABC with $A = 40^\circ, C = 65^\circ$ and $b = 8\ in$, what is a?

14. Given ΔABC with $A = \frac{\pi}{3}\ radians, C = \frac{\pi}{8}\ radians$ and $b = 12\ in$ what is a?

15. Given ΔABC with $A = \frac{\pi}{6}\ radians, C = \frac{\pi}{4}\ radians$ and $b = 20\ in$ what is a?

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.8.

5.19 Right Triangle Trigonometry

Objective

To develop an understanding of trigonometric ratios and to use the trigonometric ratios sine, cosine and tangent along with their inverses to solve right triangles.

Review Queue

1. Given that $P(A) = 0.8, P(B) = 0.5$ and $P(A \cup B) = 0.9$, determine whether events A and B are independent.

2. Events A and B are independent and $P(A) = 0.6$ and $P(B) = 0.5$, find $P(A \cup B)'$.

3. Reduce the radical expressions:

a. $\sqrt{240}$

b. $3\sqrt{48} + 5\sqrt{75}$

c. $4\sqrt{15} \cdot \sqrt{30}$

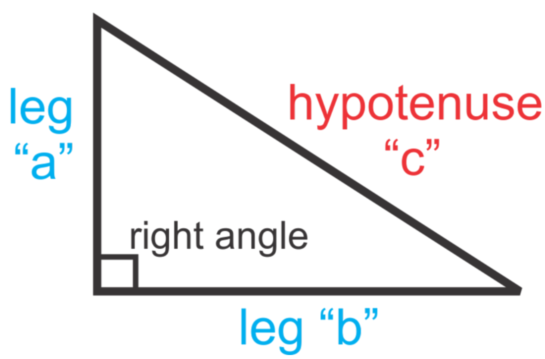
Pythagorean Theorem and its Converse

Objective

Discover, prove and apply the Pythagorean Theorem to solve for unknown sides in right triangles and prove triangles are right triangles.

Guidance

The Pythagorean Theorem refers to the relationship between the lengths of the three sides in a right triangle. It states that if a and b are the legs of the right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. For example, the lengths 3, 4, and 5 are the sides of a right triangle because $3^2 + 4^2 = 5^2 (9 + 16 = 25)$. Keep in mind that c is always the longest side.

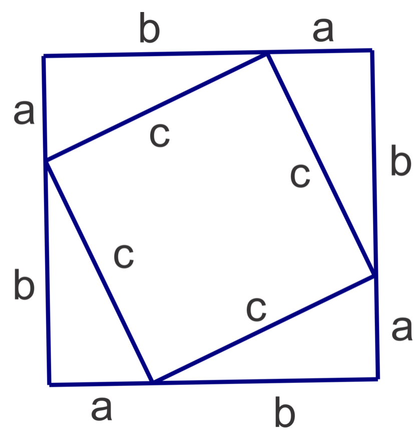


The converse of this statement is also true. If, in a triangle, c is the length of the longest side and the shorter sides have lengths a and b, and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Proof of Pythagorean Theorem

There are many proofs of the Pythagorean Theorem and here is one of them. We will be using the concept that the area of a figure is equal to the sum of the areas of the smaller figures contained within it and algebra to derive the Pythagorean Theorem.

Using the figure below (a square with a smaller square inside), first write two equations for its area, one using the lengths of the sides of the outer square and one using the sum of the areas of the smaller square and the four triangles.



Area 1: $(a+b)^2 = a^2 + 2ab + b^2$

Area 2: $c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$

Now, equate the two areas and simplify:

$$\begin{aligned} a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 &= c^2 \end{aligned}$$

Example A

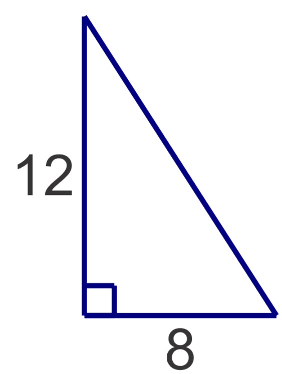
In a right triangle $a = 7$ and $c = 25$, find the length of the third side.

Solution: We can start by substituting what we know into the Pythagorean Theorem and then solve for the unknown side, b:

$$\begin{aligned} 7^2 + b^2 &= 25^2 \\ 49 + b^2 &= 625 \\ b^2 &= 576 \\ b &= 24 \end{aligned}$$

Example B

Find the length of the third side of the triangle below. Leave your answer in reduced radical form.



Solution: Since we are given the lengths of the two legs, we can plug them into the Pythagorean Theorem and find the length of the hypotenuse.

$$\begin{aligned}
8^2+12^2 &= c^2 \\
64+144 &= c^2 \\
c^2 &= 208 \\
c &= \sqrt{208} = \sqrt{16 \cdot 13} = 4\sqrt{13}
\end{aligned}$$

Example C

Determine whether a triangle with lengths 21, 28, 35 is a right triangle.

Solution: We need to see if these values will satisfy $a^2+b^2=c^2$. If they do, then a right triangle is formed. So,

$$\begin{aligned}
21^2+28^2 &= 441+784 = 1225 \\
35^2 &= 1225
\end{aligned}$$

Yes, the Pythagorean Theorem is satisfied by these lengths and a right triangle is formed by the lengths 21, 28 and 35.

Guided Practice

For the given two sides, determine the length of the third side if the triangle is a right triangle.

1. $a=10$ and $b=5$

2. $a=5$ and $c=13$

Use the Pythagorean Theorem to determine if a right triangle is formed by the given lengths.

3. 16, 30, 34

4. 9, 40, 42

5. 2, 2, 4

Answers

1. $\sqrt{10^2+5^2} = \sqrt{100+25} = \sqrt{125} = 5\sqrt{5}$

2. $\sqrt{13^2-5^2} = \sqrt{169-25} = \sqrt{144} = 12$

3.

$$\begin{aligned}
16^2+30^2 &= 256+900 = 1156 \\
34^2 &= 1156
\end{aligned}$$

Yes, this is a right triangle.

4.

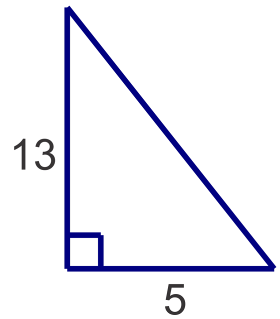
$$\begin{aligned}
9^2+40^2 &= 81+1600 = 1681 \\
42^2 &= 1764
\end{aligned}$$

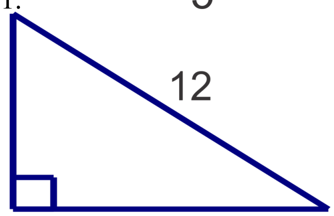
No, this is not a right triangle.

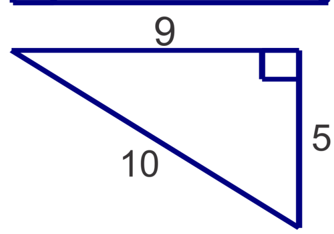
5. This one is tricky, in a triangle the lengths of any two sides must have a sum *greater* than the length of the third side. These lengths do not meet that requirement so not only do they not form a *right* triangle, they do not make a triangle at all.

Problem Set

Find the unknown side length for each right triangle below.

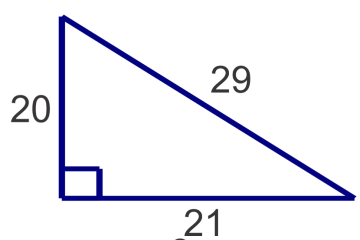
1. 

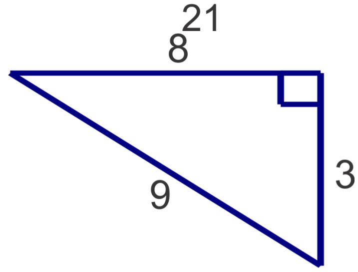
2. 

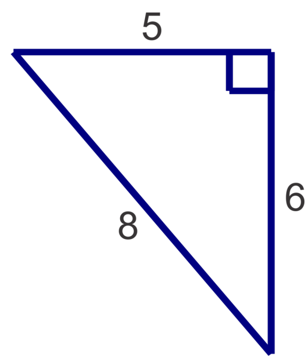
3. 

4. $a = 6, b = 8$
5. $b = 6, c = 14$
6. $a = 12, c = 18$

Determine whether the following triangles are right triangles.

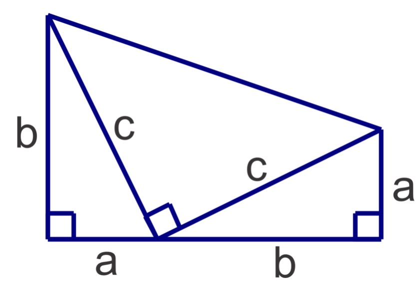
7. 

8. 

9. 

Do the lengths below form a right triangle? Remember to make sure that they form a triangle.

10. 3, 4, 5
11. 6, 6, 11
12. 11, 13, 17
13. Major General James A. Garfield (and former President of the U.S.) is credited with deriving this proof of the Pythagorean Theorem using a trapezoid. Follow the steps to recreate his proof.



(a) Find the area of the trapezoid using the trapezoid area formula: $A = \frac{1}{2}(b_1 + b_2)h$ (b) Find the sum of the areas of the three right triangles in the diagram. (c) The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

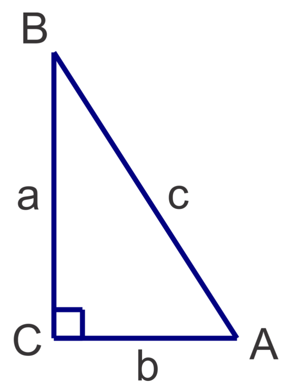
Sine, Cosine, and Tangent

Objective

Define and apply the trigonometric ratios sine, cosine and tangent to solve for the lengths of unknown sides in right triangles.

Guidance

The trigonometric ratios sine, cosine and tangent refer to the known ratios between particular sides in a right triangle based on an acute angle measure.



In this right triangle, side c is the hypotenuse.

If we consider the angle B, then we can describe each of the legs by its position relative to angle B: side a is adjacent to B; side b is opposite B

If we consider the angle A, then we can describe each of the legs by its position relative to angle A: side b is adjacent to A; side a is opposite A

Now we can define the trigonometry ratios as follows:

$$\text{Sine is } \frac{opposite}{hypotenuse} \quad \text{Cosine is } \frac{adjacent}{hypotenuse} \quad \text{Tangent is } \frac{opposite}{adjacent}$$

A shorthand way to remember these ratios is to take the letters in red above and write the phrase:

SOH CAH TOA

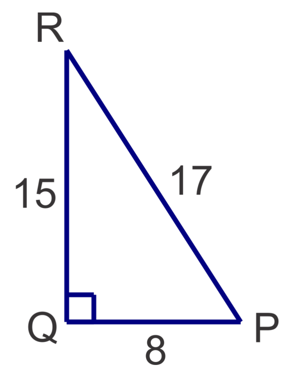
Now we can find the trigonometric ratios for each of the acute angles in the triangle above.

$$\begin{array}{ll} \sin A = \frac{a}{c} & \sin B = \frac{b}{c} \\ \cos A = \frac{b}{c} & \cos B = \frac{a}{c} \\ \tan A = \frac{a}{b} & \tan B = \frac{b}{a} \end{array}$$

It is important to understand that given a particular (acute) angle measure in a right triangle, these ratios are constant no matter how big or small the triangle. For example; if the measure of the angle is 25°, then $\sin 25^\circ \approx 0.4226$ and ratio of the opposite side to the hypotenuse is always 0.4226 no matter how big or small the triangle.

Example A

Find the trig ratios for the acute angles R and P in ΔPQR.



Solution: From angle R, $O = 8$; $A = 15$; and $H = 17$. Now the trig ratios are:

$$\sin R = \frac{8}{17};\ \cos R = \frac{15}{17};\ \tan R = \frac{8}{15}$$

From angle P, $O = 15$; $A = 8$; and $H = 17$. Now the trig ratios are:

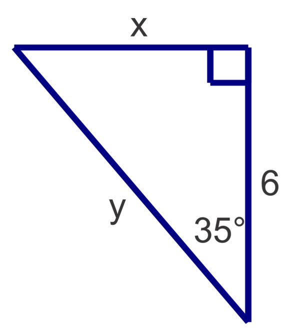
$$\sin P = \frac{15}{17};\ \cos P = \frac{8}{17};\ \tan P = \frac{15}{8}$$

Do you notice any patterns or similarities between the trigonometric ratios? The opposite and adjacent sides are switched and the hypotenuse is the same. Notice how this switch affects the ratios:

$$\sin R = \cos P \quad \cos R = \sin P \quad \tan R = \frac{1}{\tan P}$$

Example B

Use trigonometric ratios to find the x and y.



Solution: First identify or label the sides with respect to the given acute angle. So, x is opposite, y is hypotenuse (note that it is the hypotenuse because it is the side opposite the right angle, it may be adjacent to the given angle but the hypotenuse cannot be the adjacent side) and 6 is the adjacent side.

To find x, we must use the given length of 6 in our ratio too. So we are using opposite and adjacent. Since tangent is the ratio of opposite over adjacent we get:

$$\begin{aligned} \tan 35^\circ &= \frac{x}{6} \\ x &= 6\tan 35^\circ && \text{multiply both sides by 6} \\ x &\approx 4.20 && \text{Use the calculator to evaluate-type in 6TAN(35) ENTER} \end{aligned}$$

NOTE: make sure that your calculator is in DEGREE mode. To check, press the MODE button and verify that DEGREE is highlighted (as opposed to RADIAN). If it is not, use the arrow buttons to go to DEGREE and press ENTER. The default mode is radian, so if your calculator is reset or the memory is cleared it will go back to radian mode until you change it.

To find y using trig ratios and the given length of 6, we have adjacent and hypotenuse so we'll use cosine:

$$\begin{aligned}
\cos 35^\circ &= \frac{6}{y} \\
\frac{\cos 35^\circ}{1} &= \frac{6}{y} && \text{set up a proportion to solve for } y \\
6 &= y \cos 35^\circ && \text{cross multiply} \\
y &= \frac{6}{\cos 35^\circ} && \text{divide by} \cos 35^\circ \\
y &= 7.32 && \text{Use the calculator to evaluate-type in 6/TAN(35) ENTER}
\end{aligned}$$

Alternatively, we could find y using the value we found for x and the Pythagorean theorem:

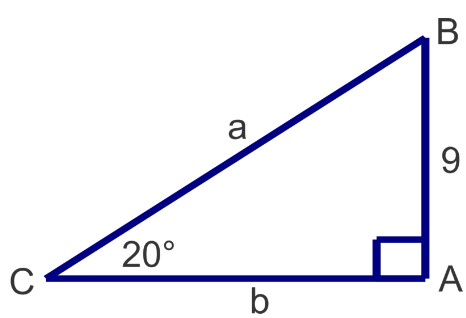
$$\begin{aligned}
4.20^2 + 6^2 &= y^2 \\
53.64 &= y^2 \\
y &\approx 7.32
\end{aligned}$$

The downside of this method is that if we miscalculated our x value, we will double down on our mistake and guarantee an incorrect y value. In general you will help avoid this kind of mistake if you use the given information whenever possible.

Example C

Given ΔABC, with $m\angle A = 90^\circ, m\angle C = 20^\circ$ and $c = 9$, find a and b.

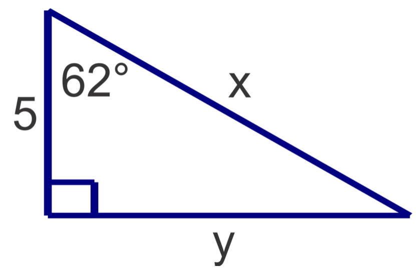
Solution: Visual learners may find it particularly useful to make a sketch of this triangle and label it with the given information:



To find a (the hypotenuse) we can use the opposite side and the sine ratio: $\sin 20^\circ = \frac{9}{a}$, solving as we did in Example B we get $a = \frac{9}{\sin 20^\circ} \approx 26.31$ To find b (the adjacent side) we can use the opposite side and the tangent ratio: $\tan 20^\circ = \frac{9}{b}$, solving for b we get $b = \frac{9}{\tan 20^\circ} \approx 24.73$.

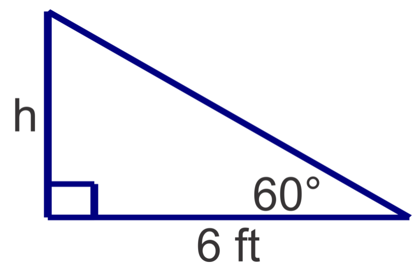
Guided Practice

1. Use trig ratios to find x and y:



2. Given ΔABC with $m\angle B = 90^\circ, m\angle A = 43^\circ$ and $a = 7$, find b and c.

3. The base of a playground slide is 6 ft from the base of the platform and the slide makes a 60° angle with the ground. To the nearest tenth of a foot, how high is the platform at the top of the slide?



Answers

1. For x:

$$\cos 62^\circ = \frac{5}{x}$$
$$x = \frac{5}{\cos 62^\circ} \approx 10.65$$

For y:

$$\tan 62^\circ = \frac{y}{5}$$
$$y = 5\tan 62^\circ \approx 9.40$$

2. For b:

$$\sin 43^\circ = \frac{7}{b}$$
$$b = \frac{7}{\sin 43^\circ} \approx 10.26$$

For c:

$$\tan 43^\circ = \frac{7}{c}$$
$$c = \frac{7}{\tan 43^\circ} \approx 7.51$$

3.

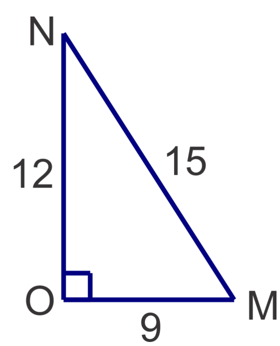
$$\tan 60^\circ = \frac{h}{6}$$
$$h = 6\tan 60^\circ \approx 10.39$$

, so the height of the platform is 10.4 ft

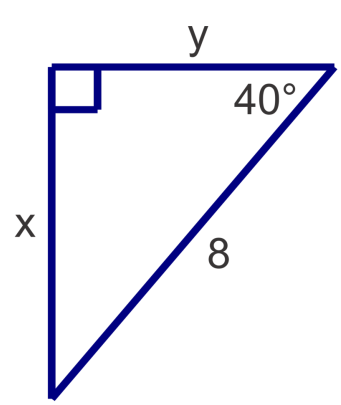
Problem Set

Use you calculator to find the following trigonometric ratios. Give answers to four decimal places.

1. $\sin 35°$
2. $\tan 72°$
3. $\cos 48°$
4. Write the three trigonometric ratios of each of the acute angles in the triangle below.



Use trigonometric ratios to find the unknown side lengths in the triangles below. Round your answers to the nearest hundredth.



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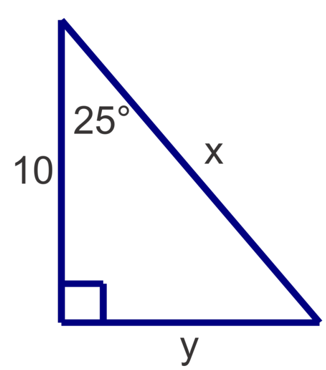

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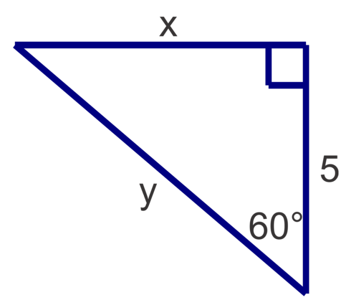
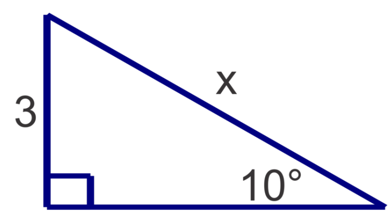


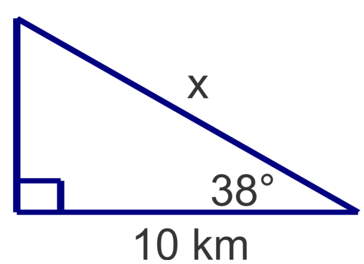
7.

For problems 8-10 use the given information about ΔABC with right angle B to find the unknown side lengths. Round your answer to the nearest hundredth.

8. $a = 12$ and $m\angle A = 43^\circ$
9. $m\angle C = 75^\circ$ and $b = 24$
10. $c = 7$ and $m\angle A = 65^\circ$
11. A ramp needs to have an angle of elevation no greater than 10 degrees. If the door is 3 ft above the sidewalk level, what is the minimum possible ramp length to the nearest tenth of a foot?



12. A ship, *Sea Dancer*, is 10 km due East of a lighthouse. A second ship, *Nelly*, is due north of the lighthouse. A spotter on the *Sea Dancer* measures the angle between the *Nelly* and the lighthouse to be 38°. How far apart are the two ships to the nearest tenth of a kilometer?



Inverse Trig Functions and Solving Right Triangles

Objective

Use the inverse trigonometric functions to find the measure of unknown acute angles in right triangles and solve right triangles.

Guidance

In the previous concept we used the trigonometric functions sine, cosine and tangent to find the ratio of particular sides in a right triangle given an angle. In this concept we will use the inverses of these functions, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, to find the angle measure when the ratio of the side lengths is known. When we type $\sin 30^\circ$ into our calculator, the calculator goes to a table and finds the trig ratio associated with 30°, which is $\frac{1}{2}$. When we use an inverse function we tell the calculator to look up the ratio and give us the angle measure. For example: $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$. On your calculator you would press 2^{ND}SIN to get SIN^{-1}(and then type in $\frac{1}{2}$, close the parenthesis and press ENTER. Your calculator screen should read SIN$^{-1}\left(\frac{1}{2}\right)$ when you press ENTER.

Example A

Find the measure of angle A associated with the following ratios. Round answers to the nearest degree.

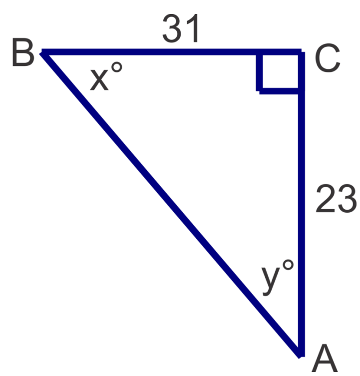
1. $\sin A = 0.8336$
2. $\tan A = 1.3527$
3. $\cos A = 0.2785$

Solution: Using the calculator we get the following:

1. $\sin^{-1}(0.8336) \approx 56^\circ$
2. $\tan^{-1}(1.3527) \approx 54^\circ$
3. $\cos^{-1}(0.2785) \approx 74^\circ$

Example B

Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



Solution: We can solve for either x or y first. If we choose to solve for x first, the 23 is opposite and 31 is adjacent so we will use the tangent ratio.

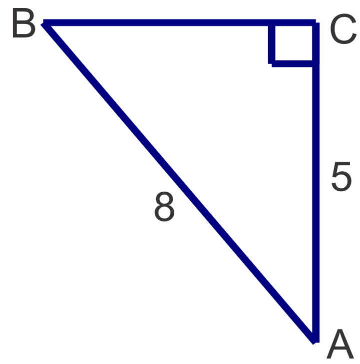
$$x = \tan^{-1}\left(\frac{23}{31}\right) \approx 37^\circ.$$

Recall that in a right triangle, the acute angles are always complementary, so $90^\circ - 37^\circ = 53^\circ$, so $y = 53^\circ$. We can also use the side lengths an a trig ratio to solve for y:

$$y = \tan^{-1}\left(\frac{31}{23}\right) \approx 53^\circ.$$

Example C

Solve the right triangle shown below. Round all answers to the nearest tenth.



Solution: We can solve for either angle A or angle B first. If we choose to solve for angle B first, then 8 is the hypotenuse and 5 is the opposite side length so we will use the sine ratio.

$$\sin B = \frac{5}{8}$$
$$m\angle B = \sin^{-1}\left(\frac{5}{8}\right) \approx 38.7^\circ$$

Now we can find A two different ways.

Method 1: We can using trigonometry and the cosine ratio:

$$\cos A = \frac{5}{8}$$
$$m\angle A = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

Method 2: We can subtract $m\angle B$ from 90°: $90^\circ - 38.7^\circ = 51.3^\circ$ since the acute angles in a right triangle are always complimentary.

Either method is valid, but be careful with Method 2 because a miscalculation of angle B would make the measure you get for angle A incorrect as well.

Guided Practice

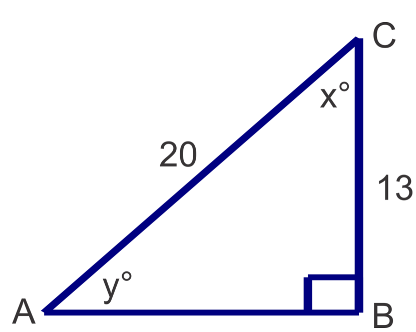
1. Find the measure of angle A to the nearest degree given the trigonometric ratios.

a. $\sin A = 0.2894$

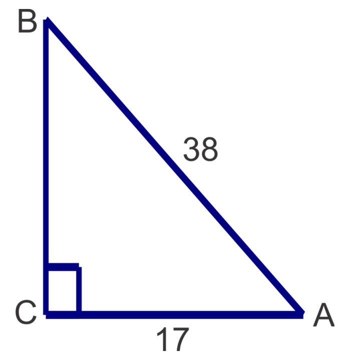
b. $\tan A = 2.1432$

c. $\cos A = 0.8911$

2. Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



3. Solve the triangle. Round side lengths to the nearest tenth and angles to the nearest degree.



Answers

1. a. $\sin^{-1}(0.2894) \approx 17°$

b. $\tan^{-1}(2.1432) \approx 65°$

c. $\cos^{-1}(0.8911) \approx 27°$

2.

$$x = \cos^{-1}\left(\frac{13}{20}\right) \approx 49°; \quad y = \sin^{-1}\left(\frac{13}{20}\right) \approx 41°$$

3.

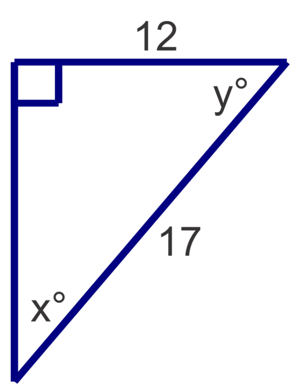
$$m\angle A = \cos^{-1}\left(\frac{17}{38}\right) \approx 63°; \quad m\angle B = \sin^{-1}\left(\frac{17}{38}\right) \approx 27°; \quad a = \sqrt{38^2 - 17^2} \approx 34.0$$

Problem Set

Use your calculator to find the measure of angle B. Round answers to the nearest degree.

1. $\tan B = 0.9523$
2. $\sin B = 0.8659$
3. $\cos B = 0.1568$

Find the measures of the unknown acute angles. Round measures to the nearest degree.



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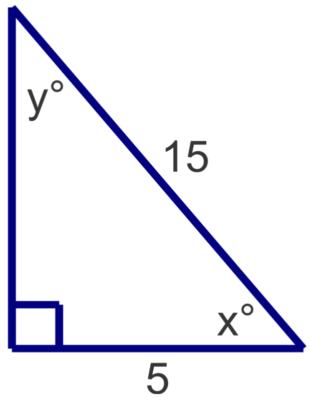
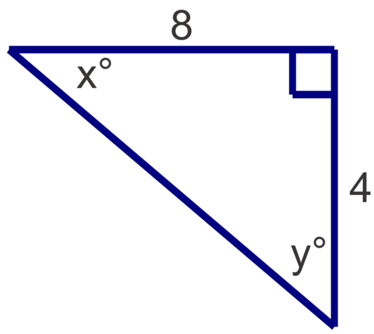
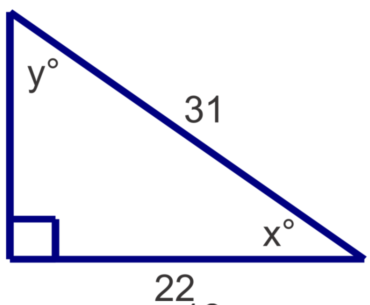
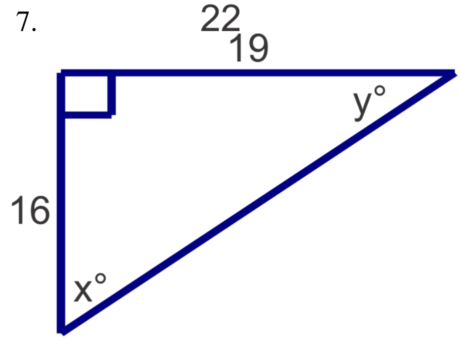
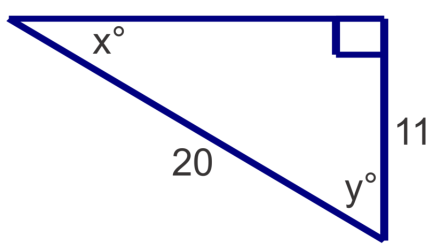
5. 

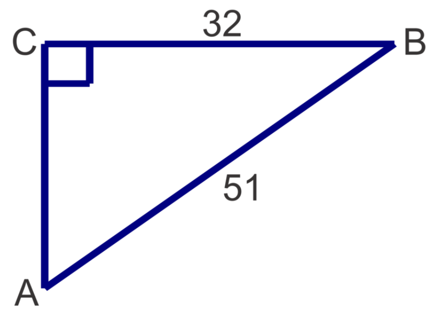
6. 

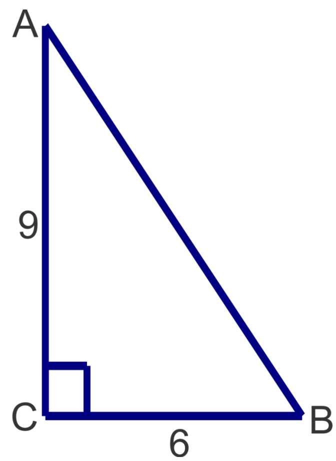
7. 

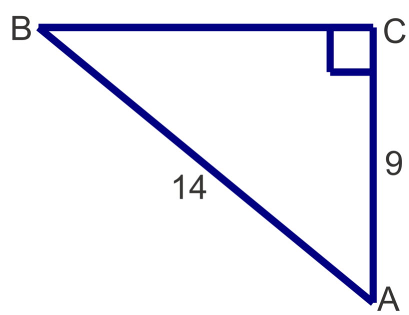
8. 

9. 

Solve the following right triangles. Round angle measures to the nearest degree and side lengths to the nearest tenth.

10. 

11. 

12. 

Application Problems

Objective

Use the Pythagorean Theorem and trigonometric ratios to solve the real world application problems.

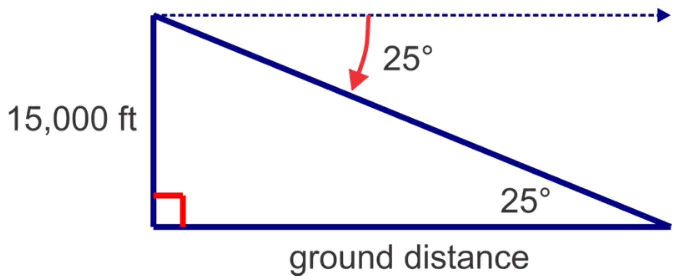
Guidance

When solving word problems, it is important to understand the terminology used to describe angles. In trigonometric problems, the terms angle of elevation and angle of depression are commonly used. Both of these angles are always measured from a horizontal line as shown in the diagrams below.

Example A

An airplane approaching an airport spots the runway at an angle of depression of 25°. If the airplane is 15,000 ft above the ground, how far (ground distance) is the plane from the runway? Give your answer to the nearest 100 ft.

Solution: Make a diagram to illustrate the situation described and then use a trigonometric ratio to solve. Keep in mind that an angle of depression is down from a horizontal line of sight-in this case a horizontal line from the pilot of the plane parallel to the ground.



Note that the angle of depression and the alternate interior angle will be congruent, so the angle in the triangle is also 25°.

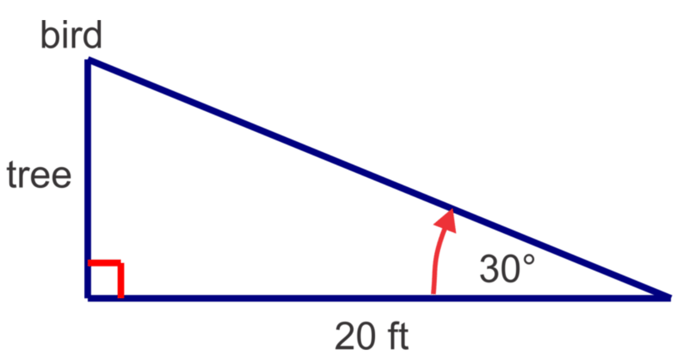
From the picture, we can see that we should use the tangent ratio to find the ground distance.

$$\tan 25^\circ = \frac{15000}{d}$$
$$d = \frac{15000}{\tan 25^\circ} \approx 32,200\ ft$$

Example B

Rachel spots a bird in a tree at an angle of elevation of 30°. If Rachel is 20 ft from the base of the tree, how high up in the tree is the bird? Give your answer to the nearest tenth of a foot.

Solution: Make a diagram to illustrate the situation. Keep in mind that there will be a right triangle and that the right angle is formed by the ground and the trunk of the tree.



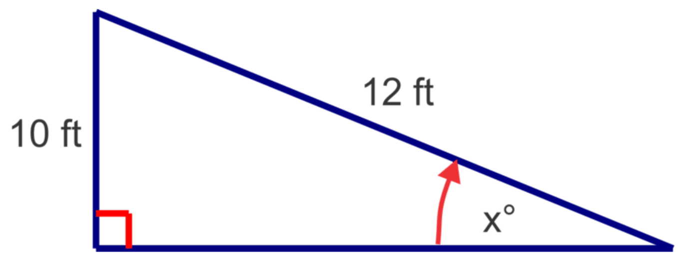
Here we can use the tangent ratio to solve for the height of the bird

$$\tan 30^\circ = \frac{h}{20}$$
$$h = 20 \tan 30^\circ \approx 11.5\ ft$$

Example C

A 12 ft ladder is leaning against a house and reaches 10 ft up the side of the house. To the nearest degree, what angle does the ladder make with the ground?

Solution: In this problem, we will need to find an angle. By making a sketch of the triangle we can see which inverse trigonometric ratio to use.



$$\sin x^\circ = \frac{10}{12}$$
$$\sin^{-1}\left(\frac{10}{12}\right) \approx 56^\circ$$

Guided Practice

Use a trigonometry to solve the following application problems.

1. A ramp makes a 20° angle with the ground. If door the ramp leads to is 2 ft above the ground, how long is the ramp? Give your answer to the nearest tenth of a foot.

2. Charlie lets out 90 ft of kite string. If the angle of elevation of the string is 70°, approximately how high is the kite? Give your answer to the nearest foot.

3. A ship's sonar spots a wreckage at an angle of depression of 32°. If the depth of the ocean is about 250 ft, how far is the wreckage (measured along the surface of the water) from the ship, to the nearest foot.

Answers

1.

$$\sin 20^\circ = \frac{2}{x}$$
$$x = \frac{2}{\sin 20^\circ} \approx 5.8\ ft$$

2.

$$\sin 70^\circ = \frac{x}{90}$$
$$x = 90 \sin 70^\circ \approx 85\ ft$$

3.

$$\tan 32^\circ = \frac{250}{x}$$
$$x = \frac{250}{\tan 32^\circ} \approx 400\ ft$$

Vocabulary

Angle of Elevation
An angle measured up from a horizontal line.

Angle of Depression
An angle measured down from a horizontal line.

Problem Set

Use the Pythagorean Theorem and/or trigonometry to solve the following word problems.

1. A square has sides of length 8 inches. To the nearest tenth of an inch, what is the length of its diagonal?
2. Layne spots a sailboat from her fifth floor balcony, about 25 m above the beach, at an angle of depression of 3°. To the nearest meter, how far out is the boat?
3. A zip line takes passengers on a 200 m ride from high up in the trees to a ground level platform. If the angle of elevation of the zip line is 10°, how high above ground is the tree top start platform? Give your answer to the nearest meter.
4. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 57°. If the building is 150 ft tall, how far away, to the nearest foot, is the fountain?
5. A playground slide platform is 6 ft above ground. If the slide is 8 ft long and the end of the slide is 1 ft above ground, what angle does the slide make with the ground? Give your answer to the nearest degree.
6. Benjamin spots a tree directly across the river from where he is standing. He then walks 27 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 73°. How wide, to the nearest foot, is the river?
7. A rectangle has sides of length 6 in and 10 in. To the nearest degree, what angle does the diagonal make with the longer side?
8. Tommy is flying his kite one afternoon and notices that he has let out the entire 130 ft of string. The angle his string makes with the ground is 48°. How high, to the nearest foot, is his kite at this time?
9. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 18° angle with the ground 21 ft from the base of the tree. What was the height of the tree to the nearest foot?
10. Upon descent an airplane is 19,000 ft above the ground. The air traffic control tower is 190 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
11. Why will the sine and cosine ratios always be less than 1?

5.20 Vectors

Learning Objectives

- Understand directed line segments, equal vectors, and absolute value in relation to vectors.
- Perform vector addition and subtraction.
- Find the resultant vector of two displacements.

In previous examples, we could simply use triangles to represent direction and distance. In real-life, there are typically other factors involved, such as the speed of the object (that is moving in the given direction and distance) and wind. We need another tool to represent not only direction but also magnitude (length) or force. This is why we need vectors. Vectors capture the interactions of real world velocities, forces and distance changes.

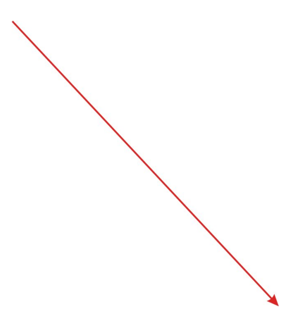
Any application in which direction is specified requires the use of vectors. A **vector** is any quantity having **direction** and **magnitude**. Vectors are very common in science, particularly physics, engineering, electronics, and chemistry in which one must consider an object's motion (either velocity or acceleration) and the direction of that motion.

In this section, we will look at how and when to use vectors. We will also explore vector addition, subtraction, and the resultant of two displacements. In addition we will look at real-world problems and application involving vectors.

Directed Line Segments, Equal Vectors, and Absolute Value

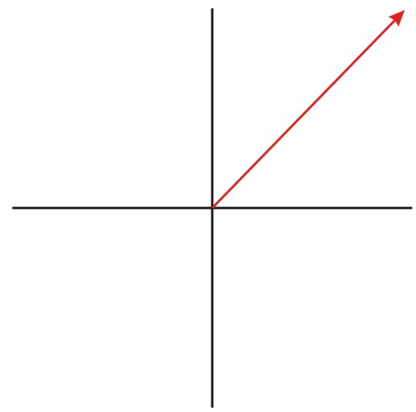
A vector is represented diagrammatically by a directed line segment or arrow. A **directed line segment** has both **magnitude** and **direction**. **Magnitude** refers to the length of the directed line segment and is usually based on a scale. The vector quantity represented, such as influence of the wind or water current may be completely invisible.

A 25 mph wind is blowing from the northwest. If 1 $cm = 5\ mph$, then the vector would look like this:



An object affected by this wind would travel in a southeast direction at 25 mph.

A vector is said to be in **standard position** if its **initial point** is at the origin. The initial point is where the vector begins and the **terminal point** is where it ends. The axes are arbitrary. They just give a place to draw the vector.



vector in standard position

If we know the coordinates of a vector's initial point and terminal point, we can use these coordinates to find the magnitude and direction of the vector.

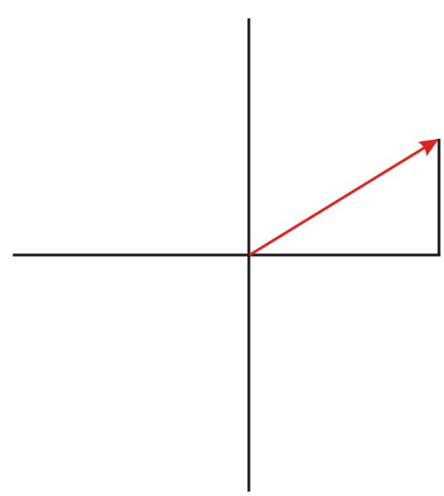
All vectors have **magnitude**. This measures the total distance moved, total velocity, force or acceleration. "Distance" here applies to the magnitude of the vector even though the vector is a measure of velocity, force, or acceleration. In order to find the magnitude of a vector, we use the distance formula. A vector can have a negative magnitude. A force acting on a block pushing it at 20 lbs north can be also written as vector acting on the block from the south with a magnitude of -20 lbs. Such negative magnitudes can be confusing; making a diagram helps. The -20 lbs south can be re-written as +20 lbs north without changing the vector. Magnitude is also called the **absolute value** of a vector.

Example 1: If we know the coordinates of the initial point and the terminal point, we can find the magnitude by using the distance formula. Initial point (0,0) and terminal point (3,5).

Solution: $|\vec{v}| = \sqrt{(3-0)^2+(5-0)^2} = \sqrt{9+25} = 5.8$ The magnitude of$\vec{v}$ is 5.8.

If we don't know the coordinates of the vector, we must use a ruler and the given scale to find the magnitude. Also notice the notation of a vector, which is usually a lower case letter (typically u, v, or w) in italics, with an arrow over it, which indicates direction. If a vector is in standard position, we can use trigonometric ratios such as sine, cosine and tangent to find the **direction** of that vector.

Example 2: If a vector is in standard position and its terminal point has coordinates of (12, 9) what is the direction?



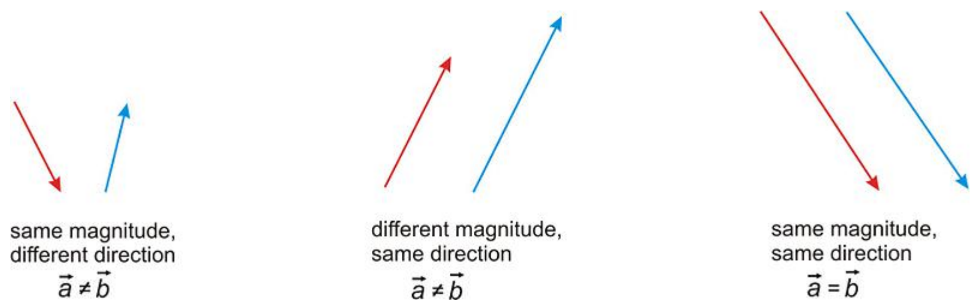
Solution: The horizontal distance is 12 while the vertical distance is 9. We can use the tangent function since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{9}{12}$$
$$\tan^{-1}\frac{9}{12} = 36.9^\circ$$

So, the direction of the vector is 36.9°.

If the vector isn't in standard position and we don't know the coordinates of the terminal point, we must a protractor to find the direction.

Two vectors are **equal** if they have the same magnitude and direction. Look at the figures below for a visual understanding of **equal vectors**.



Example 3: Determine if the two vectors are equal.

$\vec{a}$ is in standard position with terminal point (-4, 12)

$\vec{b}$ has an initial point of (7, -6) and terminal point (3, 6)

Solution: You need to determine if both the magnitude and the direction are the same.

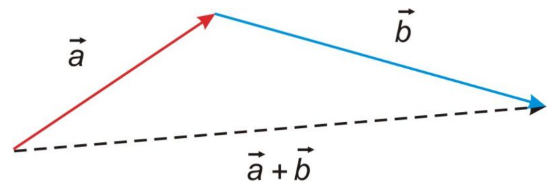
$$\begin{aligned}
\text{Magnitude :} \quad |\vec{a}| &= \sqrt{(0-(-4))^2+(0-12)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
|\vec{b}| &= \sqrt{(7-3)^2+(-6-6)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \\
\text{Direction :} \quad \vec{a} &\to \tan\theta = \frac{12}{-4} \to \theta = 108.43^\circ \\
\vec{b} &\to \tan\theta = \frac{-6-6}{7-3} = \frac{-12}{4} \to \theta = 108.43^\circ
\end{aligned}$$

Because the magnitude and the direction are the same, we can conclude that the two vectors are equal.

Vector Addition

The sum of two or more vectors is called the **resultant** of the vectors. There are two methods we can use to find the resultant: the triangle method and the parallelogram method.

The Triangle Method: To use the triangle method, we draw the vectors one after another and place the initial point of the second vector at the terminal point of the first vector. Then, we draw the resultant vector from the initial point of the first vector to the terminal point of the second vector. *This method is also referred to as the tip-to-tail method.*



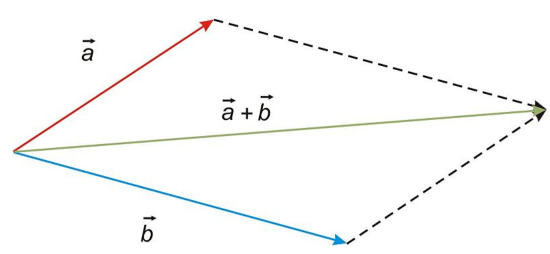
To find the sum of the resultant vector we would use a ruler and a protractor to find the magnitude and direction.

The resultant vector can be much longer than either $\vec{a}$ or $\vec{b}$, or it can be shorter. Below are some more examples of the triangle method.

Example 4:

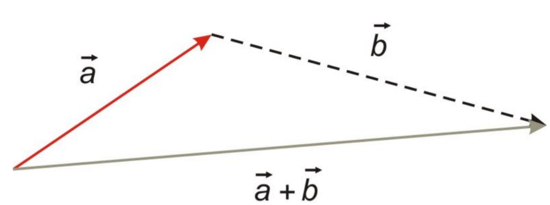
Example 5:

The Parallelogram Method: Another method we could use is the parallelogram method. To use the parallelogram method, we draw the vectors so that their initial points meet. Then, we draw in lines to form a parallelogram. The resultant is the diagonal from the initial point to the opposite vertex of the parallelogram. *It is important to note that we cannot use the parallelogram method to find the sum of a vector and itself.*



To find the sum of the resultant vector, we would again use a ruler and a protractor to find the magnitude and direction.

If you look closely, you'll notice that the parallelogram method is really a version of the triangle or tip-to-tail method. If you look at the top portion of the figure above, you can see that one side of our parallelogram is really vector *b* translated.

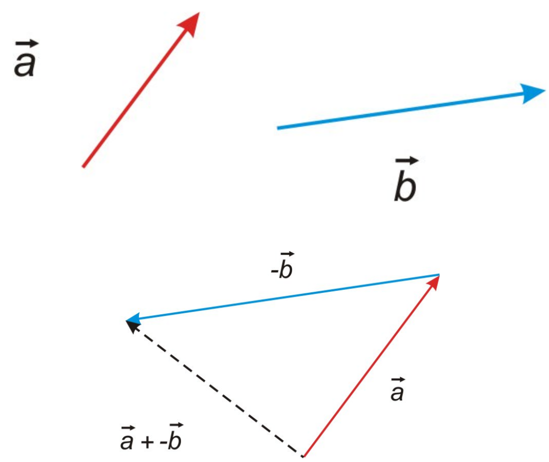


Vector Subtraction

As you know from Algebra, $A - B = A + (-B)$. When we think of vector subtraction, we must think about it in terms of adding a negative vector. A **negative** vector is the same magnitude of the original vector, but its direction is opposite.

In order to subtract two vectors, we can use either the triangle method or the parallelogram method from above. The only difference is that instead of adding vectors A and B, we will be adding A and $-B$.

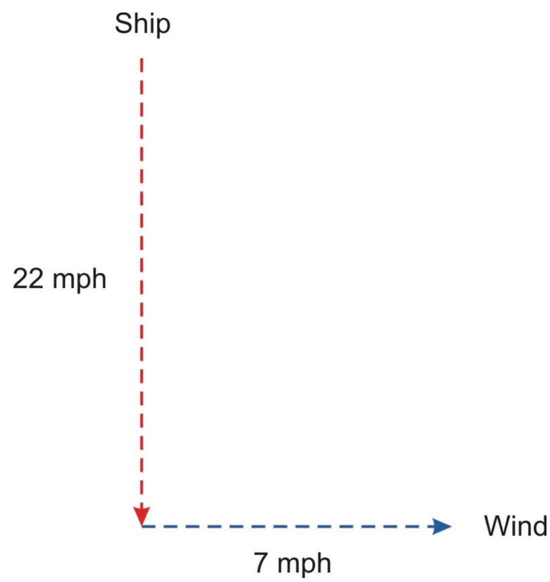
Example 6: Using the triangle method for subtraction.



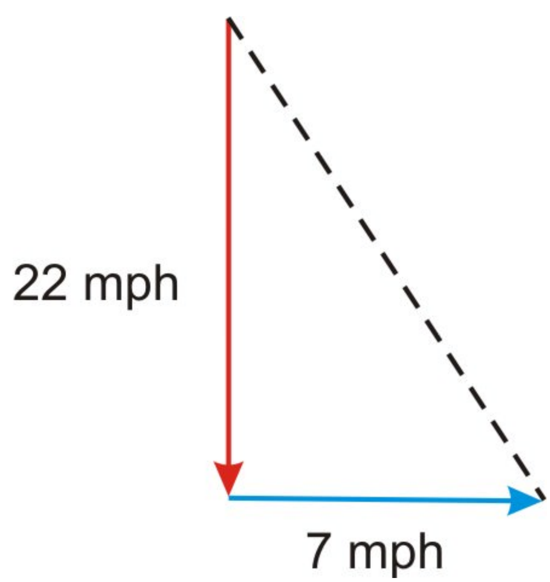
Resultant of Two Displacements

We can use vectors to find direction, velocity, and force of moving objects. In this section we will look at a few applications where we will use resultants of vectors to find speed, direction, and other quantities. A displacement is a distance considered as a vector. If one is 10 ft away from a point, then any point at a radius of 10 ft from that point satisfies the condition. If one is 28 degrees to the east of north, then only one point satisfies this.

Example 7: A cruise ship is traveling south at 22 mph. A wind is also blowing the ship eastward at 7 mph. What speed is the ship traveling at and in what direction is it moving?



Solution: In order to find the direction and the speed the boat is traveling, we must find the resultant of the two vectors representing 22 mph south and 7 mph east. Since these two vectors form a right angle, we can use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the resultant vector.



First, we will find the speed.

$$\begin{aligned} 22^2 + 7^2 &= x^2 \\ 533 &= x^2 \\ 23.1 &= x \end{aligned}$$

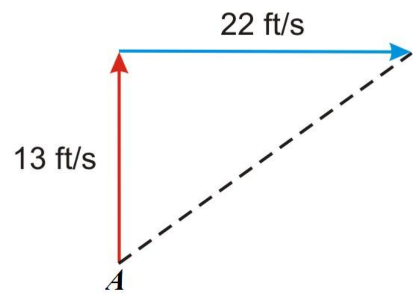
The ship is traveling at a speed of 23.1mph.

To find the direction, we will use tangent, since we know the opposite and adjacent sides of our triangle.

$$\tan\theta = \frac{7}{22}$$
$$\tan^{-1}\frac{7}{22} = 17.7^\circ$$

The ship's direction is $S17.7^\circ E$.

Example 8: A hot air balloon is rising at a rate of 13 ft/sec, while a wind is blowing at a rate of 22 ft/sec. Find the speed at which the balloon is traveling as well as its angle of elevation.



First, we will find the speed at which our balloon is rising. Since we have a right triangle, we can use the Pythagorean Theorem to find calculate the magnitude of the resultant.

$$x^2 = 13^2 + 22^2$$
$$x^2 = 653$$
$$x = 25.6\ ft/\text{sec}$$

The balloon is traveling at rate of 25.6 feet per second.

To find the angle of elevation of the balloon, we need to find the angle it makes with the horizontal. We will find the angle A in the triangle and then we will subtract it from 90°.

$$\tan A = \frac{22}{13}$$
$$A = \tan^{-1}\frac{22}{13}$$
$$A = 59.4^\circ$$

Angle with the horizontal $= 90 - 59.4 = 30.6^\circ$.

The balloon has an angle of elevation of 30.6°.

Example 9: Continuing on with the previous example, find:

a. How far from the lift off point is the balloon in 2 hours? Assume constant rise and constant wind speed. (this is *total displacement*)

b. How far must the support crew travel on the ground to get under the balloon? (*horizontal displacement*)

c. If the balloon stops rising after 2 hours and floats for another 2 hours, how far from the initial point is it at the end of the 4 hours? How far away does the crew have to go to be under the balloon when it lands?

Solution:

a. After two hours, the balloon will be 184,320 feet from the lift off point (25.6 ft/sec multiplied by 7200 seconds in two hours).

b. After two hours, the horizontal displacement will be 158,400 feet (22ft/sec multiplied by 7200 seconds in two hours).

c. After two hours, the balloon will have risen 93,600 feet. After an additional two hours of floating (horizontally only) in the 22ft/sec wind, the balloon will have traveled 316,800 feet horizontally (22ft/second times 14,400 seconds in four hours).

We must recalculate our resultant vector using Pythagorean Theorem.

$$x = \sqrt{93600^2 + 316800^2} = 330338\ ft.$$

The balloon is 330,338 feet from its initial point. The crew will have to travel 316,800 feet or 90 miles (horizontal displacement) to be under the balloon when it lands.

Points to Consider

- Is it possible to find the magnitude and direction of resultants without using a protractor and ruler and without using right triangles?
- How can we use the Law of Cosines and the Law of Sines to help us find magnitude and direction of resultants?

Review Questions

1. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition, find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if:
   a. $|\vec{m}| = 29.8 |\vec{n}| = 37.7$
   b. $|\vec{m}| = 2.8 |\vec{n}| = 5.4$
   c. $|\vec{m}| = 11.9 |\vec{n}| = 9.4$
2. For $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ below, make a diagram of each addition or subtraction. $|\vec{a}| = 6cm$, direction $= 45^\circ$ $|\vec{b}| = 3.2cm$, direction $= 30^\circ$ $|\vec{c}| = 1.3cm$, direction $= 110^\circ$ $|\vec{d}| = 4.8cm$, direction $= 80^\circ$
   a. $\vec{a} + \vec{b}$
   b. $\vec{a} + \vec{d}$
   c. $\vec{c} + \vec{d}$
   d. $\vec{a} - \vec{d}$
   e. $\vec{b} - \vec{a}$
   f. $\vec{d} - \vec{c}$
3. Does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$? Explain your answer.
4. A plane is traveling north at a speed of 225 mph while an easterly wind is blowing the plane west at 18 mph. What is the direction and the speed of the plane?
5. Two workers are pulling on ropes attached to a tree stump. One worker is pulling the stump east with 330 Newtons of forces while the second working is pulling the stump north with 410 Newtons of force. Find the magnitude and direction of the resultant force on the tree stump.
6. Assume $\vec{a}$ is in standard position. For each terminal point is given, find the magnitude and direction of each vector.

a. (12, 18)
b. (-3, 6)

7. Given the initial and terminal coordinates of $\vec{a}$, find the magnitude and direction.

a. initial (2, 4) terminal (8, 6)
b. initial (5, -2) terminal (3, 1)

8. The magnitudes of vectors $\vec{a}$ and $\vec{b}$ are given, along with the angle they make with each other, theta, when positioned tip-to-tail. Find the magnitude of the resultant and the angle it makes with a.

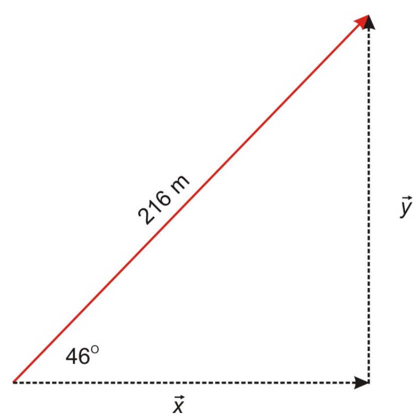
a. $|\vec{a}| = 31, |\vec{b}| = 31, \theta = 132^\circ$
b. $|\vec{a}| = 29, |\vec{b}| = 44, \theta = 26^\circ$

5.21 Component Vectors

Learning Objectives

- Perform scalar multiplication with vectors.
- Find the resultant as a sum of two components.
- Find the resultant as magnitude and direction.
- Use component vectors to solve real-world and applied problems.

A car has traveled 216 miles in a direction of 46° north of east. How far east of its initial point has it traveled? How far north has the car traveled?



The car traveled on a vector distance called a displacement. It moved in a line to a particular distance from the starting point. Having two **components** in their expression, vectors are confusing to some. A diagram helps sort out confusion. Looking at vectors by separating them into components allows us to deal with many real-world problems. The components often relate to very different elements of the problem, such as wind speed in one direction and speed supplied by a motor in another.

In order to find how far the car has traveled east and how far it has traveled north, we will need to find the horizontal and vertical components of the vector. To find $\vec{x}$, we use cosine and to find $\vec{y}$ we use sine.

$$\begin{aligned} \cos 46 &= \frac{|\vec{x}|}{216} = \frac{x}{216} \\ \cos 46 &= \frac{x}{216} \\ 216\cos 46 &= x \\ x &= 150.0 \end{aligned} \qquad \begin{aligned} \sin 46 &= \frac{|\vec{y}|}{216} = \frac{y}{216} \\ \sin 46 &= \frac{y}{216} \\ 216\sin 46 &= y \\ y &= 155.4 \end{aligned}$$

In this section, we will learn about component vectors and how to find them. We will also explore other ways of finding the magnitude and direction of a resultant of two or more vectors. We will be using many of the tools we learned in the previous sections dealing with right and oblique triangles.

Vector Multiplied by a Scalar

In working with vectors there are two kinds of quantities employed. The first is the vector, a quantity that has both magnitude and direction. The second quantity is a scalar. Scalars are just numbers. The magnitude of a vector is a scalar quantity. A vector can be multiplied by a real number. This real number is called a **scalar**. The product of a vector $\vec{a}$ and a scalar k is a vector, written $\vec{ka}$. It has the same direction as $\vec{a}$ with a magnitude of $k|\vec{a}|$ if $k > 0$. If $k < 0$, the vector has the opposite direction of $\vec{a}$ and a magnitude of $k|\vec{a}|$.

Example 1: The speed of the wind before a hurricane arrived was 20 mph from the SSE ($N22.5°W$). It quadrupled when the hurricane arrived. What is the current vector for wind velocity?

Solution: The wind is coming now at 80 mph from the same direction.

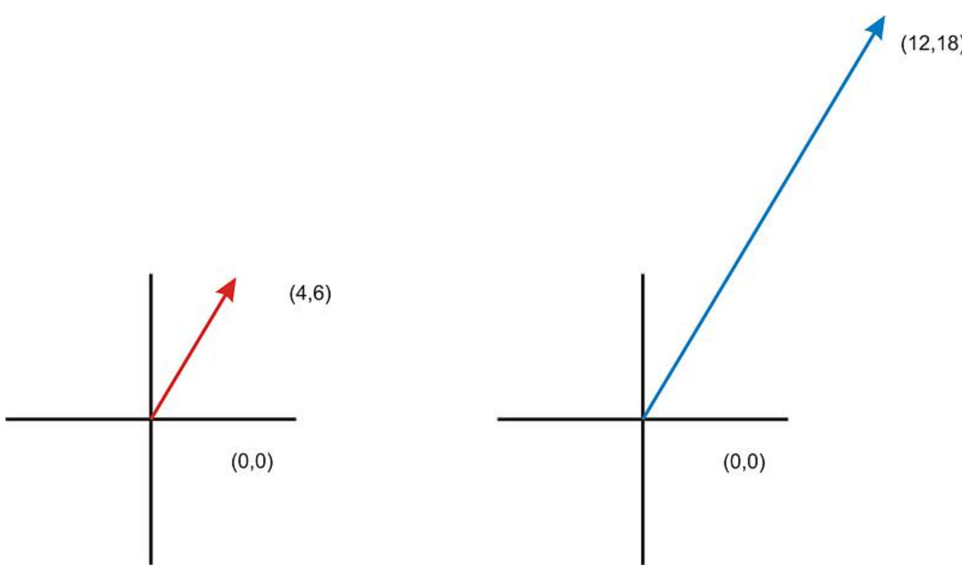
Example 2: A sailboat was traveling at 15 knots due north. After realizing he had overshot his destination, the captain turned the boat around and began traveling twice as fast due south. What is the current velocity vector of the ship?

Solution: The ship is traveling at 30 knots in the opposite direction.

If the vector is expressed in coordinates with the starting end of the vector at the origin, this is called standard form. To perform a scalar multiplication, we multiply our scalar by both the coordinates of our vector. The word scalar comes from "scale." Multiplying by a scalar just makes the vectors longer or shorter, but doesn't change their direction.

Example 3: Consider the vector from the origin to (4, 6). What would the representation of a vector that had three times the magnitude be?

Solution: Here $k = 3$ and $\vec{v}$ is the directed segment from (0,0) to (4, 6).



Multiply each of the components in the vector by 3.

$$\vec{kv} = (0,0) \ to \ (12,18)$$

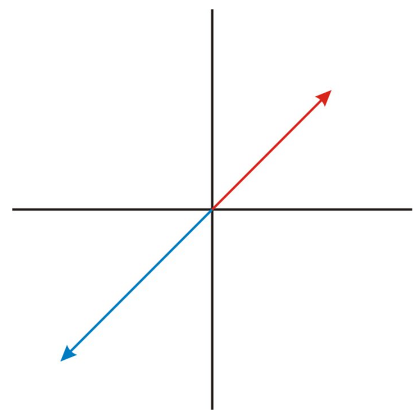
The new coordinates of the directed segment are (0, 0), (12, 18).

Example 4: Consider the vector from the origin to (3, 5). What would the representation of a vector that had -2 times the magnitude be?

Solution: Here, $k = -2$ and $\vec{v}$ is the directed segment from (0, 0) to (3, 5).

$$\vec{kv} = (-2(3), -2(5)) = (-6, -10)$$

Since $k < 0$, our result would be a directed segment that is twice and long but in the opposite direction of our original vector.



Translation of Vectors and Slope

What would happen if we performed scalar multiplication on a vector that didn't start at the origin?

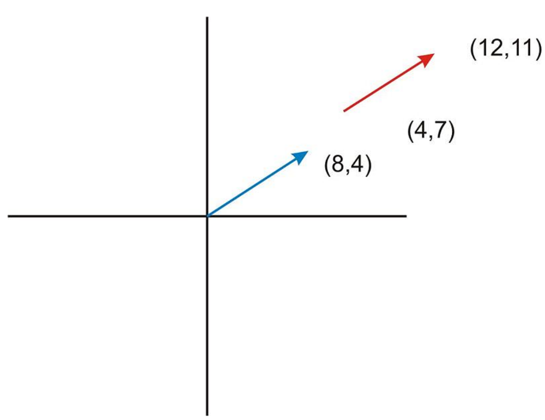
Example 5: Consider the vector from (4, 7) to (12, 11). What would the representation of a vector that had 2.5 times the magnitude be?

Solution: Here, $k = 2.5$ and $\vec{v} =$ the directed segment from (4, 7) to (12, 11).

Mathematically, two vectors are equal if their direction and magnitude are the same. The positions of the vectors do not matter. This means that if we have a vector that is not in standard position, we can translate it to the origin. The initial point of $\vec{v}$ is (4, 7). In order to **translate** this to the origin, we would need to add (-4, -7) to both the initial and terminal points of the vector.

Initial point: $(4,7)+(-4,-7) = (0,0)$

Terminal point: $(12,11)+(-4,-7) = (8,4)$



Now, to calculate $\vec{kv}$:

$$\vec{kv} = (2.5(8), 2.5(4))$$
$$\vec{kv} = (20, 10)$$

The new coordinates of the directed segment are (0, 0) and (20, 10). To translate this back to our original terminal point:

Initial point: $(0,0) + (4,7) = (4,7)$

Terminal point: $(20,10) + (4,7) = (24,17)$

The new coordinates of the directed segment are (4, 7) and (24, 17).

Vectors with the same magnitude and direction are equal. This means that the same ordered pair could represent many different vectors. For instance, the ordered pair (4, 8) can represent a vector in standard position where the initial point is at the origin and the terminal point is at (4, 8). This vector could be thought of as the resultant of a horizontal vector with a magnitude or 4 units and a vertical vector with a magnitude of 8 units. Therefore, any vector with a horizontal component of 4 and vertical component of 8 could also be represented by the ordered pair (4, 8).

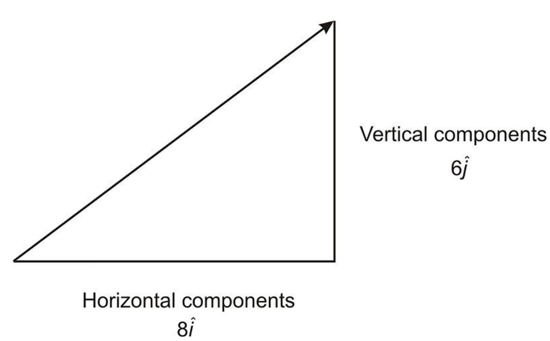
If you think back to Algebra, you know that the slope of a line is the change in y over the change in x, or the vertical change over the horizontal change. Looking at our vectors above, since they all have the same horizontal and vertical components, they all have the same slope, even though they do not all start at the origin.

Unit Vectors and Components

A **unit vector** is a vector that has a magnitude of one unit and can have any direction. Traditionally $\hat{i}$ (read "i hat") is the unit vector in the x direction and $\hat{j}$ (read "j hat") is the unit vector in the y direction. $|\hat{i}| = 1$ and $|\hat{j}| = 1$. Unit vectors on perpendicular axes can be used to express all vectors in that plane. Vectors are used to express position and motion in three dimensions with $\hat{k}$ ("k hat") as the unit vector in the z direction. We are not studying 3D space in this course. The unit vector notation may seem burdensome but one must distinguish between a vector and the components of that vector in the direction of the $x-$ or $y-$axis. The unit vectors carry the meaning for the direction of the vector in each of the coordinate directions. The number in front of the unit vector shows its magnitude or length. Unit vectors are convenient if one wishes to express a 2D or 3D vector as a sum of two or three orthogonal components, such as $x-$ and $y-$axes, or the $z-$axis. (Orthogonal components are those that intersect at right angles.)

Component vectors of a given vector are two or more vectors whose sum is the given vector. The sum is viewed as equivalent to the original vector. Since component vectors can have any direction, it is useful to have them perpendicular to one another. Commonly one chooses the x and y axis as the basis for the unit vectors. Component vectors do not have to be orthogonal.

A vector from the origin (0, 0) to the point (8, 0) is written as $8\hat{i}$. A vector from the origin to the point (0, 6) is written as $6\hat{j}$.



The reason for having the component vectors perpendicular to one another is that this condition allows us to use the Pythagorean Theorem and trigonometric ratios to find the magnitude and direction of the components. One can solve vector problems without use of unit vectors if specific information about orientation or direction in space such as N, E, S or W is part of the problem.

Resultant as the Sum of Two Components

We can look at any vector as the resultant of two perpendicular components. If we generalize the figure above, $|\vec{r}|\hat{i}$ is the horizontal component of a vector $\vec{q}$ and $|\vec{s}|\hat{j}$ is the vertical component of $\vec{q}$. Therefore $\vec{r}$ is a magnitude, $|\vec{r}|$, times the unit vector in the x direction and $\vec{s}$ is its magnitude, $|\vec{s}|$, times the unit vector in the y direction. The sum of $\vec{r}$ plus $\vec{s}$ is: $\vec{r}+\vec{s}=\vec{q}$. This addition can also be written as $|\vec{r}|\hat{i}+|\vec{s}|\hat{j}=\vec{q}$.

If we are given the vector $\vec{q}$, we can find the components of $\vec{q}, \vec{r}$, and $\vec{s}$ using trigonometric ratios if we know the magnitude and direction of $\vec{q}$.

Example 6: If $|\vec{q}|= 19.6$ and its direction is 73°, find the horizontal and vertical components.

Solution: If we know an angle and a side of a right triangle, we can find the other remaining sides using trigonometric ratios. In this case, $\vec{q}$ is the hypotenuse of our triangle, $\vec{r}$ is the side adjacent to our 73° angle, $\vec{s}$ is the side opposite our 73° angle, and $\vec{r}$ is directed along the $x-$axis.

To find $\vec{r}$, we will use cosine and to find $\vec{s}$ we will use sine. Notice this is a scalar equation so all quantities are just numbers. It is written as the quotient of the magnitudes, not the vectors.

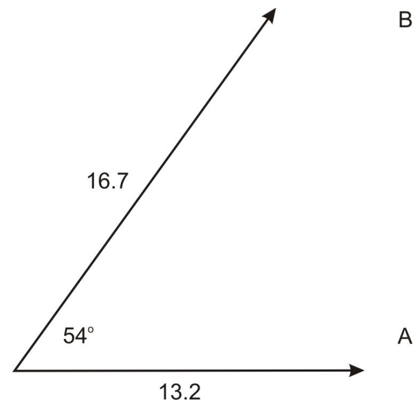
$$\begin{aligned}
\cos 73 &= \frac{|\vec{r}|}{|\vec{q}|}=\frac{r}{q} & \sin 73 &= \frac{|\vec{s}|}{|\vec{q}|}=\frac{s}{q}\\
\cos 73 &= \frac{r}{19.6} & \sin 73 &= \frac{s}{19.6}\\
r &= 19.6\cos 73 & s &= 19.6\sin 73\\
r &= 5.7 & s &= 18.7
\end{aligned}$$

The horizontal component is 5.7 and the vertical component is 18.7. One can rewrite this in vector notation as $5.7\hat{i}+18.7\hat{j}=\vec{q}$. The components can also be written $\vec{q}=\langle 5.7, 18.7\rangle$, with the horizontal component first, followed by the vertical component. Be careful not to confuse this with the notation for plotted points.

Resultant as Magnitude and Direction

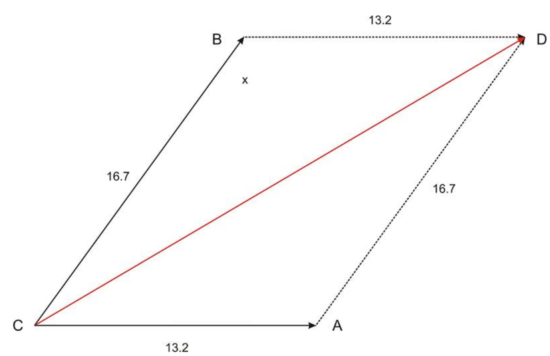
If we don't have two perpendicular vectors, we can still find the magnitude and direction of the resultant without a graphic estimate with a construction using a compass and ruler. This can be accomplished using both the Law of Sines and the Law of Cosines.

Example 7: $\vec{A}$ makes a 54° angle with $\vec{B}$. The magnitude of $\vec{A}$ is 13.2. The magnitude of $\vec{B}$ is 16.7. Find the magnitude and direction the resultant makes with the smaller vector.



There is no preferred orientation such as a compass direction or any necessary use of x and y coordinates. The problem can be solved without the use of unit vectors.

Solution: In order to solve this problem, we will need to use the parallelogram method. Since vectors only have magnitude and direction, one can move them on the plane to any position one wishes, as long as the magnitude and direction remain the same. First, we will complete the parallelogram: Label the vectors. Move $\vec{b}$ so its tail is on the tip of $\vec{a}$. Move $\vec{a}$ so its tail is on the tip of $\vec{b}$. This makes a parallelogram because the angles did not change during the translation. Put in labels for the vertices of the parallelogram.



Since opposite angles in a parallelogram are congruent, we can find angle A.

$$\begin{aligned}
\angle CBD + \angle CAD + \angle ACB + \angle BDA &= 360 \\
2\angle CBD + 2\angle ACB &= 360 \\
\angle ACB &= 54^\circ \\
2\angle CBD &= 360 - 2(54) \\
\angle CBD &= \frac{360 - 2(54)}{2} = 126
\end{aligned}$$

Now, we know two sides and the included angle in an oblique triangle. This means we can use the Law of Cosines to find the magnitude of our resultant.

$$\begin{aligned}
x^2 &= 13.2^2 + 16.7^2 - 2(13.2)(16.7)\cos 126 \\
x^2 &= 712.272762 \\
x &= 26.7
\end{aligned}$$

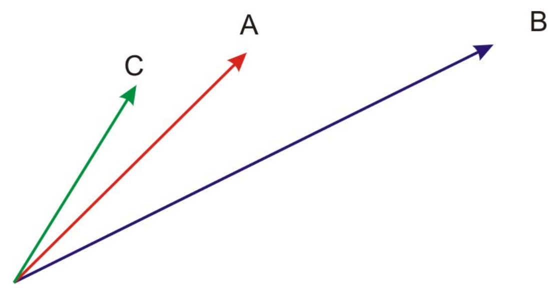
To find the direction, we can use the Law of Sines since we now know an angle and a side across from it. We choose the Law of Sines because it is a proportion and less computationally intense than the Law of Cosines.

$$\begin{aligned}
\frac{\sin\theta}{16.7} &= \frac{\sin 126}{26.7}\\
\sin\theta &= \frac{16.7\sin 126}{26.7}\\
\sin\theta &= 0.5060143748\\
\theta &= \sin^{-1}\ 0.5060 = 30.4^\circ
\end{aligned}$$

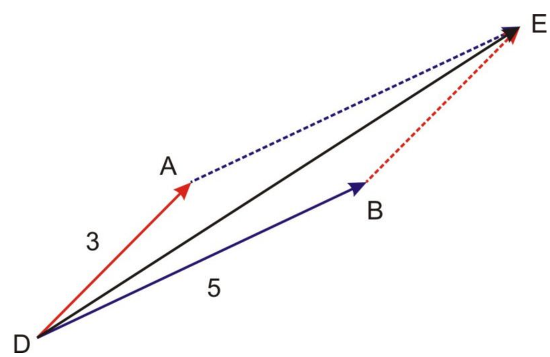
The magnitude of the resultant is 26.7 and the direction it makes with the smaller vector is 30.4° counterclockwise.

We can use a similar method to add three or more vectors.

Example 8: Vector A makes a 45° angle with the horizontal and has a magnitude of 3. Vector B makes a 25° angle with the horizontal and has a magnitude of 5. Vector C makes a 65° angle with the horizontal and has a magnitude of 2. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.



Solution: To begin this problem, we will find the resultant using Vector A and Vector B. We will do this using the parallelogram method like we did above.



Since Vector A makes a 45° angle with the horizontal and Vector B makes a 25° angle with the horizontal, we know that the angle between the two ($\angle ADB$) is 20°.

To find $\angle DBE$:

$$\begin{aligned}
2\angle ADB + 2\angle DBE &= 360\\
\angle ADB &= 20^\circ\\
2\angle DBE &= 360 - 2(20)\\
\angle DBE &= \frac{360-2(20)}{2} = 160
\end{aligned}$$

Now, we will use the Law of Cosines to find the magnitude of DE.

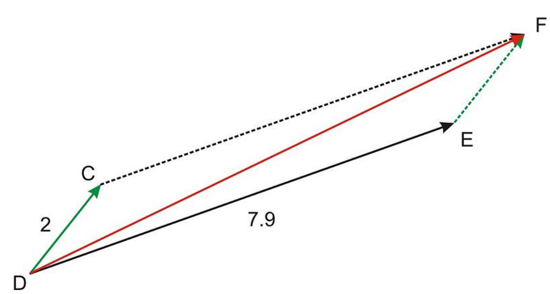
$$\begin{aligned}DE^2 &= 3^2+5^2-2(3)(5)\cos 160\\DE^2 &= 62\\DE &= 7.9\end{aligned}$$

Next, we will use the Law of Sines to find the measure of angle EDB.

$$\begin{aligned}\frac{\sin 160}{7.9} &= \frac{\sin\angle EDB}{3}\\\sin\angle EDB &= \frac{3\sin 160}{7.9}\\\sin\angle EDB &= .1299\\\angle EDB &= \sin^{-1}\ 0.1299 = 7.46^\circ\end{aligned}$$

We know that Vector B forms a 25° angle with the horizontal so we add that value to the measure of $\angle EDB$ to find the angle DE makes with the horizontal. Therefore, DE makes a 32.46° angle with the horizontal.

Next, we will take DE, and we will find the resultant vector of DE and Vector C from above. We will repeat the same process we used above.



Vector C makes a 65° angle with the horizontal and DE makes a 32° angle with the horizontal. This means that the angle between the two ($\angle CDE$) is 33°. We will use this information to find the measure of $\angle DEF$.

$$\begin{aligned}2\angle CDE+2\angle DEF &= 360\\\angle CDE &= 33^\circ\\2\angle DEF &= 360-2(33)\\\angle DEF &= \frac{360-2(33)}{2} = 147\end{aligned}$$

Now we will use the Law of Cosines to find the magnitude of DF.

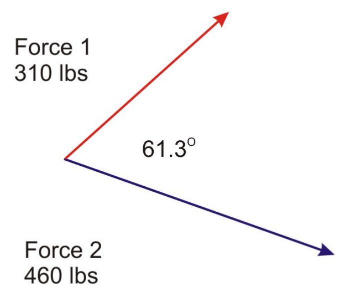
$$\begin{aligned}DF^2 &= 7.9^2+2^2-2(7.9)(2)\cos 147\\DF^2 &= 92.9\\DF &= 9.6\end{aligned}$$

Next, we will use the Law of Sines to find $\angle FDE$.

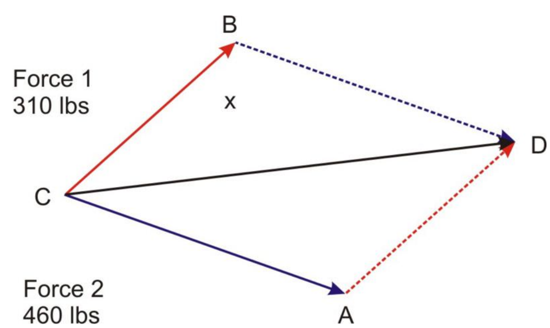
$$\begin{aligned}
\frac{\sin 147}{9.6} &= \frac{\sin \angle FDE}{2} \\
\sin \angle FDE &= \frac{2\sin 147}{9.6} \\
\sin \angle FDE &= .1135 \\
\angle FDE &= \sin^{-1} 0.1135 = 6.5^\circ = 7^\circ
\end{aligned}$$

Finally, we will take the measure of $\angle FDE$ and add it to the 32° angle that DE forms with the horizontal. Therefore, DF forms a 39° angle with the horizontal.

Example 9: Two forces of 310 lbs and 460 lbs are acting on an object. The angle between the two forces is 61.3°. What is the magnitude of the resultant? What angle does the resultant make with the smaller force?



Solution: We do not need unit vectors here as there is no preferred direction like a compass direction or a specific axis. First, to find the magnitude we will need to figure out the other angle in our parallelogram.



$$\begin{aligned}
2\angle ACB + 2\angle CAD &= 360 \\
\angle ACB &= 61.3^\circ \\
2\angle CAD &= 360 - 2(61.3) \\
\angle CAD &= \frac{360 - 2(61.3)}{2} = 118.7
\end{aligned}$$

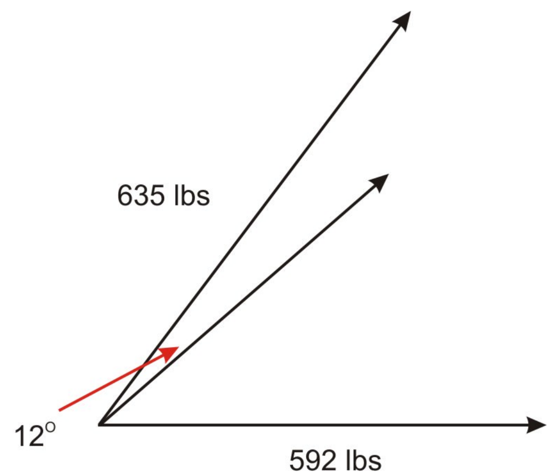
Now that we know the other angle, we can find the magnitude using the Law of Cosines.

$$\begin{aligned}
x^2 &= 460^2 + 310^2 - 2(460)(310)\cos 118.7^\circ \\
x^2 &= 444659.7415 \\
x &= 667
\end{aligned}$$

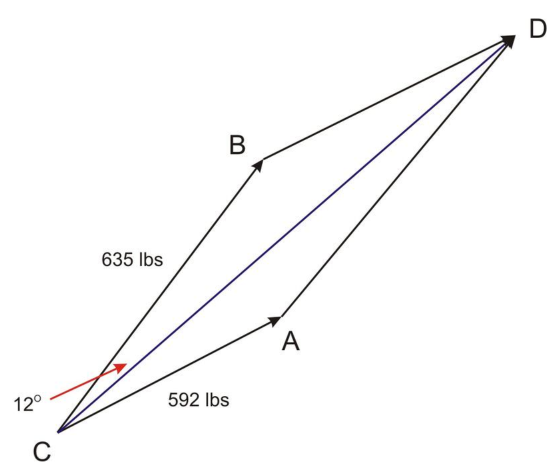
To find the angle the resultant makes with the smaller force, we will use the Law of Sines.

$$\begin{aligned}
\frac{\sin\theta}{460} &= \frac{\sin 118.7}{666.8} \\
\sin\theta &= \frac{460\sin 118.7}{666.8} \\
\sin\theta &= .6049283888 \\
\theta &= \sin^{-1}\ 0.6049 = 37.2^\circ
\end{aligned}$$

Example 10: Two trucks are pulling a large chunk of stone. Truck 1 is pulling with a force of 635 lbs at a 53° angle from the horizontal while Truck 2 is pulling with a force of 592 lbs at a 41° angle from the horizontal. What is the magnitude and direction of the resultant force?



Solution: Since Truck 1 has a direction of 53° and Truck 2 has a direction of 41°, we can see that the angle between the two forces is 12°. We need this angle measurement in order to figure out the other angles in our parallelogram.



$$\begin{aligned}
2\angle ACB + 2\angle CAD &= 360 \\
\angle ACB &= 12^\circ \\
2\angle CAD &= 360 - 2(12) \\
\angle CAD &= \frac{360 - 2(12)}{2} = 168
\end{aligned}$$

Now, use the Law of Cosines to find the magnitude of the resultant.

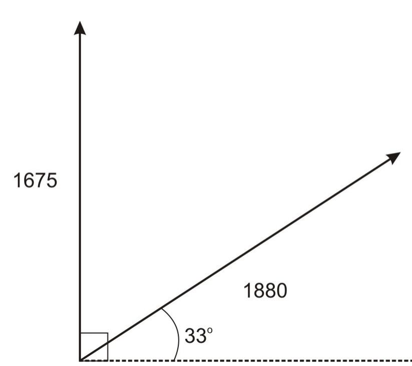
$$\begin{aligned}
x^2 &= 635^2 + 592^2 - 2(635)(592)\cos 168^\circ \\
x^2 &= 1489099 \\
x &= 1220.3 \ lbs
\end{aligned}$$

Now to find the direction we will use the Law of Sines.

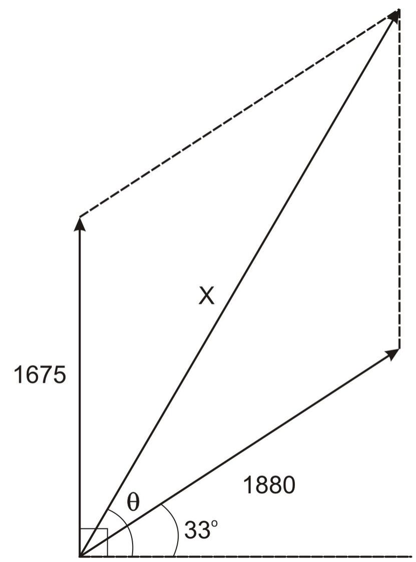
$$\begin{aligned}
\frac{\sin\theta}{635} &= \frac{\sin 168}{1220.3} \\
\sin\theta &= \frac{635 \sin 168}{1220.3} \\
\sin\theta &= 0.1082 \\
\theta &= \sin^{-1}\ 0.1082 = 6^\circ
\end{aligned}$$

Since we want the direction we need to add the 6° to the 41° from the smaller force. The magnitude is 1220 lbs and 47° counterclockwise from the horizontal.

Example 11: Two tractors are being used to pull down the framework of an old building. One tractor is pulling on the frame with a force of 1675 pounds and is headed directly north. The second tractor is pulling on the frame with a force of 1880 pounds and is headed 33° north of east. What is the magnitude of the resultant force on the building? What is the direction of the result force?

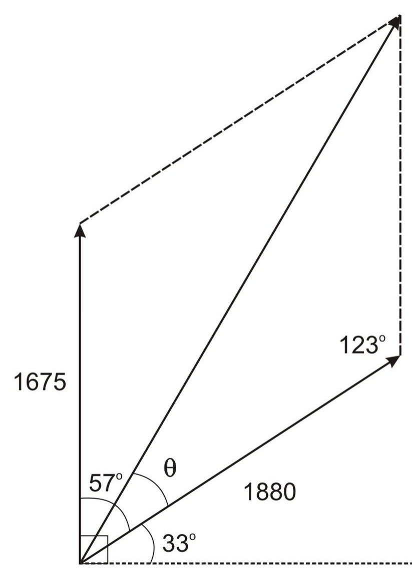


Solution: We are asked to find the resultant force and direction, which means we are dealing with vectors. In order to complete our diagram, we will need to connect our two vectors and draw in our resultant. We will refer to the magnitude of our resultant as x and the direction of our resultant as θ.



When finding the resultant of two vectors, we can choose from either the triangle method or the parallelogram method. We will solve this problem using the parallelogram method. Looking at the diagram, we can see that the two vectors form an angle of 57, (90 - 33). This means that the angle opposite the angle formed by our two vectors is also 57. To find the other two angles in our parallelogram, we know that the sum of all the angles must add up to 360 and that opposite angles must be congruent, $\frac{360-(57+57)}{2} = 123$.

Now, we can use two sides of our parallelogram and our resultant to form a triangle in which we know two sides and the included angle (SAS).



This means that we can use the Law of Cosines to find the magnitude (x) of the resultant.

$$\begin{aligned}
x^2 &= 1675^2 + 1880^2 - 2(1675)(1880)\cos 123 \\
x^2 &= 9770161.643 \\
x &= 3125.7
\end{aligned}$$

To find the direction (θ), we can use the Law of Sines since we now know an angle and the side opposite it.

$$\begin{aligned}\frac{\sin 123}{3125.7} &= \frac{\sin\theta}{1675}\\ \frac{1675\sin 123}{3125.7} &= \sin\theta\\ 0.449427 &= \sin\theta\\ 26.71 &= \theta\end{aligned}$$

Now that we know θ, in order to find the angle of the resultant, we must add the 33° from the $x-$axis to $\theta, 33^\circ + 26.71^\circ = 59.71^\circ$.

Points to Consider

- How you can verify if your answers to problems involving vectors that are not perpendicular are correct?
- In what ways are solving problems with oblique triangles and solving problems involving vectors similar?
- In what ways are they different?
- When is it appropriate to use vectors instead of oblique triangles to solve problems?
- When is it helpful to use unit vectors? When can one solve a problem without explicitly using them?

Review Questions

1. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (0,0)\ to\ (5,4)$ and $\vec{c} = (0,0)\ to\ (-3,7)$.
   a. $\vec{a} = 2\vec{b}$
   b. $\vec{a} = -\frac{1}{2}\vec{c}$
   c. $\vec{a} = 0.6\vec{b}$
   d. $\vec{a} = -3\vec{b}$
2. Find the magnitude of the horizontal and vertical components of the following vectors given that the coordinates of their initial and terminal points.
   a. initial $= (-3,8)$ terminal $= (2,-1)$
   b. initial $= (7,13)$ terminal $= (11,19)$
   c. initial $= (4.2,-6.8)$ terminal $= (-1.3,-9.4)$
3. Find the magnitude of the horizontal and vertical components if the resultant vector's magnitude and direction are given.
   a. magnitude $= 75$ direction $= 35^\circ$
   b. magnitude $= 3.4$ direction $= 162^\circ$
   c. magnitude $= 15.9$ direction $= 12^\circ$
4. Two forces of 8.50 Newtons and 32.1 Newtons act on an object at right angles. Find the magnitude of the resultant and the angle that it makes with the smaller force.
5. Forces of 140 Newtons and 186 Newtons act on an object. The angle between the forces is 43°. Find the magnitude of the resultant and the angle it makes with the larger force.
6. An incline ramp is 12 feet long and forms an angle of 28.2° with the ground. Find the horizontal and vertical components of the ramp.

7. An airplane is traveling at a speed of 155 km/h. It's heading is set at 83° while there is a 42.0 km/h wind from 305°. What is the airplane's actual heading?
8. A speedboat is capable of traveling at 10.0 mph, but is in a river that has a current of 2.00 mph. In order to cross the river at right angle, in what direction should the boat be heading?
9. If $\overrightarrow{AB}$ is any vector, what is $\overrightarrow{AB}+\overrightarrow{BA}$?

Chapter 6 Further Topics in Pre-Calculus

Chapter Outline

6.1 Polynomial Expansion and Pascal's Triangle

Learning Objectives

Here you will explore patterns with binomial and polynomial expansion and find out how to get coefficients using Pascal's Triangle.

The expression $(2x+3)^5$ would take a while to multiply out. Is there a pattern you can use?

Expansion with Pascal's Triangle

Expanding a Binomial

Pascal was a French mathematician in the 17^{th} century, but the triangle now named **Pascal's Triangle** was studied long before Pascal used it. The pattern was used around the 10^{th} century in Persia, India and China as well as many other places.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
\vdots
\end{array}
$$

The primary purpose for using this triangle is to introduce how to expand binomials.

$$
\begin{aligned}
&(x+y)^0 = 1 \\
&(x+y)^1 = x+y \\
&(x+y)^2 = x^2+2y+y^2 \\
&(x+y)^3 = x^3+3x^2y+3xy^2+y^3
\end{aligned}
$$

Notice that the coefficients for the x and y terms on the right hand side line up exactly with the numbers from Pascal's triangle. This means that given $(x+y)^n$ for any power n you can write out the expansion using the coefficients from the triangle. Note that to write the coefficients for any power n, you need to look at row $n+1$ to find the coefficients.

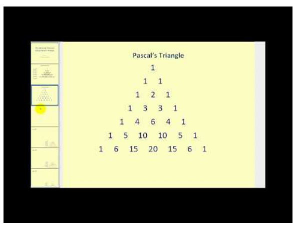
If you were asked to expand $(3x-2)^4$ using Pascal's Triangle, you would look at the 5th row to find the coefficients. The coefficients will be 1, 4, 6, 4, 1; however, since there are already coefficients with the x and the constant term you must be particularly careful.

$1\cdot(3x)^4+4\cdot(3x)^3\cdot(-2)+6\cdot(3x)^2\cdot(-2)^2+4\cdot(3x)\cdot(-2)^3+1\cdot(-2)^4$

Then it is only a matter of multiplying out and keeping track of negative signs.

$81x^4-216x^3+216x^2-96x+16$

When you study how to count with combinations then you will be able to calculate the value of any coefficient without writing out the whole triangle.



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Patterns and Pascal's Triangle

There are many patterns in the triangle. Here are just a few.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

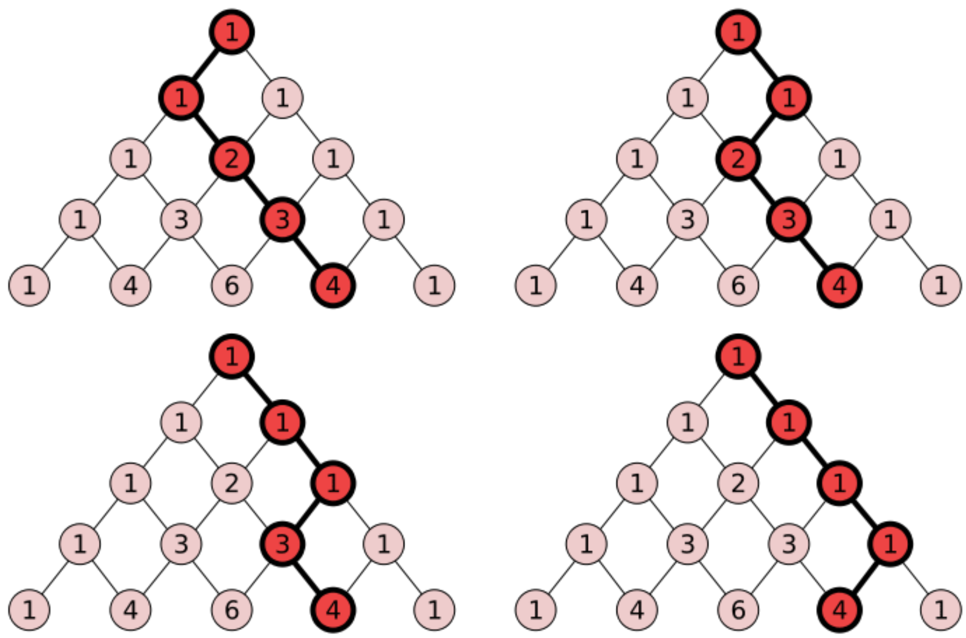
1 5 10 10 5 1

1 6 15 20 15 6 1

⋮

1. Notice the way each number is created by summing the two numbers above on the left and right hand side.
2. As you go further down the triangle the values in a row approach a bell curve. This is closely related to the normal distribution in statistics.

3. For any row that has a second term that is prime, all the numbers besides 1 in that row are divisible by that prime number.
4. In the game Plinko where an object is dropped through a triangular array of pegs, the probability (which corresponds proportionally to the values in the triangle) of landing towards the center is greater than landing towards the edge. This is because every number in the triangle indicates the number of ways a falling object can get to that space through the preceding numbers.



Examples

Example 1

Earlier, you were asked to multiply out $(2x+3)^5$. Pascal's triangle allows you to identify that the coefficients of $(2x+3)^5$ will be 1, 5, 10, 10, 5, 1. By carefully substituting, the expansion will be:

$$1\cdot(2x)^5+5\cdot(2x)^4\cdot 3+10\cdot(2x)^3\cdot 3^2+10\cdot(2x^2)\cdot 3^3+5(2x)^1\cdot 3^4+3^5$$

Simplifying is a matter of arithmetic, but most of the work is done thanks to the patterns of Pascal's Triangle.

Example 2

Factor the following polynomial by recognizing the coefficients.

$x^4+4x^3+6x^2+4x+1$

The coefficients are 1, 4, 6, 4, and 1 and those coefficients are on the 5th row. The first row of Pascal's Triangle shows the coefficients for the 0th power so the 5th row shows the coefficients for the 4th power. Thus, the factored form is:

$(x+1)^4$

Example 3

Factor the following polynomial by recognizing the coefficients.

$8x^3 - 12x^2 + 6x - 1$

Since the first coefficient is not 1, you need to take the appropriate root of the first term of the expression to find the first term of the binomial. In this case, the first term of the binomial must be $2x$. Also, the last term must be -1 and the power must be 3. Now all that remains is to check.

$(2x-1)^3 = (2x)^3 + 3(2x)^2 \cdot (-1) + 3(2x)^1(-1)^2 + (-1)^3 = 8x^3 - 12x^2 + 6x - 1$

Example 4

Expand the following trinomial: $(x+y+z)^4$

Unfortunately, Pascal's triangle does not apply to trinomials. Instead of thinking of a two dimensional triangle, you would need to calculate a three dimensional pyramid which is called Pascal's Pyramid. The sum of all the terms below is your answer.

$$
\begin{aligned}
&1x^4 + 4x^3z + 6x^2z^2 + 4xz^3 + 1z^4 \\
&4x^3y + 12x^2yz + 12xyz^2 + 4yz^3 \\
&6x^2y^2 + 12xy^2z + 6y^2z^2 \\
&4xy^3 + 4y^3z \\
&1y^4
\end{aligned}
$$

Notice how many patterns exist in the coefficients of this layer of the pyramid.

Example 5

Expand the following binomial: $\left(\frac{1}{2}x - 3\right)^5$

You know that the coefficients will be 1, 5, 10, 10, 5, 1.

$$1\left(\frac{1}{2}x\right)^5 + 5\left(\frac{1}{2}x\right)^4(-3) + 10\left(\frac{1}{2}x\right)^3(-3)^2 + 10\left(\frac{1}{2}x\right)^2(-3)^3 + 5\left(\frac{1}{2}x\right)(-3)^4 + 1\cdot(-3)^5$$

$$= \frac{x^5}{32} - \frac{15x^4}{16} + \frac{90x^3}{8} - \frac{270x^2}{4} + \frac{405x}{2} - 243$$

Remember to simplify fractions.

$$= \frac{x^5}{32} - \frac{15x^4}{16} + \frac{45x^3}{4} - \frac{135x^2}{2} + \frac{405x}{2} - 243$$

Review

Factor the following polynomials by recognizing the coefficients.

1. $x^2+2xy+y^2$
2. x^3+3x^2+3x+1
3. $x^5+5x^4+10x^3+10x^2+5x+1$
4. $27x^3-27x^2+9x-1$
5. $x^3+12x^2+48x+64$

Expand the following binomials using Pascal's Triangle.

6. $(2x-3)^3$
7. $(3x+4)^4$
8. $(x-y)^7$
9. $(a+b)^{10}$
10. $(2x+5)^5$
11. $(4x-1)^4$
12. $(5x+2)^3$
13. $(x+y)^6$
14. $(3x+2y)^3$
15. $(5x-2y)^4$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.3.

6.2 Limit Notation

Learning Objectives

Here you will write and read limit notation and use limit notation to describe the behavior of a function at a point and at infinity.

When learning about the end behavior of a rational function you described the function as either having a horizontal asymptote at zero or another number, or going to infinity. Limit notation is a way of describing this end behavior mathematically.

You already know that as x gets extremely large then the function $f(x) = \frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ goes to $\frac{8}{3}$ because the greatest powers are equal and $\frac{8}{3}$ is the ratio of the leading coefficients. How is this statement represented using limit notation?

Introduction to Limits

Limit notation is a way of stating an idea that is a little more subtle than simply saying $x = 5$ or $y = 3$.

$\lim_{x \to a} f(x) = b$

"The limit of f of x as x approaches a is b"

The letter a can be any number or infinity. The function $f(x)$ is any function of x. The letter b can be any number. If the function goes to infinity, then instead of writing "$= \infty$" you should write that the limit does not exist or " *DNE*". This is because infinity is not a number. If a function goes to infinity then it has no limit.

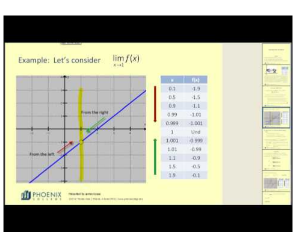
Take the following limit:

The limit of $y = 4x^2$ as x approaches 2 is 16

In limit notation, this would be:

$\lim_{x \to 2} 4x^2 = 16$

While a function may never actually reach a height of b it will get arbitrarily close to b. One way to think about the concept of a limit is to use a physical example. Stand some distance from a wall and then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.



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Examples

Example 1

Earlier, you were asked how to write the statement "The limit of $\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ as x approaches infinity is $\frac{8}{3}$" in limit notation.

This can be written using limit notation as:

$$\lim_{x\to\infty}\left(\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}\right)=\frac{8}{3}$$

Example 2

Translate the following mathematical statement into words.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i=1$$

The limit of the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ as the number of terms approaches infinity is 1.

Example 3

Use limit notation to represent the following mathematical statement.

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i=\frac{1}{2}$$

Example 4

Describe the end behavior of the following rational function at infinity and negative infinity using limits.

$$f(x)=\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}$$

Since the function has equal powers of x in the numerator and in the denominator, the end behavior is $-\frac{1}{2}$ as x goes to both positive and negative infinity.

$$\lim_{x\to\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=\lim_{x\to-\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=-\frac{1}{2}$$

Example 5

Translate the following limit expression into words. What do you notice about the limit expression?

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)=x$$

The limit of the ratio of the difference between f of quantity x plus h and f of x and h as h approaches 0 is x.

You should notice that $h\to 0$ does not mean $h=0$ because if it did then you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as h approaches zero. Calculus will enable you to deal with problems that seem to look like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Review

Describe the end behavior of the following rational functions at infinity and negative infinity using limits.

1. $f(x) = \frac{2x^4+4x^2-1}{5x^4+3x+9}$

2. $g(x) = \frac{8x^3+4x^2-1}{2x^3+4x+7}$

3. $f(x) = \frac{x^2+2x^3-3}{5x^3+x+4}$

4. $f(x) = \frac{4x+4x^2-5}{2x^2+3x+3}$

5. $f(x) = \frac{3x^2+4x^3+4}{6x^3+3x^2+6}$

Translate the following statements into limit notation.

6. The limit of $y = 2x^2+1$ as x approaches 3 is 19.

7. The limit of $y = e^x$ as x approaches negative infinity is 0.

8. The limit of $y = \frac{1}{x}$ as x approaches infinity is 0.

Use limit notation to represent the following mathematical statements.

9. $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots = \frac{1}{3}$

10. The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

11. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 2$

12. $\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots = 1$

Translate the following mathematical statements into words.

13. $\lim_{x\to 0}\frac{5x^2-4}{x+1} = -4$

14. $\lim_{x\to 1}\frac{x^3-1}{x-1} = 3$

15. If $\lim_{x\to a} f(x) = b$, is it possible that $f(a) = b$? Explain.

Review (Answers)

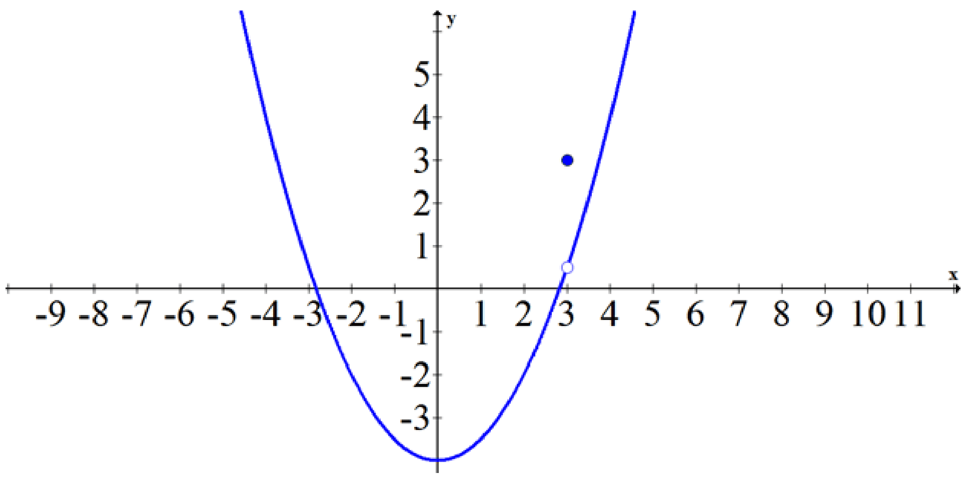
To see the Review answers, open this PDF file and look for section 14.1.

6.3 Graphs to Find Limits

Learning Objectives

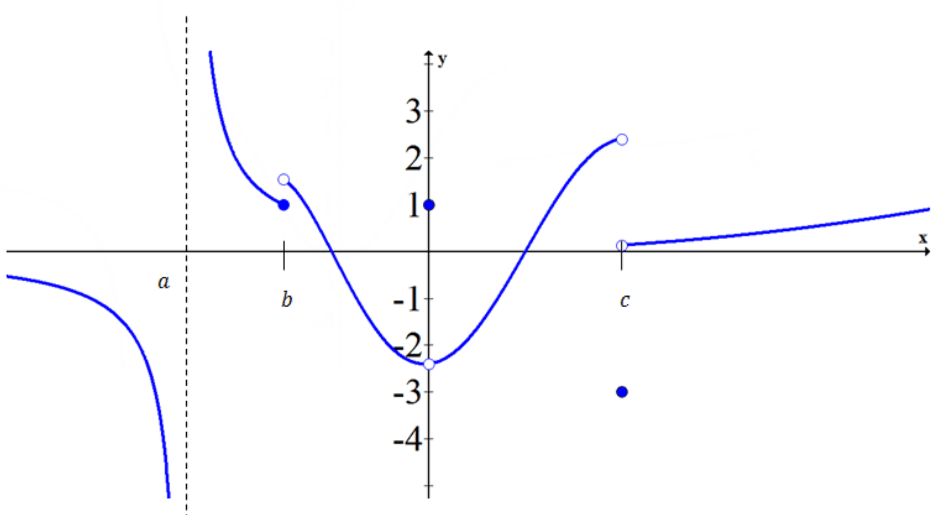
Here you will use graphs to help you evaluate limits and refine your understanding of what a limit represents.

A limit can describe the **end behavior** of a function. This is called a limit at infinity or negative infinity. A limit can also describe the limit at any normal x value. Sometimes this is simply the height of the function at that point. Other times this is what you would expect the height of the function to be at that point even if the height does not exist or is at some other point. In the following graph, what are $f(3)$, $\lim_{x\to 3} f(x)$, $\lim_{x\to\infty} f(x)$?



Using Graphs to Find Limits

When evaluating the limit of a function from its graph, you need to distinguish between the function evaluated at the point and the limit around the point.



Functions like the one above with discontinuities, asymptotes and holes require you to have a very solid understanding of how to evaluate and interpret limits.

At $x = a$, the function is undefined because there is a vertical asymptote. You would write:

$f(a) = DNE,\ \lim_{x \to a} f(x) = DNE$

At $x = b$, the function is defined because the filled in circle represents that it is the height of the function. This appears to be at about 1. However, since the two sides do not agree, the limit does not exist here either.

$f(b) = 1,\ \lim_{x \to b} f(x) = DNE$

At $x = 0$, the function has a discontinuity in the form of a hole. It is as if the point $(0, -2.4)$ has been lifted up and placed at $(0, 1)$. You can evaluate both the function and the limit at this point, however these quantities will not match. When you evaluate the function you have to give the actual height of the function, which is 1 in this case. When you evaluate the limit, you have to give what the height of the function is supposed to be based solely on the **neighborhood** around 0. By neighborhood around 0, we mean what is happening on the lines around $x = 0$, not the at the point. Since the function appears to reach a height of -2.4 from both the left and the right, the limit does exist.

$f(0) = 1,\ \lim_{x \to 0} f(x) = -2.4$

At $x = c$, the limit does not exist because the left and right hand neighborhoods do not agree on a height. On the other hand, the filled in circle represents that the function is defined at $x = c$ to be -3.

$f(c) = -3,\ \lim\limits_{x \to c} f(x) = DNE$

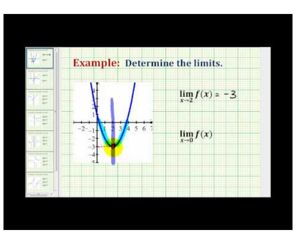
At $x \to \infty$ you may only discuss the limit of the function since it is not appropriate to evaluate a function at infinity (you cannot find $f(\infty)$). Since the function appears to increase without bound, the limit does not exist.

$\lim\limits_{x \to \infty} f(x) = DNE$

At $x \to -\infty$ the graph appears to flatten as it moves to the left. There is a horizontal asymptote at $y = 0$ that this function approaches as $x \to -\infty$.

$\lim\limits_{x \to -\infty} f(x) = 0$

When evaluating limits graphically, your main goal is to determine whether the limit exists. The limit only exists when the left and right sides of the functions meet at a specific height. Whatever the function is doing at that point does not matter for the sake of limits. The function could be defined at that point, could be undefined at that point, or the point could be defined at some other height. Regardless of what is happening at that point, when you evaluate limits graphically, you only look at the neighborhood to the left and right of the function at the point.



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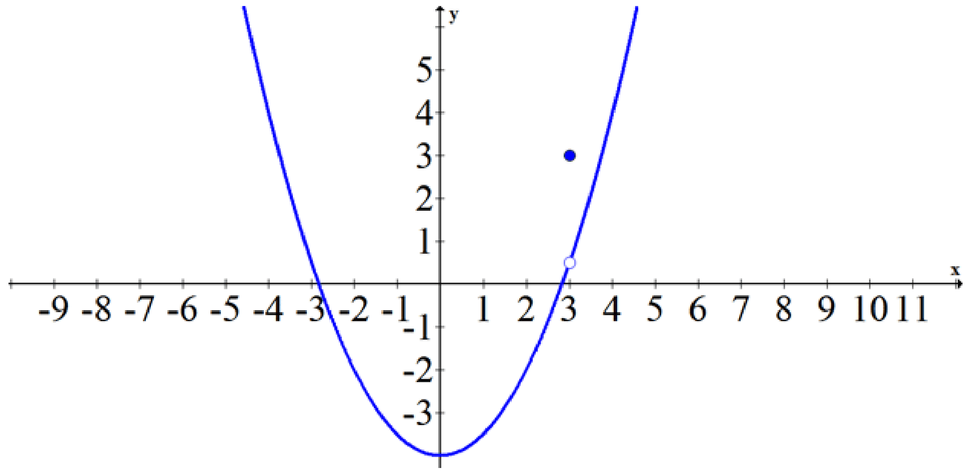
Examples

Example 1

Earlier, you were asked to find $f(3),\ \lim_{x \to 3} f(x),\ \lim_{x \to \infty} f(x)$ given the graph of the function $f(x)$ to be:



$\lim_{x \to 3} f(x) = \frac{1}{2}$

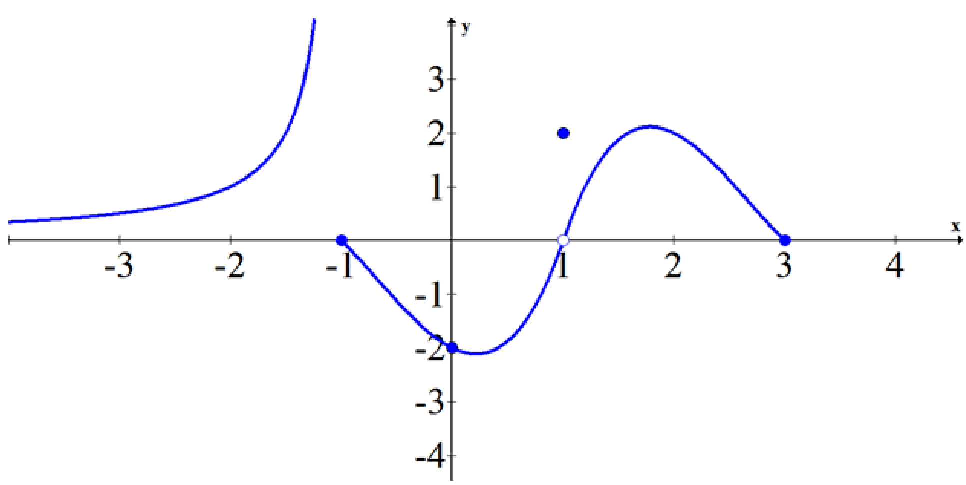
$f(3) = 3$

$\lim_{x \to \infty} f(x) = DNE$

Example 2

Evaluate the following expressions using the graph of the function $f(x)$.

1. $\lim_{x \to -\infty} f(x)$
2. $\lim_{x \to -1} f(x)$
3. $\lim_{x \to 0} f(x)$
4. $\lim_{x \to 1} f(x)$
5. $\lim_{x \to 3} f(x)$
6. $f(-1)$
7. $f(2)$
8. $f(1)$
9. $f(3)$



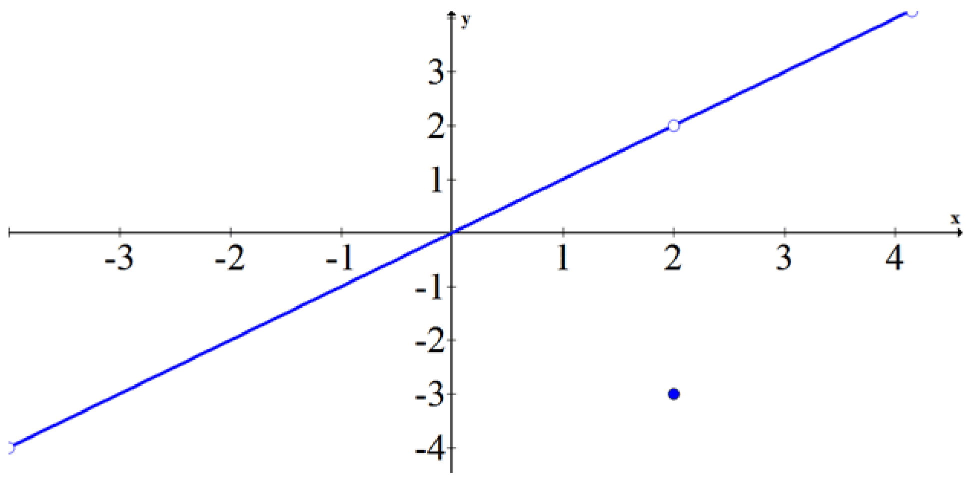
1. $\lim_{x \to -\infty} f(x) = 0$
2. $\lim_{x \to -1} f(x) = DNE$
3. $\lim_{x \to 0} f(x) = -2$

4. $\lim_{x \to 1} f(x) = 0$
5. $\lim_{x \to 3} f(x) = DNE$ (This is because only one side exists and a regular limit requires both left and right sides to agree)
6. $f(-1) = 0$
7. $f(0) = -2$
8. $f(1) = 2$
9. $f(3) = 0$

Example 3

Sketch a graph that has a limit at $x = 2$, but that limit does not match the height of the function.

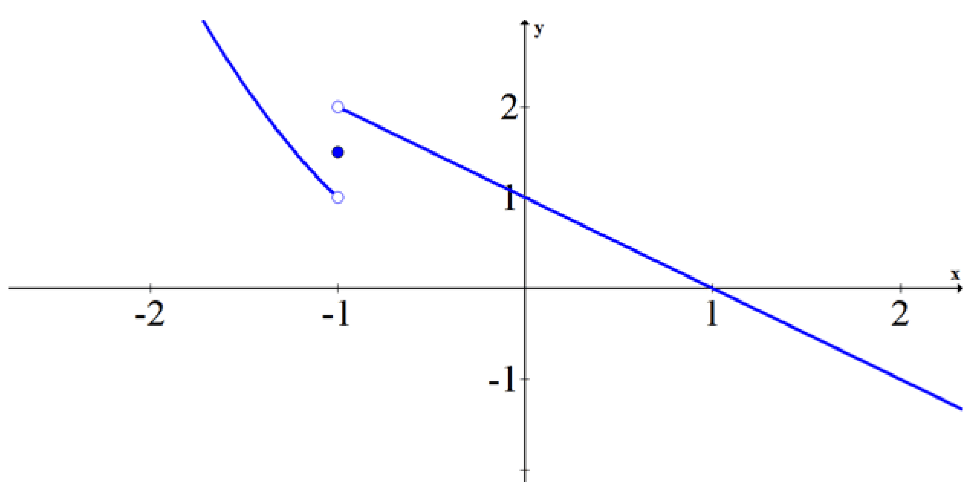
While there are an infinite number of graphs that fit this criteria, you should make sure your graph has a removable discontinuity at $x = 2$.



Example 4

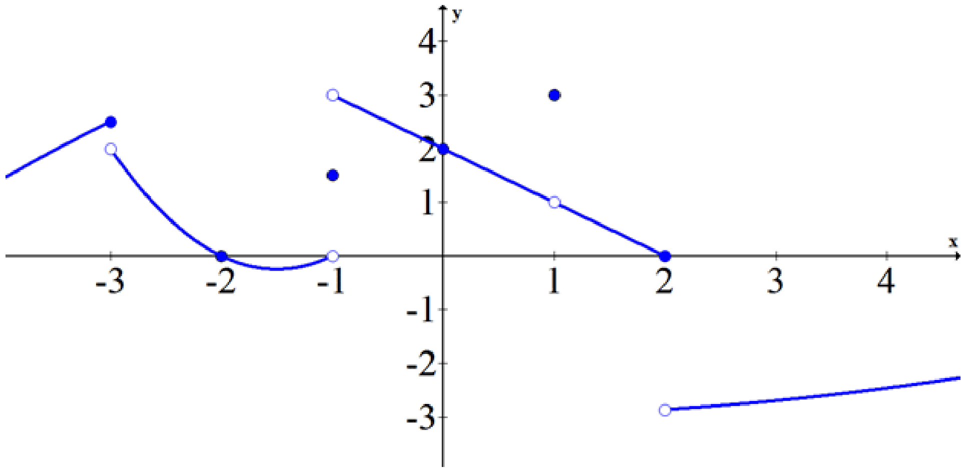
Sketch a graph that is defined at $x = -1$ but $\lim_{x \to -1} f(x)$ does not exist.

The graph must have either a jump or an infinite discontinuity at $x = -1$ and also have a solid hole filled in somewhere on that vertical line.



Example 5

Evaluate and explain how to find the limits as x approaches 0 and 1 for the graph below:

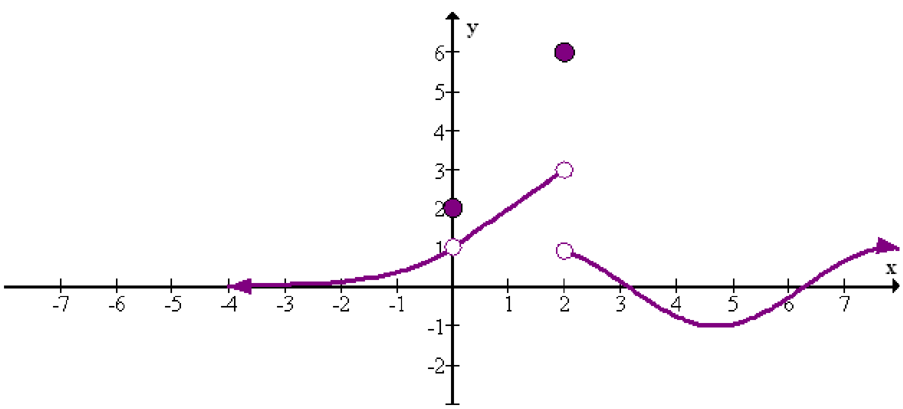


$\lim_{x\to 0} f(x) = 2, \lim_{x\to 1} f(x) = 1$

Both of these limits exist because the left hand and right hand neighborhoods of these points seem to approach the same height. In the case of the point $(0,2)$ the function happened to be defined there. In the case of the point $(1,1)$ the function happened to be defined elsewhere, but that does not matter. You only need to consider what the function does right around the point.

Review

Use the graph of $f(x)$ below to evaluate the expressions in 1-6.



$\lim_{x\to -\infty} f(x)$1.

$\lim_{x\to \infty} f(x)$2.

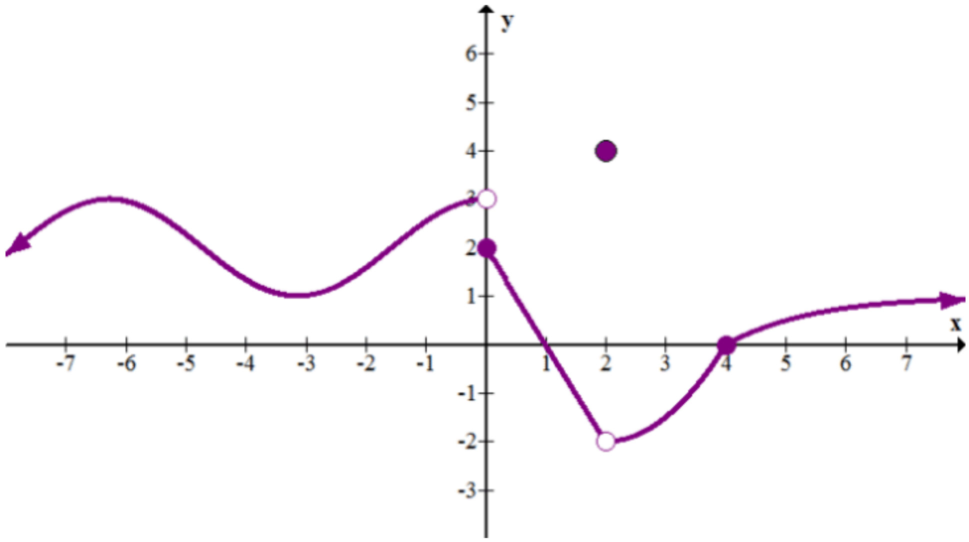
$\lim_{x\to 2} f(x)$3.

$\lim_{x\to 0} f(x)$4.

$f(0)$5.

$f(2)$6.

$g(x)$ below to evaluate the expressions in 7-13.Use the graph of



7. $\lim_{x\to-\infty} g(x)$

8. $\lim_{x\to\infty} g(x)$

9. $\lim_{x\to 2} g(x)$

10. $\lim_{x\to 0} g(x)$

11. $\lim_{x\to 4} g(x)$

12. $g(0)$

13. $g(2)$

14. Sketch a function $h(x)$ such that $h(2) = 4$, but $\lim_{x\to 2} h(x) = DNE$.

15. Sketch a function $j(x)$ such that $j(2) = 4$, but $\lim_{x\to 2} j(x) = 3$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.2.

6.4 Tables to Find Limits

Learning Objectives

Here you will estimate limits using tables.

Calculators such as the TI-84 have a table view that allows you to make extremely educated guesses as to what the limit of a function will be at a specific point, even if the function is not actually defined at that point.

How could you use a table to calculate the following limit?

$\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$

Using Tables to Find Limits

If you were given the following information organized in a table, how would you fill in the center column?

TABLE 6.1:

3.9	3.99	3.999		4.001	4.01	4.1
12.25	12.01	12.00001		11.99999	11.99	11.75

It would be logical to see the symmetry and notice how the top row approaches the number 4 from the left and the right. It would also be logical to notice how the bottom row approaches the number 12 from the left and the right. This would lead you to the conclusion that the limit of the function represented by this table is 12 as the top row approaches 4. It would not matter if the value at 4 was undefined or defined to be another number like 17, the pattern tells you that the limit at 4 is 12.

Using tables to help evaluate limits numerically requires this type of logic. **Numerically** is a term used to describe one of several different representations in mathematics. It refers to tables where the actual numbers are visible.

To estimate the limit $\lim\limits_{x \to 2} \frac{x-2}{x^2-x-2}$, complete the table:

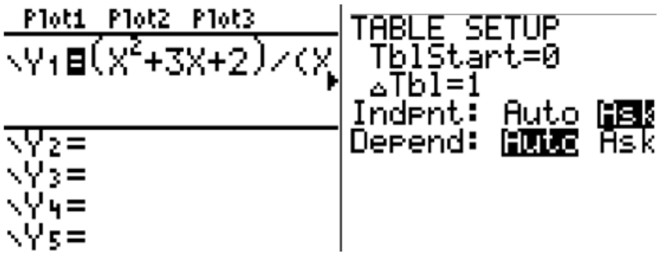
TABLE 6.2:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

While it is not necessary to use the table feature in the calculator, it is very efficient.

To use a table on your calculator to evaluate a limit:

1. Enter the function on the $y =$ screen
2. Go to table set up and highlight "ask" for the independent variable
3. Go to the table and enter values close to the number that x approaches

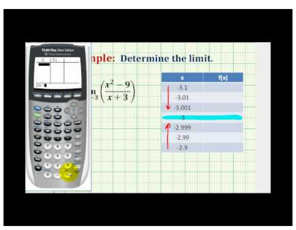


Another option is to substitute the given x values into the expression $\frac{x-2}{x^2-x-2}$ and record your results. Either way, the completed table is as follows.

TABLE 6.3:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.34483	0.33445	0.33344	0.33322	0.33223	0.32258

The evidence suggests that the limit is $\frac{1}{3}$.



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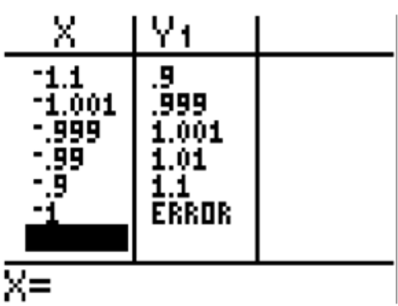
Examples

Example 1

Earlier, you were asked to find the limit of $\lim_{x\to -1}\frac{x^2+3x+2}{x+1}$. When you enter values close to -1 in the table you get y values that are increasingly close to the number 1. This implies that the limit as x approaches -1 is 1. Notice that when you evaluate the function at -1, the calculator produces an error. This should lead you to the conclusion that while the function is not defined at $x=-1$, the limit does exist.



Example 2

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-4}$

TABLE 6.4:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

You can trick the calculator into giving a very exact answer by typing in 1.999999999999 because then the calculator rounds instead of producing an error.

TABLE 6.5:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.25063	0.25006	0.24994	0.24938	0.2439

The evidence suggests that the limit is $\frac{1}{4}$.

Example 3

Complete the table and use the result to estimate the limit.

$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

TABLE 6.6:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

TABLE 6.7:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.29112	0.28892	0.2887	0.28865	0.28843	0.28631

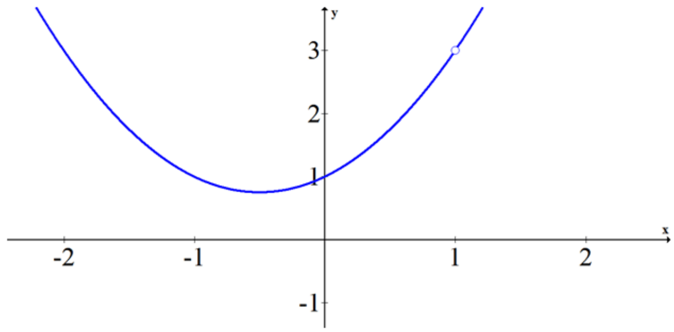
The evidence suggests that the limit is a number between 0.2887 and 0.28865. When you learn to find the limit analytically, you will know that the exact limit is $\frac{1}{2} \cdot 3\frac{1}{2} \approx 0.2886751346$.

Example 4

Graph the following function and the use a table to verify the limit as x approaches 1.

$f(x) = \frac{x^3-1}{x-1}, x \neq 1$

$\lim_{x \to 1} f(x) = 3$. This is because when you factor the numerator and cancel common factors, the function becomes a quadratic with a hole at the point $(1,3)$.



You can verify the limit in the table.

TABLE 6.8:

x	$f(x)$
.75	2.3125
.9	2.71
.99	2.9701
.999	2.997
1	Error
1.001	3.003
1.01	3.0301
1.1	3.31
1.25	3.8125

Example 5

Estimate the limit numerically.

$\lim_{x\to 0} \frac{\left[\frac{4}{x+2}\right]-2}{x}$

TABLE 6.9:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.20526	0.02005	0.002	-0.002	-0.02	-0.1952

$\lim_{x\to 0} f(x) = 0$

Review

Estimate the following limits numerically.

1. $\lim_{x\to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x\to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+2}$
4. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$
5. $\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{9}{x^2-9}\right)$

6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
13. $\lim_{x \to 2} \frac{\sqrt{x+3}-2}{x-1}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.3.

6.5 General Sequences

Objective

Recognize patterns in sequences of numbers, describe the pattern and determine additional terms in the sequence.

Review Queue

Find the domain and range of the functions below.

1. $y = 3x - 4$
2. $y = x^2 - 2$
3. $y = \frac{x(x-1)}{2}$

Finding the Next Term in a Sequence

Objective

Observe and continue the pattern apparent in a **sequence**, or arrangement of numerical values.

Guidance

When looking at a sequence of numbers, consider the following possibilities.

1. There could be a **common difference** (the same value is added or subtracted) to progress from each term to the next.

Example: $5, 8, 11, 14, \ldots$ (add 3)

2. There could be a **common ratio** (factor by which each term is multiplied) to progress from one term to the next.

Example: $9, 3, 1, \frac{1}{3} \ldots \left(\text{ multiply by } \frac{1}{3} \right)$

3. If the terms are fractions, perhaps there is a pattern in the numerator and a different pattern in the denominators.

Example: $\frac{1}{9}, \frac{3}{8}, \frac{5}{7}, \frac{7}{6}, \ldots$ (numerator (+2), denominator (-1))

4. If the terms are growing rapidly, perhaps the difference between the term values is increasing by some constant factor.

Example: $2, 5, 9, 14, \ldots$ (add 3, add 4, add 5, ...)

5. The terms may represent a particular type of number such as prime numbers, perfect squares, cubes, etc.

Example: $2, 3, 5, 7, \ldots$ (prime numbers)

6. Consider whether each term is the result of performing an operation on the two prior terms.

Example: $2, 5, 7, 12, 19, \ldots$ (add the previous two terms)

7. Consider the possibility that the value is connected to the term number:

Example: $0, 2, 6, 12, \ldots$

In this example $(0 \times 1) = 0, (1 \times 2) = 2, (2 \times 3) = 6, (3 \times 4) = 12, \ldots$

This list is not intended to be a comprehensive list of all possible patterns that may be present in a sequence but they are a good place to start when looking for a pattern.

Example A

Find the next two terms in the sequence: $160, 80, 40, 20, \_\_\_, \_\_\_$

Solution: Each term is the result of multiplying the previous term by $\frac{1}{2}$. Therefore, the next terms are:

$\frac{1}{2}(20) = 10$ and $\frac{1}{2}(10) = 5$

Example B

Find the next two terms in the sequence: $0, 3, 7, 12, 18, \_\_\_, \_\_\_$

Solution: The difference between the first two terms $(3-0)$ is 3, the difference between the second and third terms $(7-3)$ is 4, the difference between the third and fourth terms $(12-7)$ is 5 and the difference between the fourth and fifth terms $(18-12)$ is 6. Each time we add one more to get the next term. The next difference will be 7, so $18+7=25$ for the sixth term. To get the seventh term, we add 8, so $25+8=33$.

Example C

Find the next two terms in the sequence: $9, 5, 4, 1, 3, \_\_\_, \_\_\_$

Solution: This sequence requires that we look at the previous two terms. To get the third term, the second term was subtracted from the first: $9-5=4$. To get the fourth term, the third term is subtracted from the second: $5-4=1$. Similarly: $4-1=3$. Now, to get the next terms, continue the pattern:

$1-3=-2$ and $3-(-2)=5$

Guided Practice

Find the next two terms in each of the following sequences:

1. $-5, -1, 3, 7, \_\_\_, \_\_\_$

2. $\frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{5}{6}\_\_\_, \_\_\_$

3. $1, 4, 9, 16, \_\_\_, \_\_\_$

Answers

1. Each term is the previous term plus 4. Therefore, the next two terms are 11 and 15.

2. The pattern here is somewhat hidden because some of the fractions have been reduced. If we "unreduced" the second and fourth terms we get the sequence: $\frac{1}{3}, \frac{4}{6}, \frac{7}{9}, \frac{10}{12}, \_\_\_, \_\_\_$. Now the pattern can be observed to be that the numerator and denominator each increase by 3. So the next two terms are $\frac{13}{15}$ and $\frac{16}{18}$. Reducing the last term gives us the final answer of $\frac{13}{15}$ and $\frac{8}{9}$.

3. This sequence is the set of perfect squares or the term number squared. Therefore the 5^{th} and 6^{th} terms will be $5^2=25$ and $6^2=36$.

Vocabulary

Sequence
An arrangement of numbers which follows a pattern.

Common Difference
The constant value which is repeatedly added to each term in an arithmetic sequence to obtain the next term.

Common Ratio
The constant value which is multiplied by each term in a geometric sequence to obtain the next term.

Problem Set

Find the next three terms in each sequence.

1. $15, 21, 27, 33, \_\_\_, \_\_\_, \_\_\_$
2. $-4, 12, -36, 108, \_\_\_, \_\_\_, \_\_\_$
3. $51, 47, 43, 39, \_\_\_, \_\_\_, \_\_\_$
4. $100, 10, 1, 0.1, \_\_\_, \_\_\_, \_\_\_$
5. $1, 2, 4, 8, \_\_\_, \_\_\_, \_\_\_$
6. $\frac{7}{2}, \frac{5}{3}, \frac{3}{4}, \frac{1}{5}, \_\_\_, \_\_\_, \_\_\_$

Find the missing terms in the sequences.

7. $1, 4, \_\_\_, 16, 25, \_\_\_$
8. $\frac{2}{3}, \frac{3}{4}, \_\_\_, \frac{5}{6}, \_\_\_$
9. $0, 2, \_\_\_, 9, 14, \_\_\_$
10. $1, \_\_\_, 27, 64, 125, \_\_\_$
11. $5, \_\_\_, 11, 17, 28, \_\_\_, 73$
12. $3, 8, \_\_\_, 24, \_\_\_, 48$

Describing the Pattern and Writing a Recursive Rule for a Sequence

Objective

Recognize and describe the pattern and write a recursive rule for a sequence.

Guidance

A **recursive rule** for a sequence is a formula which tells us how to progress from one term to the next in a sequence. Generally, the variable n is used to represent the term number. In other words, n takes on the values 1 (first term), 2 (second term), 3 (third term), etc. The variable, a_n represents the n^{th} term and a_{n-1} represents the term preceding a_n.

Example sequence: $4, 7, 11, 16, \ldots, a_{n-1}, a_n$

In the above sequence, $a_1 = 4$, $a_2 = 7$, $a_3 = 11$ and $a_4 = 16$.

Example A

Describe the pattern and write a recursive rule for the sequence: $9, 11, 13, 15, \ldots$

Solution: First we need to determine what the pattern is in the sequence. It appears that 2 is added to each term to obtain the following term in the sequence. We can use a_{n-1} and a_n to write a recursive rule as follows: $a_n = a_{n-1} + 2$

Example B

Write a recursive rule for the sequence: $3, 9, 27, 81, \ldots$

Solution: In this sequence, each term is multiplied by 3 to get the next term. We can write a recursive rule: $a_n = 3a_{n-1}$

Example C

Write a recursive rule for the sequence: $1, 1, 2, 3, 5, 8, \ldots$

Solution: This is a special sequence called the Fibonacci sequence. In this sequence each term is the sum of the previous two terms. We can write the recursive rule for this sequence as follows: $a_n = a_{n-2} + a_{n-1}$.

Guided Practice

Write the recursive rules for the following sequences.

1. $1, 2, 4, 8, \ldots$

2. $1, -2, -5, -8, \ldots$

3. $1, 2, 4, 7, \ldots$

Answers

1. In this sequence each term is double the previous term so the recursive rule is: $a_n = 2a_{n-1}$

2. This time three is subtracted each time to get the next term: $a_n = a_{n-1} - 3$.

3. This one is a little trickier to express. Try looking at each term as shown below:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 1 \\ a_3 &= a_2 + 2 \\ a_4 &= a_3 + 3 \\ &\vdots \\ a_n &= a_{n-1} + (n-1) \end{aligned}$$

Vocabulary

Recursive Rule
A rule that can be used to calculate a term in a sequence given the previous term(s).

Problem Set

Describe the pattern and write a recursive rule for the following sequences.

1. $\frac{1}{4}, -\frac{1}{2}, 1, -2 \ldots$
2. $5, 11, 17, 23, \ldots$
3. $33, 28, 23, 18, \ldots$
4. $1, 4, 16, 64, \ldots$
5. $21, 30, 39, 48, \ldots$
6. $100, 75, 50, 25, \ldots$
7. $243, 162, 108, 72, \ldots$
8. $128, 96, 72, 54, \ldots$
9. $1, 5, 10, 16, 23, \ldots$
10. $0, 2, 2, 4, 6, \ldots$
11. $3, 5, 8, 12, \ldots$
12. $0, 2, 6, 12, \ldots$

Using and Writing N[U+1D57]^h Term Rules for Sequences

Objective

Use an n^{th} term rule or general rule for a sequence to find terms and write a general rule for a given sequence.

Guidance

In the previous concept we wrote a recursive rule to find the next term in a sequence. Recursive rules can help us generate multiple sequential terms in a sequence but are not helpful in determining a particular single term. Consider the sequence: $3, 5, 7, \ldots, a_n$. The recursive rule for this sequence is $a_n = a_{n-1} + 2$. What if we want to find the 100^{th} term? The recursive rule only allows us to find a term in the sequence if we know the previous term. An n^{th} **term or general rule,** however, will allow us to find the 100^{th} term by replacing n in the formula with 100.

Example A

Write the first three terms, the 15^{th} term and the 40^{th} term of the sequence with the general rule: $a_n = n^2 - 1$.

Solution: We can find each of these terms by replacing n with the appropriate term number:

$$
\begin{aligned}
a_1 &= (1)^2 - 1 = 0 \\
a_2 &= (2)^2 - 1 = 3 \\
a_3 &= (3)^2 - 1 = 8 \\
a_{15} &= (15)^2 - 1 = 224 \\
a_{40} &= (40)^2 - 1 = 1599
\end{aligned}
$$

Calculator: These terms can also be found using a graphing calculator. First press 2^{nd} **STAT** (to get to the **List** menu). Arrow over to **OPS,** select option **5: seq(** and type in (expression, variable, begin, end). For this particular problem, the calculator yields the following:

$seq\left(x^2 - 1, x, 1, 3\right) = \{0\ 3\ 8\}$ for the first three terms

$seq\left(x^2 - 1, x, 15, 15\right) = \{224\}$ for the 15^{th} term

$seq\left(x^2 - 1, x, 40, 40\right) = \{1599\}$ for the 40^{th} term

Example B

Write a general rule for the sequence: $5, 10, 15, 20, \ldots$

Solution: The previous example illustrates how a general rule maps a term number directly to the term value. Another way to say this is that the general rule expresses the n^{th} term as a function of n. Let's put the terms in the above sequence in a table with their term numbers to help identify the rule.

Looking at the terms and term numbers together helps us to see that each term is the result of multiplying the term number by 5. The general rule is $a_n = 5n$

TABLE 6.10:

n	1	2	3	4
a	5	10	15	20

Example C

Find the n^{th} term rule for the sequence: $0, 2, 6, 12, \ldots$

Solution: Let's make the table again to begin to analyze the relationship between the term number and the term value.

TABLE 6.11:

n	1	2	3	4
a_n	0	2	6	12
$n(?)$	(1)(0)	(2)(1)	(3)(2)	(4)(3)

This time the pattern is not so obvious. To start, write each term as a product of the term number and a second factor. Then it can be observed that the second factor is always one less that the term number and the general rule can be written as $a_n = n(n-1)$

Guided Practice

1. Given the general rule: $a_n = 3n - 13$, write the first five terms, 25^{th} term and the 200^{th} term of the sequence.

2. Write the general rule for the sequence: $4, 5, 6, 7, \ldots$

3. Write the general rule and find the 35^{th} term of the sequence: $-1, 0, 3, 8, 15, \ldots$

Answers

1. Plug in the term numbers as shown:

$$\begin{aligned}
a_1 &= 3(1) - 13 = -10\\
a_2 &= 3(2) - 13 = -7\\
a_3 &= 3(3) - 13 = -4\\
a_4 &= 3(4) - 13 = -1\\
a_5 &= 3(5) - 13 = 2\\
a_{25} &= 3(25) - 13 = 62\\
a_{200} &= 3(200) - 13 = 587
\end{aligned}$$

2. Put the values in a table with the term numbers and see if there is a way to write the term as a function of the term number.

TABLE 6.12:

n	1	2	3	4
a_n	4	5	6	7
$n \pm (?)$	$(1)+3$	$(2)+3$	$(3)+3$	$(4)+3$

Each term appears to be the result of adding three to the term number. Thus, the general rule is $a_n = n+3$

3. Put the values in a table with the term numbers and see if there is a way to write the term as a function of the term number.

TABLE 6.13:

n	1	2	3	4	5
a_n	-1	10	3	8	15
$n(?)$	$(1)(-1)$	$(2)(0)$	$(3)(1)$	$(4)(2)$	$(5)(3)$

Each term appears to be the result of multiplying the term number by two less than the term number. Thus, the general rule is $a_n = n(n-2)$

Vocabulary

N^{th} term or general rule
: A formula which relates the term to the term number and thus can be used to calculate any term in a sequence whether or not any terms are known.

Problem Set

Use the n^{th} term rule to generate the indicated terms in each sequence.

1. $2n+7$, terms $1-5$ and the 10^{th} term.
2. $-5n-1$, terms $1-3$ and the 50^{th} term.
3. $2^n - 1$, terms $1-3$ and the 10^{th} term.
4. $\left(\frac{1}{2}\right)^n$, terms $1-3$ and the 8^{th} term.

5. $\frac{n(n+1)}{2}$, terms $1-4$ and the 20^{th} term.

Use your calculator to generate the first 5 terms in each sequence. Use **MATH >FRAC**, on your calculator to convert decimals to fractions.

6. $4n-3$
7. $-\frac{1}{2}n+5$
8. $\left(\frac{2}{3}\right)^n+1$
9. $2n(n-1)$
10. $\frac{n(n+1)(2n+1)}{6}$

Write the n^{th} term rule for the following sequences.

11. $3,5,7,9,\ldots$
12. $1,7,25,79,\ldots$
13. $6,14,24,36,\ldots$
14. $6,5,4,3,\ldots$
15. $2,5,9,14,\ldots$

Series and Summation Notation

Objective

Write the terms of a series and find the sum of a finite series.

Guidance

A **series** is the sum of the terms in a sequence. A series is often expresses in summation notation(also called sigma notation) which uses the capital Greek letter $\sum$, sigma. Example: $\sum_{n=1}^{5} n=1+2+3+4+5=15$. Beneath the sigma is the index (in this case n) which tells us what value to plug in first. Above the sigma is the upper limit which tells us the upper limit to plug into the rule.

Example A

Write the terms and find the sum of the series: $\sum_{n=1}^{6} 4n+1$

Solution: Begin by replacing n with the values 1 through 6 to find the terms in the series and then add them together.

$$\begin{aligned}&(4(1)-1)+(4(2)-1)+(4(3)-1)+(4(4)-1)+(4(5)-1)+(4(6)-1)\\&3+7+11+15+19+23\\&=78\end{aligned}$$

Calculator: The graphing calculator can also be used to evaluate this sum. We will use a compound function in which we will sum a sequence. Go to 2^{nd} **STAT** (to get to the **List** menu) and arrow over to **MATH**. Select option **5: sum(** then return to the **List** menu, arrow over to **OPS** and select option **5: seq(** to get sum(seq(on your screen. Next, enter in (expression, variable, begin, end)just as we did in the previous topic to list the terms in a sequence. By including the sum(command, the calculator will sum the terms in the sequence for us. For this particular problem the expression and result on the calculator are:

$$sum(seq(4x-1,x,1,6)) = 78$$

To obtain a list of the terms, just use $seq\,(4x-1,x,1,6) = \{3 \;\; 7 \;\; 11 \;\; 15 \;\; 19 \;\; 23\}$.

Example B

Write the terms and find the sum of the series: $\sum\limits_{n=9}^{11} \frac{n(n+1)}{2}$

Solution: Replace n with the values 9, 10 and 11 and sum the resulting series.

$$\frac{9(9-1)}{2} + \frac{10(10-1)}{2} + \frac{11(11-1)}{2}$$
$$36+45+55$$
$$136$$

Using the calculator: $sum(seq(x(x-1)/2,x,9,11)) = 136$.

More Guidance

There are a few special series which are used in more advanced math classes, such as calculus. In these series, we will use the variable, i, to represent the index and n to represent the upper bound (the total number of terms) for the sum.

$\sum\limits_{i=1}^{n} 1 = n$

Let $n = 5$, now we have the series $\sum\limits_{i=1}^{5} 1 = 1+1+1+1+1 = 5$. Basically, in the series we are adding 1 to itself n times (or calculating $n \times 1$) so the resulting sum will always be n.

$\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$

If we let $n = 5$ again we get $\sum\limits_{i=1}^{n} i = 1+2+3+4+5 = 15 = \frac{5(5+1)}{2}$. This one is a little harder to derive but can be illustrated using different values of n. This rule is closely related to the rule for the sum of an arithmetic series and will be used to prove the sum formula later in the chapter.

$\sum\limits_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6}$

Let $n = 5$ once more. Using the rule, the sum is $\frac{5(5+1)(2(5)+1)}{6} = \frac{5(6)(11)}{6} = 55$

If we write the terms in the series and find their sum we get $1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25 = 55$.

The derivation of this rule is beyond the scope of this course.

Example C

Use one of the rules above to evaluate $\sum\limits_{i=1}^{15} i^2$.

Solution: Using the rule $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, we get $\frac{15(15+1)(2(15)+1)}{6} = \frac{15(16)(31)}{6} = 1240$

Guided Practice

Evaluate the following. First without a calculator, then use the calculator to check your result.

1. $\sum\limits_{n=3}^{7} 2(n-3)$

2. $\sum_{n=1}^{7} \frac{1}{2}n+1$

3. $\sum_{n=1}^{4} 3n^2-5$

Answers

1.

$$\begin{aligned}\sum_{n=3}^{7} 2(n-3) &= 2(3-3)+2(4-3)+2(5-3)+2(6-3)+2(7-3)\\ &= 2(0)+2(1)+2(2)+2(3)+2(4)\\ &= 0+2+4+6+8\\ &= 20\end{aligned}$$

$sum(seq(2(x-3),x,3,7) = 20$

2.

$$\begin{aligned}\sum_{n=1}^{7} \frac{1}{2}n+1 &= \frac{1}{2}(1)+1+\frac{1}{2}(2)+1+\frac{1}{2}(3)+1+\frac{1}{2}(4)+1+\frac{1}{2}(5)+1+\frac{1}{2}(6)+1+\frac{1}{2}(7)+1\\ &= \frac{1}{2}+1+1+1+\frac{3}{2}+1+2+1+\frac{5}{2}+1+3+1+\frac{7}{2}+1\\ &= \frac{16}{2}+13\\ &= 8+13\\ &= 21\end{aligned}$$

$sum(seq(1/2x+1,x,1,7) = 21$

3.

$$\begin{aligned}\sum_{n=1}^{4} 3n^2-5 &= 3(1)^2-5+3(2)^2-5+3(3)^2-5+3(4)^2-5\\ &= 3-5+12-5+27-5+48-5\\ &= 90-20\\ &= 70\end{aligned}$$

$sum(seq(3x^2-5,x,1,4) = 70$

Vocabulary

Series
The sum of the terms in a sequence.

Problem Set

Write out the terms and find the sum of the following series.

1. $\sum_{n=1}^{5} 2n$
2. $\sum_{n=5}^{8} n+3$

3. $\sum_{n=10}^{15} n(n-3)$
4. $\sum_{n=3}^{7} \frac{n(n-1)}{2}$
5. $\sum_{n=1}^{6} 2^{n-1}+3$

Use your calculator to find the following sums.

6. $\sum_{n=10}^{15} \frac{1}{2}n+3$
7. $\sum_{n=0}^{50} n-25$
8. $\sum_{n=1}^{5} \left(\frac{1}{2}\right)^{n-5}$
9. $\sum_{n=5}^{12} \frac{n(2n+1)}{2}$
10. $\sum_{n-1}^{100} \frac{1}{2}n$

In problems 11 and 12, write out the terms in each of the series (*a* and *b*) and explain why the sums are equal.

1. $\sum_{n=1}^{5} 2n+3$
2. $3(5)+\sum_{n=1}^{5} 2n$

1. $\sum_{n=1}^{5} \frac{n(n+1)}{2}$
2. $\frac{1}{2}\sum_{n=1}^{5} n(n+1)$

6.6 Arithmetic Sequences and Series

Objective

Identify arithmetic sequences, find the n^{th} term rule for an arithmetic sequence and find the sum of a finite arithmetic sequence.

Review Queue

Describe the pattern in each sequence below.

1. $3, 5, 7, 9, \ldots$
2. $1, 4, 8, 13, \ldots$
3. Find $\sum_{n=1}^{5} 7n+3$

Arithmetic Sequences and Finding the N[U+1D57]^h Term Given the Common Difference and a Term

Objective

Identify an arithmetic sequence and its common difference and write an n^{th} term rule given the common difference and a term.

Guidance

In this concept we will begin looking at a specific type of sequence called an **arithmetic sequence**. In an arithmetic sequence the difference between any two consecutive terms is constant. This constant difference is called the **common difference**. For example, question one in the Review Queue above is an arithmetic sequence. The difference between the first and second terms is $(5-3) = 2$, the difference between the second and third terms is $(7-5) = 2$ and so on. We can generalize this in the equation below:

$a_n - a_{n-1} = d$, where a_{n-1} and a_n represent two consecutive terms and d represents the common difference.

Since the same value, the common difference, d, is added to get each successive term in an arithmetic sequence we can determine the value of any term from the first term and how many time we need to add d to get to the desired term as illustrated below:

Given the sequence: $22, 19, 16, 13, \ldots$ in which $a_1 = 22$ and $d = -3$

$$
\begin{aligned}
a_1 &= 22 \ or\ 22+(1-1)(-3) = 22+0 = 22 \\
a_2 &= 19 \ or\ 22+(2-1)(-3) = 22+(-3) = 19 \\
a_3 &= 16 \ or\ 22+(3-1)(-3) = 22+(-6) = 16 \\
a_4 &= 13 \ or\ 22+(4-1)(-3) = 22+(-9) = 13 \\
&\vdots \\
a_n &= 22+(n-1)(-3) \\
a_n &= 22-3n+3 \\
a_n &= -3n+25
\end{aligned}
$$

Now we can generalize this into a rule for the n^{th} term of any arithmetic sequence:

$$a_n = a_1 + (n-1)d$$

Example A

Find the common difference and n^{th} term rule for the arithmetic sequence: $2, 5, 8, 11\ldots$

Solution: To find the common difference we subtract consecutive terms.

$$\begin{aligned} 5-2&=3 \\ 8-5&=3 \text{ ,thus the common difference is 3.} \\ 11-8&=3 \end{aligned}$$

Now we can put our first term and common difference into the n^{th} term rule discovered above and simplify the expression.

$$\begin{aligned} a_n &= 2+(n-1)(3) \\ &= 2+3n-3 \quad \text{,so } a_n = 3n-1. \\ &= 3n-1 \end{aligned}$$

Example B

Find the n^{th} term rule and thus the 100^{th} term for the arithmetic sequence in which $a_1 = -9$ and $d = 2$.

Solution: We have what we need to plug into the rule:

$$\begin{aligned} a_n &= -9+(n-1)(2) \\ &= -9+2n-2 \quad \text{,thus the } n^{th} \text{ term rule is } a_n = 2n-11. \\ &= 2n-11 \end{aligned}$$

Now to find the 100^{th} term we can use our rule and replace n with 100: $a_{100} = 2(100) - 11 = 200 - 11 = 189$.

Example C

Find the n^{th} term rule and thus the 100^{th} term for the arithmetic sequence in which $a_3 = 8$ and $d = 7$.

Solution: This one is a little less straightforward as we will have to first determine the first term from the term we are given. To do this, we will replace a_n with $a_3 = 8$ and use 3 for n in the formula to determine the unknown first term as shown:

$$\begin{aligned} a_1+(3-1)(7) &= 8 \\ a_1+2(7) &= 8 \\ a_1+14 &= 8 \\ a_1 &= -6 \end{aligned}$$

Now that we have the first term and the common difference we can follow the same process used in the previous example to complete the problem.

$$\begin{aligned} a_n &= -6+(n-1)(7) \\ &= -6+7n-7 \quad ,\text{thus } a_n = 7n-13. \\ &= 7n-13 \end{aligned}$$

Now we can find the 100^{th} term: $a_{100} = 7(100) - 13 = 687$.

Guided Practice

1. Find the common difference and the n^{th} term rule for the sequence: $5, -3, -11, \ldots$

2. Write the n^{th} term rule and find the 45^{th} term for the arithmetic sequence with $a_{10} = 1$ and $d = -6$.

3. Find the 62^{nd} term for the arithmetic sequence with $a_1 = -7$ and $d = \frac{3}{2}$.

Answers

1. The common difference is $-3-5 = -8$. Now $a_n = 5+(n-1)(-8) = 5-8n+8 = -8n+13$.

2. To find the first term:

$$\begin{aligned} a_1 + (10-1)(-6) &= 1 \\ a_1 - 54 &= 1 \\ a_1 &= 55 \end{aligned}$$

Find the n^{th} term rule: $a_n = 55+(n-1)(-6) = 55-6n+6 = -6n+61$.

Finally, the 45^{th} term: $a_{45} = -6(45)+61 = -209$.

3. This time we will not simplify the n^{th} term rule, we will just use the formula to find the 62^{rd} term: $a_{62} = -7+(62-1)\left(\frac{3}{2}\right) = -7+61\left(\frac{3}{2}\right) = -\frac{14}{2}+\frac{183}{2} = \frac{169}{2}$.

Vocabulary

Arithmetic Sequence
A sequence in which the difference between any two consecutive terms is constant.

Common Difference
The value of the constant difference between any two consecutive terms in an arithmetic sequence.

Problem Set

Identify which of the following sequences is arithmetic. If the sequence is arithmetic find the n^{th} term rule.

1. $2, 3, 4, 5, \ldots$
2. $6, 2, -1, -3, \ldots$
3. $5, 0, -5, -10, \ldots$
4. $1, 2, 4, 8, \ldots$
5. $0, 3, 6, 9, \ldots$
6. $13, 12, 11, 10, \ldots$

Write the n^{th} term rule for each arithmetic sequence with the given term and common difference.

7. $a_1 = 15$ and $d = -8$

8. $a_1 = -10$ and $d = \frac{1}{2}$
9. $a_3 = 24$ and $d = -2$
10. $a_5 = -3$ and $d = 3$
11. $a_{10} = -15$ and $d = -11$
12. $a_7 = 32$ and $d = 7$

Finding the N[U+1D57]^h Term Given Two Terms

Objective

Write an n^{th} term rule for an arithmetic sequence given any two terms in the sequence.

Guidance

In the last concept we were given the common difference directly or two consecutive terms from which we could determine the common difference. In this concept we will find the common difference and write n^{th} term rule given any two terms in the sequence.

Example A

Find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_7 = 17$ and $a_{20} = 82$.

Solution: We will start by using the n^{th} term rule for an arithmetic sequence to create two equations in two variables:

$a_7 = 17$, so $a_1 + (7-1)d = 17$ or more simply: $a_1 + 6d = 17$

$a_{20} = 82$, so $a_1 + (20-1)d = 82$ or more simply: $a_1 + 19d = 82$

Solve the resulting system:

$$\begin{array}{rcr} a_1 + 6d = 17 & & \cancel{a}_1 + 6d \ = 17 \\ \underline{-1(a_1 + 19d = 82)} & \Rightarrow & \underline{-\cancel{a}_1 - 19d = -82} \\ & & -13d = -65 \\ & & d = 5 \end{array}$$

, replacing d with 5 in one of the equations we get

$$\begin{aligned} a_1 + 6(5) &= 7 \\ a_1 + 30 &= 17 \\ a_1 &= -13 \end{aligned}$$

.

Using these values we can find the n^{th} term rule:

$$\begin{aligned} a_n &= -13 + (n-1)(5) \\ a_n &= -13 + 5n - 5 \\ a_n &= 5n - 18 \end{aligned}$$

Example B

Find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_{11} = -13$ and $a_{40} = -71$.

Solution: Though this is exactly the same type of problem as Example A, we are going to use a different approach. We discovered in the last concept that the n^{th} term rule is really just using the first term and adding d to it $n-1$ times to find the n^{th} term. We are going to use that idea to find the common difference. To get from the 11^{th} term to the 40^{th} term, the common difference is added $40-11$ or 29 times. The difference in the term values is $-71-(-13)$ or -58. What must be added 29 times to create a difference of -58? We can subtract the terms and divide by the difference in term number to determine the common difference.

$$\frac{-71-(-13)}{40-11} = \frac{-71+13}{29} = \frac{-58}{29} = -2. \text{ So } d = -2.$$

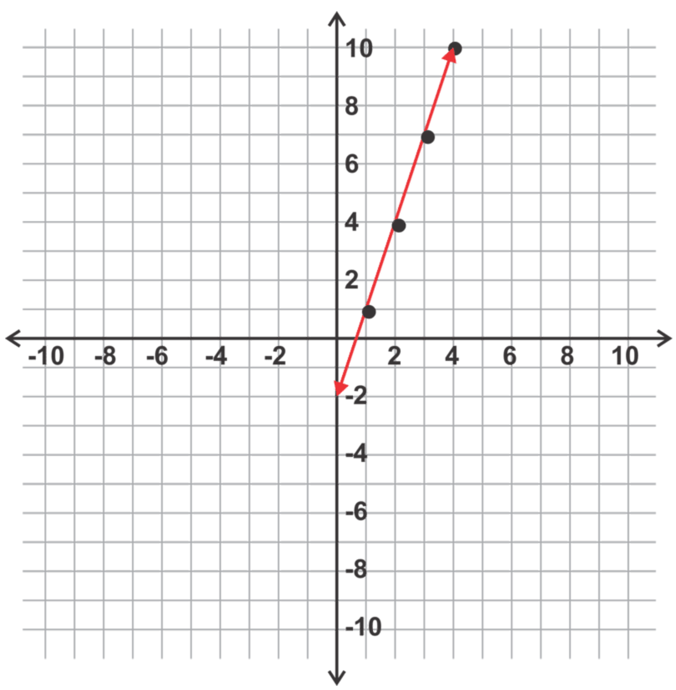
Now we can use the common difference and one of the terms to find the first term as we did previously.

$$\begin{aligned} a_1 + (11-1)(-2) &= -13 \\ a_1 + (-20) &= -13 \\ a_1 &= 7 \end{aligned}$$

Writing the n^{th} term rule we get: $a_n = 7+(n-1)(-2) = 7-2n+2 = -2n+9$.

More Guidance

Before we look at the final example for this concept, we are going to connect the n^{th} term rule for an arithmetic sequence to the equation of a line. Have you noticed that the simplified n^{th} term rule, $a_n = pn+q$, where p and q represent constants, looks a little like $y = mx+b$, the slope-intercept form of the equation of a line? Let's explore why this is the case using the arithmetic sequence $1,4,7,10,\ldots$ If we create points by letting the x –coordinate be the term number and the y –coordinate be the term, we get the following points and can plot them in the coordinate plane as shown below,



The points are: $(1,1),(2,4),(3,7),(4,10)$

Notice, that all of these points lie on the same line. This happens because for each increase of one in the term number (x), the term value (y) increases by 3. This common difference is actually the slope of the line.

We can find the equation of this line using the slope, 3, and the point $(1,1)$ in the equation $y = mx + b$ as follows:

$$\begin{aligned} 1 &= 3(1) + b \\ 1 &= 2 + b \quad \text{, so the equation of the line is } y = 3x - 1 \\ -1 &= b \end{aligned}$$

The n^{th} term rule for the sequence is thus: $a_n = 3n - 1$.

Example C

Find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_{10} = -50$ and $a_{32} = -182$.

Solution: This time we will use the concept that the terms in an arithmetic sequence are actually points on a line to write an equation. In this case our points are $(10,-50)$ and $(32,-182)$. We can find the slope and the equation as shown.

$$m = \frac{-182 - (-50)}{32 - 10} = \frac{-132}{22} = -6$$

Use the point $(10,-50)$ so find the y-intercept:

$$\begin{aligned} -50 &= -6(10) + b \\ -50 &= -60 + b \\ 10 &= b \end{aligned}$$

, so $y = -6x + 10$ and $a_n = -6n + 10$.

Guided Practice

1. Use the method in Example A to find the n^{th} term rule for the arithmetic sequence with $a_6 = -13$ and $a_{15} = -40$.

2. Use the method in Example B to find the n^{th} term rule for the arithmetic sequence with $a_6 = 13$ and $a_{22} = 77$.

3. Use the method in Example C to find the n^{th} term rule for the arithmetic sequence with $a_7 = -75$ and $a_{25} = -273$.

Answers

1. From $a_6 = -13$ we get the equation $a_1 + (6-1)d = a_1 + 5d = -13$.

From $a_{15} = -40$ we get the equation $a_1 + (15-1)d = a_1 + 14d = -40$.

Use the two equations to solve for a_1 and d:

$$\begin{aligned} a_1 + 5d &= -13 & a_1 + 5(-3) &= -13 \\ \underline{-a_1 - 14d = 40} &\ \text{Use } d \text{ to find } a_1 \Rightarrow & a_1 - 15 &= -13. \\ -9d &= 27 & a_1 &= 2 \\ d &= -3 \end{aligned}$$

Find the n^{th} term rule: $a_n = 2 + (n-1)(-3) = 2 - 3n + 3 = -3n + 5$.

2. The common difference is $\frac{77-13}{22-6} = \frac{64}{16} = 4$. The first term can be found using $a_6 = 13$:

$$\begin{aligned} a_1 + (6-1)(4) &= 13 \\ a_1 + 20 &= 13 \\ a_1 &= -7 \end{aligned}$$

. Thus $a_n = -7 + (n-1)(4) = -7 + 4n - 4 = 4n - 11$.

3. From $a_7 = -75$ we get the point $(7, -75)$. From $a_{25} = -273$ we get the point $(25, -273)$. The slope between these points is $\frac{-273-(-75)}{25-7} = \frac{-198}{18} = -11$. The y-intercept can be found next using the point $(7, -75)$:

$$\begin{aligned} -75 &= -11(7) + b \\ -75 &= -77 + b \\ 2 &= b \end{aligned}$$

The final equation is $y = -11x + 2$ and the n^{th} term rule is $a_n = -11n + 2$.

Problem Set

Use the two given terms to find an n^{th} term rule for the sequence.

1. $a_7 = -17$ and $a_{25} = -71$
2. $a_{11} = 23$ and $a_{42} = 85$
3. $a_3 = -6$ and $a_{12} = -3$
4. $a_8 = 24$ and $a_2 = 9$
5. $a_6 = -27$ and $a_{10} = -47$
6. $a_4 = 37$ and $a_{12} = 85$
7. $a_{13} = -20$ and $a_{30} = -54$
8. $a_3 = 23$ and $a_9 = 65$
9. $a_{30} = -31$ and $a_{45} = -46$
10. $a_5 = 25$ and $a_{11} = 73$
11. $a_{10} = -2$ and $a_{25} = -14$
12. $a_{16} = 14$ and $a_{28} = 23$

Finding the Sum of a Finite Arithmetic Series

Objective

Find the sum of an arithmetic series using the formula and the calculator.

Guidance

In the concept **Series and Summation Notation** we explored how to use the calculator to evaluate the sum of a series. This method can be used to find the sum of an arithmetic series as well. However, in this concept we will explore an algebraic method unique to arithmetic series. As we discussed earlier in the unit a series is simply the sum of a sequence so an arithmetic series is a sum of an arithmetic sequence. Let's look at an example to illustrate this and develop a formula to find the sum of a finite arithmetic series.

Example A

Find the sum of the arithmetic series: $1 + 3 + 5 + 7 + 9 + 11 + \ldots + 35 + 37 + 39$

Solution: Now, while we could just add up all of the terms to get the sum, if we had to sum a large number of terms that would be very time consuming. A famous German mathematician, Johann Carl Friedrich Gauss, used the method described here to determine the sum of the first 100 integers in grade school. First, we can write out all the numbers twice, in ascending and descending order, and observe that the sum of each pair of numbers is the same:

$$\begin{array}{cccccccccccc} 1 & 3 & 5 & 7 & 9 & 11 & \ldots & 35 & 37 & 39 \\ 39 & 37 & 35 & 33 & 31 & 29 & \ldots & 5 & 3 & 1 \\ & & & & & \vdots & & & & \\ 40 & 40 & 40 & 40 & 40 & 40 & \ldots & 40 & 40 & 40 \end{array}$$

Notice that the sum of the corresponding terms in reverse order is always equal to 40, which is the sum of the first and last terms in the sequence.

What Gauss realized was that this sum can be multiplied by the number of terms and then divided by two (since we are actually summing the series twice here) to get the sum of the terms in the original sequence. For the problem he was given in school, finding the sum of the first 100 integers, he was able to just use the first term, $a_1 = 1$, the last term, $a_n = 100$, and the total number of terms, $n = 100$, in the following formula:

$$\frac{n(a_1 + a_n)}{2} = \frac{100(1+100)}{2} = 5050$$

In our example we know the first and last terms but how many terms are there? We need to find n to use the formula to find the sum of the series. We can use the first and last terms and the n^{th} term to do this.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ 39 &= 1 + 2(n-1) \\ 38 &= 2(n-1) \\ 19 &= n-1 \\ 20 &= n \end{aligned}$$

Now the sum is $\frac{20(1+39)}{2} = 400$.

More Guidance-Proof of the Arithmetic Sum Formula

The rule for finding the n^{th} term of an arithmetic sequence and properties of summations that were explored in the problem set in the concept **Series and Summation Notation** can be used to prove the formula algebraically. First, we will start with the n^{th} term rule $a_n = a_1 + (n-1)d$. We need to find the sum of numerous n^{th} terms (n of them to be exact) so we will use the index, i, in a summation as shown below:

$\sum_{i=1}^{n} [a_1 + (i-1)d]$ Keep in mind that a_1 and d are constants in this expression.

We can separate this into two separate summations as shown: $\sum_{i=1}^{n} a_1 + \sum_{i=1}^{n} (i-1)d$

Expanding the first summation, $\sum_{i=1}^{n} a_1 = a_1 + a_1 + a_1 + \ldots + a_1$ such that a_1 is added to itself n times. We can simplify this expression to $a_1 n$.

In the second summation, d can be brought out in front of the summation and the difference inside can be split up as we did with the addition to get: $d\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$. Using rules from the concept **Series and Summation Notation**, $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ and $\sum_{i=1}^{n} 1 = n$. Putting it all together, we can write an expression without any summation symbols and simplify.

$$
\begin{aligned}
& a_1n+d\left[\frac{1}{2}n(n+1)-n\right] \\
& =a_1n+\frac{1}{2}dn(n+1)-dn && \text{Distribute } d \\
& =\frac{1}{2}n\left[2a_1+d(n+1)-2d\right] && \text{Factor out } \frac{1}{2}n \\
& =\frac{1}{2}n\left[2a_1+dn+d-2d\right] \\
& =\frac{1}{2}n\left[2a_1+dn-d\right] \\
& =\frac{1}{2}n\left[2a_1+d(n-1)\right] && \leftarrow \text{This version of the equation is very useful if you don't know the } n^{th} \text{ term.} \\
& =\frac{1}{2}n\left[a_1+(a_1+d(n-1))\right] \\
& =\frac{1}{2}n(a_1+a_n)
\end{aligned}
$$

Example B

Find the sum of the first 40 terms in the arithmetic series $35+31+27+23+\ldots$

Solution: For this particular series we know the first term and the common difference, so let's use the rule that doesn't require the n^{th} term: $\frac{1}{2}n\left[2a_1+d(n-1)\right]$, where $n=40, d=-4$ and $a_1=35$.

$$\frac{1}{2}(40)\left[2(35)+(-4)(40-1)\right]=20\left[70-156\right]=-1720$$

We could also find the n^{th} term and use the rule $\frac{1}{2}n(a_1+a_n)$, where $a_n=a_1+d(n-1)$.

$a_{40}=35+(-4)(40-1)=35-156=-121$, so the sum is $\frac{1}{2}(40)(35-121)=20(-86)=-1720$.

Example C

Given that in an arithmetic series $a_{21}=165$ and $a_{35}=277$, find the sum of terms 21 to 35.

Solution: This time we have the "first" and "last" terms of the series, but not the number of terms or the common difference. Since our series starts with the 21^{st} term and ends with the 35^{th} term, there are 15 terms in this series. Now we can use the rule to find the sum as shown.

$$\frac{1}{2}(15)(165+277)=3315$$

Example D

Find the sum of the arithmetic series $\sum_{i=1}^{8}(12-3i)$

Solution: From the summation notation, we know that we need to sum 8 terms. We can use the expression $12-3i$ to find the first and last terms as and the use the rule to find the sum.

First term: $12-3(1)=9$

Last term: $12-3(8)=-12$

$\sum_{i=1}^{8}(12-3i)=\frac{1}{2}(8)(9-12)=4(-3)=-12.$

We could use the calculator in this problem as well: $sum(seq(12-3x,x,1,8)) = -12$

Guided Practice

1. Find the sum of the series $87+79+71+63+\ldots+-105$.

2. Find $\sum_{i=10}^{50}(3i-90)$.

3. Find the sum of the first 30 terms in the series $1+6+11+16+\ldots$

Answers

1. $d=8$, so

$$\begin{aligned}-105 &= 87+(-8)(n-1)\\ -192 &= -8n+8\\ -200 &= -8n\\ n &= 25\end{aligned}$$

and then use the rule to find the sum is $\frac{1}{2}(25)(87-105) = -225$

2. 10^{th} term is $3(10)-90=-60$, 50^{th} term is $3(50)-90=60$ and $n=50-10+1=41$ (add 1 to *include* the 10^{th} term). The sum of the series is $\frac{1}{2}(41)(-60+60)=0$. Note that the calculator is a great option for this problem: $sum(seq(3x-90,x,10,50))=0$.

3. $d=5$, use the sum formula, $\frac{1}{2}n(2a_1+d(n-1))$, to get $\frac{1}{2}(30)\left[2(1)+5(30-1)\right] = 15\left[2+145\right] = 2205$

Problem Set

Find the sums of the following arithmetic series.

1. $-6+-1+4+\ldots+119$
2. $72+60+48+\ldots+-84$
3. $3+5+7+\ldots+99$
4. $25+21+17+\ldots+-23$
5. Find the sum of the first 25 terms of the series $215+200+185+\ldots$
6. Find the sum of the first 14 terms in the series $3+12+21+\ldots$
7. Find the sum of the first 32 terms in the series $-70+-65+-60+\ldots$
8. Find the sum of the first 200 terms in $-50+-49+-48+\ldots$

Evaluate the following summations.

9. $\sum_{i=4}^{10}(5i-22)$
10. $\sum_{i=2}^{25}(-3i+37)$
11. $\sum_{i=11}^{48}(i-20)$
12. $\sum_{i=5}^{40}(50-2i)$

Find the sum of the series bounded by the terms given. Include these terms in the sum.

13. $a_7=39$ and $a_{23}=103$
14. $a_8=1$ and $a_{30}=-43$
15. $a_4=-15$ and $a_{17}=24$

16. How many cans are needed to make a triangular arrangement of cans if the bottom row has 35 cans and successive row has one less can than the row below it?
17. Thomas gets a weekly allowance. The first week it is one dollar, the second week it is two dollars, the third week it is three dollars and so on. If Thomas puts all of his allowance in the bank, how much will he have at the end of one year?

6.7 Geometric Sequences and Series

Objective

Identify geometric sequences, find the n^{th} term rule for a geometric sequence and find the sum of a finite geometric sequence.

Review Queue

1. Find the n^{th} term rule for the arithmetic sequence with $a_8 = 1$ and $a_{32} = 13$.
2. Find $\sum_{i=1}^{15} (2i+7)$.
3. Find the sum of the series $9+7+5+3+\ldots+-37+-39$.

Geometric Sequences and Finding the N[U+1D57]ʰ Term Given the Common Ratio and the First Term

Objective

Identify a geometric sequence and its common ratio and write an n^{th} term rule given the common ratio and a term.

Guidance

A **geometric sequence** is a sequence in which the ratio between any two consecutive terms, $\frac{a_n}{a_{n-1}}$, is constant. This constant value is called the **common ratio**. Another way to think of this is that each term is multiplied by the same value, the common ratio, to get the next term.

Example A

Consider the sequence $2, 6, 18, 54, \ldots$

Is this sequence geometric? If so, what is the common difference?

Solution: If we look at each pair of successive terms and evaluate the ratios, we get $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$ which indicates that the sequence is geometric and that the common ratio is 3.

More Guidance

Now let's see if we can develop a general rule (n^{th} term) for this sequence. Since we know that each term is multiplied by 3 to get the next term, let's rewrite each term as a product and see if there is a pattern.

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= a_1(3) = 2(3) = 2(3)^1 \\
a_3 &= a_2(3) = 2(3)(3) = 2(3)^2 \\
a_4 &= a_3(3) = 2(3)(3)(3) = 2(3)^3
\end{aligned}$$

This illustrates that the general rule is $a_n = a_1(r)^{n-1}$, where r is the common ratio. This even works for the first term since $a_1 = 2(3)^0 = 2(1) = 2$.

Example B

Write a general rule for the geometric sequence $64, 32, 16, 8, \ldots$

Solution: From the general rule above we can see that we need to know two things: the first term and the common ratio to write the general rule. The first term is 64 and we can find the common ratio by dividing a pair of successive terms, $\frac{32}{64} = \frac{1}{2}$. The n^{th} term rule is thus $a_n = 64\left(\frac{1}{2}\right)^{n-1}$.

Example C

Find the n^{th} term rule for the sequence $81, 54, 36, 24, \ldots$ and hence find the 12^{th} term.

Solution: The first term here is 81 and the common ratio, r, is $\frac{54}{81} = \frac{2}{3}$. The n^{th} term rule is $a_n = 81\left(\frac{2}{3}\right)^{n-1}$. Now we can find the 12^{th} term $a_{12} = 81\left(\frac{2}{3}\right)^{12-1} = 81\left(\frac{2}{3}\right)^{11} = \frac{2048}{2187}$. Use the graphing calculator for the last step and **MATH >Frac** your answer to get the fraction. We could also use the calculator and the general rule to generate terms $seq(81(2/3)\wedge(x-1), x, 12, 12)$. Reminder: the $seq(\ )$ function can be found in the **LIST** (2^{nd} **STAT**) Menu under **OPS**. Be careful to make sure that the entire exponent is enclosed in parenthesis.

Example D

Randall deposits \$1000 in a savings account which earns 3% interest per year. What is the value of his investment after 15 years?

Solution: This is actually a geometric sequence. The value of the investment at the end of each year is the terms in the sequence which corresponds to that year. Our common difference will be 1.03 (103%) because we are maintaining the original amount (100%) and adding another 3%. Let's look at the terms:

Year	1	2	3	...	14	15
Value	$1000(1.03)$	$1000(1.03)^2$	$1000(1.03)^3$	...	$1000(1.03)^{14}$	$1000(1.03)^{15}$

The general rule for the terms in this sequence (i.e. the value of the investment after a prescribed number of years) is $a_n = 1000(1.03)^n$. Note that since we are looking at the value at the end of a certain number of years and we will get interest for all of those years, the exponent is n rather than $(n-1)$. The value at the end of five years can be calculated using $a_{15} = 1000(1.03)^{15} = 1557.967417 \approx \$1557.97$

Guided Practice

1. Identify which of the following are geometric sequences. If the sequence is geometric, find the common ratio.

a. $5, 10, 15, 20, \ldots$

b. $1, 2, 4, 8, \ldots$

c. $243, 49, 7, 1, \ldots$

2. Find the general rule and the 20^{th} term for the sequence $3, 6, 12, 24, \ldots$

3. Find the n^{th} term rule and list terms 5 thru 11 using your calculator for the sequence $-1024, 768, -432, -324, \ldots$

4. Find the value of a 10 year old car if the purchase price was \$22,000 and it depreciates at a rate of 9% per year.

Answers

1. a. arithmetic

b. geometric, $r = 2$

c. geometric, $r = \frac{1}{7}$

2. The first term is 3 and the common ratio is $r = \frac{6}{3} = 2$ so $a_n = 3(2)^{n-1}$.

The 20^{th} term is $a_{20} = 3(2)^{19} = 1,572,864$.

3. The first term is -1024 and the common ratio is $r = \frac{768}{-1024} = -\frac{3}{4}$ so $a_n = -1024\left(-\frac{3}{4}\right)^{n-1}$.

Using the calculator sequence function to find the terms and **MATH >Frac**,

$seq\ (-1024(-3/4)\hat{}(x-1), x, 5, 11) = \{-324 \quad 243 \quad -\frac{729}{4} \quad \frac{2187}{16} \quad -\frac{6561}{256} \quad \frac{19683}{256} \quad -\frac{59049}{1024}\}$

4. The first term (value of the car after 0 years) is \$22,000. The common ratio is $1 - .09$ or 0.91. The value of the car after n years can be determined by $a_n = 22,000(0.91)^n$. For 10 years we get $a_{10} = 22,000(0.91)^{10} = 8567.154599 \approx$ \$8567.

Vocabulary

Geometric Sequence
A sequence in which the ratio of any two consecutive terms is constant.

Common Ratio
The value of the constant ratio between any two consecutive terms in a geometric sequence. Also, the value by which you multiply a term in the sequence to get the next term.

Problem Set

Identify which of the following sequences are arithmetic, geometric or neither.

1. $2, 4, 6, 8, \ldots$
2. $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \ldots$
3. $1, 2, 4, 7, \ldots$
4. $24, -16, \frac{32}{3}, -\frac{64}{9}, \ldots$
5. $10, 5, 0, -5, \ldots$
6. $3, 4, 7, 11, \ldots$

Given the first term and common ratio, write the n^{th} term rule and use the calculator to generate the first five terms in each sequence.

7. $a_1 = 32$ and $r = \frac{3}{2}$
8. $a_1 = -81$ and $r = -\frac{1}{3}$
9. $a_1 = 7$ and $r = 2$
10. $a_1 = \frac{8}{125}$ and $r = -\frac{5}{2}$

Find the n^{th} term rule for each of the following geometric sequences.

11. $162, 108, 72, \ldots$
12. $-625, -375, -225, \ldots$
13. $\frac{9}{4}, -\frac{3}{2}, 1, \ldots$
14. $3, 15, 75, \ldots$
15. $5, 10, 20, \ldots$
16. $\frac{1}{2}, -2, 8, \ldots$

Use a geometric sequence to solve the following word problems.

17. Rebecca inherited some land worth \$50,000 that has increased in value by an average of 5% per year for the last 5 years. If this rate of appreciation continues, about how much will the land be worth in another 10 years?
18. A farmer buys a new tractor for \$75,000. If the tractor depreciates in value by about 6% per year, how much will it be worth after 15 years?

Finding the N[U+1D57]h Term Given the Common Ratio and any Term or Two Terms

Objective

Write an n^{th} term rule for a geometric sequence given the common ratio and any term or any two terms in the sequence.

Guidance

We will be using the general rule for the n^{th} term in a geometric sequence and the given term(s) to determine the first term and write a general rule to find any other term.

Example A

Consider the geometric sequence in which the common ratio is $-\frac{4}{5}$ and $a_5 = 1280$. Find the first term in the sequence and write the general rule for the sequence.

Solution: We will start by using the term we know, the common ratio and the general rule, $a_n = a_1 r^{n-1}$. By plugging in the values we know, we can then solve for the first term, a_1.

$$\begin{aligned}
a_5 &= a_1\left(-\frac{4}{5}\right)^4 \\
1280 &= a_1\left(-\frac{4}{5}\right)^4 \\
\frac{1280}{\left(-\frac{4}{5}\right)^4} &= a_1 \\
3125 &= a_1
\end{aligned}$$

Now, the n^{th} term rule is $a_n = 3125\left(-\frac{4}{5}\right)^{n-1}$.

Example B

Find the n^{th} term rule for a sequence in which $a_1 = 16$ and $a_7 = \frac{1}{4}$

Solution: Since $a_7 = \frac{1}{4}$ and we know the first term, we can write the equation $\frac{1}{4} = 16r^6$ and solve for the common ratio:

$$\begin{aligned}
\frac{1}{4} &= 16r^6 \\
\frac{1}{64} &= r^6 \\
\sqrt[6]{\frac{1}{64}} &= \sqrt[6]{r^6} \\
\frac{1}{2} &= r
\end{aligned}$$

The n^{th} term rule is $a_n = 16\left(\frac{1}{2}\right)^{n-1}$

Example C

Find the n^{th} term rule for the geometric sequence in which $a_5 = 8$ and $a_{10} = \frac{1}{4}$.

Solution: Using the same method at the previous example, we can solve for r and a_1. Then, write the general rule.

Equation 1: $a_5 = 8$, so $8 = a_1 r^4$, solving for a_1 we get $a_1 = \frac{8}{r^4}$.

Equation 2: $a_{10} = \frac{1}{4}$, so $\frac{1}{4} = a_1 r^9$, solving for a_1 we get $a_1 = \frac{\frac{1}{4}}{r^9}$.

$$\begin{aligned}
\frac{8}{r^4} &= \frac{\frac{1}{4}}{r^9} \\
8r^9 &= \frac{1}{4}r^4 \\
\frac{8r^9}{8r^4} &= \frac{\frac{1}{4}r^4}{8r^4} \\
r^5 &= \frac{1}{32} \\
\sqrt[5]{r^5} &= \sqrt[5]{\frac{1}{32}} \\
r &= \frac{1}{2}
\end{aligned}$$

Thus, $a_1 = \frac{8}{\left(\frac{1}{2}\right)^4} = \frac{8}{\frac{1}{16}} = \frac{8}{1} \cdot \frac{16}{1} = 128$.

The n^{th} term rule is $a_n = \left(\frac{3}{8}\right)(2)^{n-1}$.

* Note: In solving the equation above for r we divided both sides by r^4. In general it is not advisable to divide both sides of an equation by the variable because we may lose a possible solution, $r = 0$. However, in this case, $r \neq 0$ since it is the common ratio in a geometric sequence.

Guided Practice

1. Find the first term and the n^{th} term rule for the geometric sequence given that $r = -\frac{1}{2}$ and $a_6 = 3$.

2. Find the common ratio and the n^{th} term rule for the geometric sequence given that $a_1 = -\frac{16}{625}$ and $a_6 = -\frac{5}{2}$.

3. Find the n^{th} term rule for the geometric sequence in which $a_5 = 6$ and $a_{13} = 1536$.

Answers

1. Use the known quantities in the general form for the n^{th} term rule to find a_1.

$$\begin{aligned}
3 &= a_1 \left(-\frac{1}{2}\right)^5 \\
\left(-\frac{32}{1}\right) \cdot 3 &= a_1 \left(-\frac{1}{32}\right) \cdot \left(-\frac{32}{1}\right) \\
a_1 &= -96
\end{aligned}$$

Thus, $a_n = -96\left(-\frac{1}{2}\right)^{n-1}$

2. Again, substitute in the known quantities to solve for r.

$$
\begin{aligned}
-\frac{5}{2} &= \left(-\frac{16}{625}\right) r^5 \\
-\frac{5}{2}\left(-\frac{625}{16}\right) &= r^5 \\
\frac{3125}{32} &= r^5 \\
\sqrt[5]{\frac{3125}{32}} &= \sqrt[5]{r^5} \\
r &= \frac{5}{2}
\end{aligned}
$$

So, $a_n = -\frac{16}{625}\left(\frac{5}{2}\right)^{n-1}$

3. This time we have two unknowns, the first term and the common ratio. We will need to solve a system of equations using both given terms.

Equation 1: $a_5 = 6$, so $6 = a_1 r^4$, solving for a_1 we get $a_1 = \frac{6}{r^4}$.

Equation 2: $a_{13} = 1536$, so $1536 = a_1 r^{12}$, solving for a_1 we get $a_1 = \frac{1536}{r^{12}}$.

Now that both equations are solved for a_1 we can set them equal to each other and solve for r.

$$
\begin{aligned}
\frac{6}{r^4} &= \frac{1536}{r^{12}} \\
6r^{12} &= 1536r^4 \\
\frac{6r^{12}}{6r^4} &= \frac{1536r^4}{6r^4} \\
r^8 &= 256 \\
\sqrt[8]{r^8} &= \sqrt[8]{256} \\
r &= 2
\end{aligned}
$$

Now use r to find a_1: $a_1 = \frac{6}{(2^4)} = \frac{6}{16} = \frac{3}{8}$.

The n^{th} term rule is $a_n = \left(\frac{3}{8}\right)(2)^{n-1}$.

Problem Set

Use the given information to find the n^{th} term rule for each geometric sequence.

1. $r = \frac{2}{3}$ and $a_8 = \frac{256}{81}$
2. $r = -\frac{3}{4}$ and $a_5 = \frac{405}{8}$
3. $r = \frac{6}{5}$ and $a_4 = 3$
4. $r = -\frac{1}{2}$ and $a_7 = 5$
5. $a_1 = \frac{11}{8}$ and $a_7 = 88$
6. $a_1 = 24$ and $a_4 = 81$
7. $a_1 = 36$ and $a_4 = \frac{3}{4}$
8. $a_1 = \frac{343}{216}$ and $a_5 = \frac{6}{7}$
9. $a_6 = 486$ and $a_{10} = 39366$
10. $a_5 = 648$ and $a_{10} = \frac{19683}{4}$
11. $a_3 = \frac{2}{3}$ and $a_5 = \frac{3}{2}$
12. $a_5 = \frac{4}{3}$ and $a_9 = -\frac{128}{3}$

Use a geometric sequence to solve the following word problems.

13. Ricardo's parents want to have \$100,000 saved up to pay for college by the time Ricardo graduates from high school (16 years from now). If the investment vehicle they choose to invest in claims to yield 7% growth per year, how much should they invest today? Give your answer to the nearest one thousand dollars.
14. If a piece of machinery depreciates (loses value) at a rate of 6% per year, what was its initial value if it is 10 years old and worth \$50,000? Give your answer to the nearest one thousand dollars.

Finding the Sum of a Finite Geometric Series

Objective

Find the sum of a geometric series using the formula and the calculator.

Guidance

We have discussed in previous sections how to use the calculator to find the sum of any series provided we know the n^{th} term rule. For a geometric series, however, there is a specific rule that can be used to find the sum algebraically. Let's look at a finite geometric sequence and derive this rule.

Given $a_n = a_1 r^{n-1}$

The sum of the first n terms of a geometric sequence is: $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$

Now, factor out a_1 to get $a_1(1 + r^2 + r^3 + \ldots + r^{n-2} + r^{n-1})$. If we isolate what is in the parenthesis and multiply this sum by $(1-r)$ as shown below we can simplify the sum:

$$\begin{aligned}(1-r)S_n &= (1-r)(1+r+r^2+r^3+\ldots+r^{n-2}+r^{n-1})\\ &= (1+r+r^2+r^3+\ldots+r^{n-2}+r^{n-1}-r-r^2-r^3-r^4-\ldots-r^{n-1}-r^n)\\ &= (1+r+r^2+r^3+\ldots+r^{n-2}+r^{n-1}-r-r^2-r^3-r^4-\ldots-r^{n-1}-r^n)\\ &= (1-r)^n\end{aligned}$$

By multiplying the sum by $1-r$ we were able to cancel out all of the middle terms. However, we have changed the sum by a factor of $1-r$, so what we really need to do is multiply our sum by $\frac{1-r}{1-r}$, or 1.

$a_1(1 + r^2 + r^3 + \ldots + r^{n-2} + r^{n-1})\frac{1-r}{1-r} = \frac{a_1(1-r^n)}{1-r}$, which is the sum of a finite geometric series.

So, $S_n = \frac{a_1(1-r^n)}{1-r}$

Example A

Find the sum of the first ten terms of the geometric sequence $a_n = \frac{1}{32}(-2)^{n-1}$. This could also be written as, "Find $\sum_{n=1}^{10} \frac{1}{32}(-2)^{n-1}$."

Solution: Using the formula, $a_1 = \frac{1}{32}$, $r = -2$, and $n = 10$.

$$S_{10} = \frac{\frac{1}{32}(1-(-2)^{10})}{1-(-2)} = \frac{\frac{1}{32}(1-1024)}{3} = -\frac{341}{32}$$

We can also use the calculator as shown below.

$$sum(seq(1/32(-2)^{x-1}, x, 1, 10)) = -\frac{341}{32}$$

Example B

Find the first term and the n^{th} term rule for a geometric series in which the sum of the first 5 terms is 242 and the common ratio is 3.

Solution: Plug in what we know to the formula for the sum and solve for the first term:

$$\begin{aligned}
242 &= \frac{a_1(1-3^5)}{1-3} \\
242 &= \frac{a_1(-242)}{-2} \\
242 &= 121a_1 \\
a_1 &= 2
\end{aligned}$$

The first term is 2 and $a_n = 2(3)^{n-1}$.

Example C

Charlie deposits \$1000 on the first of each year into his investment account. The account grows at a rate of 8% per year. How much money is in the account on the first day on the 11^{th} year.

Solution: First, consider what is happening here on the first day of each year. On the first day of the first year, \$1000 is deposited. On the first day of the second year \$1000 is deposited and the previously deposited \$1000 earns 8% interest or grows by a factor of 1.08 (108%). On the first day of the third year another \$1000 is deposited, the previous year's deposit earns 8% interest and the original deposit earns 8% interest for two years (we multiply by 1.08^2):

$$\begin{aligned}
&\text{Sum Year 1}: 1000 \\
&\text{Sum Year 2}: 1000+1000(1.08) \\
&\text{Sum Year 3}: 1000+1000(1.08)+1000(1.08)^2 \\
&\text{Sum Year 4}: 1000+1000(1.08)+1000(1.08)^2+1000(1.08)^3 \\
&\qquad\qquad\qquad\vdots \\
&\text{Sum Year 11}: 1000+1000(1.08)+1000(1.08)^2+1000(1.08)^3+\ldots+1000(1.08)^9+1000(1.08)^{10}
\end{aligned}$$

* There are 11 terms in this series because on the first day of the 11^{th} year we make our final deposit and the original deposit earns interest for 10 years.

This series is geometric. The first term is 1000, the common ratio is 1.08 and $n = 11$. Now we can calculate the sum using the formula and determine the value of the investment account at the start of the 11^{th} year.

$$s_{11} = \frac{1000\left(1-1.08^{11}\right)}{1-1.08} = 16645.48746 \approx \$16,645.49$$

Guided Practice

1. Evaluate $\sum\limits_{n=3}^{8} 2(-3)^{n-1}$.

2. If the sum of the first seven terms in a geometric series is $\frac{215}{8}$ and $r=-\frac{1}{2}$, find the first term and the n^{th} term rule.

3. Sam deposits \$50 on the first of each month into an account which earns 0.5% interest each month. To the nearest dollar, how much is in the account right after Sam makes his last deposit on the first day of the fifth year (the 49^{th} month).

Answers

1. Since we are asked to find the sum of the 3^{rd} through 8^{th} terms, we will consider a_3 as the first term. The third term is $a_3 = 2(-3)^2 = 2(9) = 18$. Since we are starting with term three, we will be summing 6 terms, $a_3+a_4+a_5+a_6+a_7+a_8$, in total. We can use the rule for the sum of a geometric series now with $a_1 = 18$, $r=-3$ and $n=6$ to find the sum:

$$\sum_{n=3}^{8} 2(-3)^{n-1} = \frac{18(1-(-3)^6)}{1-(-3)} = -3276$$

2. We can substitute what we know into the formula for the sum of a geometric series and solve for a_1.

$$\begin{aligned}
\frac{215}{8} &= \frac{a_1\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)} \\
\frac{215}{8} &= a_1\left(\frac{43}{64}\right) \\
a_1 &= \left(\frac{64}{43}\right)\left(\frac{215}{8}\right) = 40
\end{aligned}$$

The n^{th} term rule is $a_n = 40\left(-\frac{1}{2}\right)^{n-1}$

3. The deposits that Sam make and the interest earned on each deposit generate a geometric series,

$$\begin{array}{cccccccc}
S_{49} = & 50 & +50(1.005)^1+50(1.005)^2+50(1.005)^3+\ldots+50(1.005)^{47}+ & 50(1.005)^{48}, \\
& \uparrow & & \uparrow \\
& \text{last deposit} & & \text{first deposit}
\end{array}$$

Note that the first deposit earns interest for 48 months and the final deposit does not earn any interest. Now we can find the sum using $a_1 = 50$, $r=1.005$ and $n=49$.

$$S_{49} = \frac{50(1-(1.005)^{49})}{(1-1.005)} \approx \$2768$$

Problem Set

Use the formula for the sum of a geometric series to find the sum of the first five terms in each series.

1. $a_n = 36\left(\frac{2}{3}\right)^{n-1}$
2. $a_n = 9(-2)^{n-1}$
3. $a_n = 5(-1)^{n-1}$
4. $a_n = \frac{8}{25}\left(\frac{5}{2}\right)^{n-1}$

Find the indicated sums using the formula and then check your answers with the calculator.

5. $\sum_{n=1}^{4}(-1)\left(\frac{1}{2}\right)^{n-1}$
6. $\sum_{n=2}^{8}(128)\left(\frac{1}{4}\right)^{n-1}$
7. $\sum_{n=2}^{7}\frac{125}{64}\left(\frac{4}{5}\right)^{n-1}$
8. $\sum_{n=5}^{11}\frac{1}{32}(-2)^{n-1}$

Given the sum and the common ratio, find the n^{th} term rule for the series.

9. $\sum_{n=1}^{6} a_n = -63$ and $r = -2$
10. $\sum_{n=1}^{4} a_n = 671$ and $r = \frac{5}{6}$
11. $\sum_{n=1}^{5} a_n = 122$ and $r = -3$
12. $\sum_{n=2}^{7} a_n = -\frac{63}{2}$ and $r = -\frac{1}{2}$

Solve the following word problems using the formula for the sum of a geometric series.

13. Sapna's grandparents deposit \$1200 into a college savings account on her 5^{th} birthday. They continue to make this birthday deposit each year until making the final deposit on her 18^{th} birthday. If the account earns 5% interest annually, how much is there after the final deposit?
14. Jeremy wants to have save \$10,000 in five years. If he makes annual deposits on the first of each year and the account earns 4.5% interest annually, how much should he deposit each year in order to have \$10,000 in the account after the final deposit on the first of the 6^{th} year. Round your answer to the nearest \$100.

6.8 Infinite Series

Objective

Evaluate partial sums of infinite series and determine the sum of convergent infinite geometric series.

Review Queue

Evaluate the following sums.

1. $\sum_{n=1}^{8} 4\left(\frac{1}{2}\right)^{n-1}$
2. $\sum_{n=4}^{10} 5n-1$
3. $\sum_{n=12}^{20} -2n+50$

Partial Sums

Objective

Determine partial sums of various types of series and observe the behavior of the sequences formed by these sums.

Guidance

An **infinite series** is a series with an infinite number of terms. In other words, the value of n increases without bound as shown in the series below.

$$\sum_{n=1}^{\infty} 3n+1 = 4+7+10+13+\ldots$$
$$\sum_{n=1}^{\infty} 4(2)^{n-1} = 4+8+16+32+\ldots$$
$$\sum_{n=1}^{\infty} 8\left(\frac{1}{2}\right)^{n-1} = 8+4+2+1+\frac{1}{2}+\ldots$$

These sums continue forever and can increase without bound.

Since we cannot find the sums of these series by adding all the terms, we can analyze their behavior by observing patterns within their **partial sums**. A partial sum is a sum of a finite number of terms in the series. We can look at a series of these sums to observe the behavior of the infinite sum. Each of these partial sums is denoted by S_n where n denotes the index of the last term in the sum. For example, S_6 is the sum of the first 6 terms in an infinite series.

Example A

Find the first five partial sums of $\sum_{n=1}^{\infty} 2n-1$ and make an observation about the sum of the infinite series.

Solution: The first five partial sums are S_1, S_2, S_3, S_4 and S_5. To find each of these sums we will need the first five terms of the sequence: 1, 3, 5, 7, 9. No we can find the partial sums as shown:

$$\begin{aligned}
S_1 &= a_1 = 1 \\
S_2 &= a_1 + a_2 = 1 + 3 = 4 \\
S_3 &= a_1 + a_2 + a_3 = 1 + 3 + 5 = 9 \\
S_4 &= a_1 + a_2 + a_3 + a_4 = 1 + 3 + 5 + 7 = 16 \\
S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 5 + 7 + 9 = 25
\end{aligned}$$

Notice that each sum can also be found by adding the n^{th} term to the previous sum: $S_n = S_{n-1} + a_n$.

For example: $S_5 = S_4 + a_4 = 16 + 9 = 25$

The sequence of the first five partial sums is 1, 4, 9, 16, 25. This pattern will continue and the terms will continue to grow without bound. In other words, the partial sums continue to grow and the infinite sum cannot be determined as it is infinitely large.

Example B

Find the first five partial sums of $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ and make an observation about the sum of the infinite series.

Solution: The first five terms of this sequence are: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. The partial sums are thus:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1.5 \\
S_3 &= 1.75 \\
S_4 &= 1.875 \\
S_5 &= 1.9375
\end{aligned}$$

Consider what happens with each subsequent term: We start with 1 and add $\frac{1}{2}$ putting us halfway between 1 and 2. Then we add $\frac{1}{4}$, putting us halfway between 1.5 and 2. Each time we add another term, we are cutting the distance between our current sum and 2 in half. If this pattern is continued, we will get ever closer to 2 but never actually reach two. Therefore, the sum is said to "converge to" or "approach" 2.

To support our conjecture further, we can use the calculator to find the 50^{th} partial sum: $S_{50} = 2$. Eventually, if we sum enough terms, the calculator will give us the value to which the sum approaches due to rounding.

Example C

Find the first five partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$, the "harmonic series" and make an observation about the sum of the infinite series. (You may need to find addition partial sums to see the behavior of the infinite series.)

Solution: Use the calculator to find the following sums:

$$\begin{aligned}
S_1 &= sum(seq(1/x, x, 1, 1)) = 1 \\
S_2 &= sum(seq(1/x, x, 1, 2)) = 1.5 \\
S_3 &= sum(seq(1/x, x, 1, 3)) = 1.833 \\
S_4 &= sum(seq(1/x, x, 1, 4)) = 2.083 \\
S_5 &= sum(seq(1/x, x, 1, 5)) = 2.283
\end{aligned}$$

In this series, the behavior is not quite as clear. Consider some additional partial sums:

$$S_{50} = 4.499$$
$$S_{100} = 5.187$$
$$S_{500} = 6.793$$

In this case, the partial sums don't seem to have a bound. They will continue to grow and therefore there is no finite sum.

Guided Practice

Find the first five partial sums of the infinite series below and additional partial sums if needed to determine the behavior of the infinite series. Use the calculator to find the partial sums as shown in Example C.

1. $\sum_{n=1}^{\infty} 4\left(\frac{3}{2}\right)^{n-1}$

2. $\sum_{n=1}^{\infty} 500\left(\frac{2}{3}\right)^{n-1}$

3. $\sum_{n=1}^{\infty} \frac{5}{6n}$

Answers

1. $S_1 = 4$; $S_2 = 10$; $S_3 = 19$; $S_4 = 32.5$; $S_5 = 52.75$; The partial sums are growing with increasing speed and thus the infinite series will have no bound.

2. $S_1 = 500$; $S_2 = 833.333$; $S_3 = 1055.556$; $S_4 = 1203.704$; $S_5 = 1302.469$; Here the sums seems to be growing by smaller amounts each time. Look at the sum additional partial sums to see if there is an apparent upper limit to their growth. $S_{50} = 1499.9999\ldots = 1500$; $S_{100} = 1500$. The sum is clearly approaching 1500 and thus the infinite series has a finite sum.

3. $S_1 = 0.833$; $S_2 = 1.25$; $S_3 = 1.528$; $S_4 = 1.736$; $S_5 = 1.903$; This sequence of partial sums is growing slowly, but will it approach a finite value or continue to grow? Look at additional partial sums: $S_{50} = 3.749$; $S_{100} = 4.323$; $S_{500} = 5.661$. In this case, the sums continue to grow without bound so the infinite series will have no bound.

Vocabulary

Infinite Series
A series in which the index increases without end. There are an infinite number of terms.

Partial Sum
The sum of a finite number of terms in an infinite series.

Problem Set

Find the first five partial sums and additional partial sums as needed to discuss the behavior of each infinite series. Use your calculator to find the partial sums.

1. $\sum_{n=1}^{\infty} 10(0.9)^{n-1}$
2. $\sum_{n=1}^{\infty} 8(1.03)^{n-1}$
3. $\sum_{n=1}^{\infty} \frac{1}{2}n$
4. $\sum_{n=1}^{\infty} \frac{10}{n}$

5. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$
6. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
7. $\sum_{n=1}^{\infty} 6(0.1)^{n-1}$
8. $\sum_{n=1}^{\infty} 0.01n+5$
9. $\sum_{n=1}^{\infty} 2\left(\frac{7}{8}\right)^{n-1}$
10. Which of the series above are arithmetic? Do any of them have a finite sum? Can you explain why?
11. Which of the series above are geometric? Do any of them have a finite sum? Can you explain why?

Finding the Sum of an Infinite Geometric Series

Objective

Indentify infinite geometric series for which a sum can be determined and find the sum.

Guidance

In the previous concept we explored partial sums of various infinite series and observed their behavior as n became large to see if the sum of the infinite series was finite. Now we will focus our attention on geometric series. Look at the partial sums of the infinite geometric series below:

TABLE 6.14:

Series	$\sum_{n=1}^{\infty} 3(1)^{n-1}$	$\sum_{n=1}^{\infty} 10\left(\frac{3}{4}\right)^{n-1}$	$\sum_{n=1}^{\infty} 5\left(\frac{6}{5}\right)^{n-1}$	$\sum_{n=1}^{\infty} (-2)^{n-1}$	$\sum_{n=1}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1}$
S_5	15	30.508	37.208	11	1.506
S_{10}	30	37.747	129.793	-341	1.5
S_{50}	150	40	227485.954	-3.753×10^{14}	1.5
S_{100}	300	40	2070449338	-4.226×10^{29}	1.5

From the table above, we can see that the two infinite geometric series which have a finite sum are $\sum_{n=1}^{\infty} 10\left(\frac{3}{4}\right)^{n-1}$ and $\sum_{n=1}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1}$. The two series both have a common ratio, r, such that $|r| < 1$ or $-1 < r < 1$.

Take a look at the formula for the sum of a finite geometric series: $S_n = \frac{a_1(1-r^n)}{1-r}$. What happens to r^n if we let n get very large for an r such that $|r| < 1$? Let's take a look at some examples.

TABLE 6.15:

r **values**	r^5	r^{25}	r^{50}	...	r^n **or** r^∞
$\frac{5}{6}$	0.40188	0.01048	0.00011		0
$-\frac{4}{5}$	-0.32768	-0.00378	0.00001		0
1.1	1.61051	10.83471	117.39085		keeps growing
$-\frac{1}{3}$	-0.00412	-1.18024×10^{-12}	1.39296×10^{-24}		0

This table shows that when $|r| < 1$, $r^n = 0$, for large values of n. Therefore, for the sum of an infinite geometric series in which $|r| < 1, S_\infty = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(1-0)}{1-r} = \frac{a_1}{1-r}$.

Example A

Find the sum of the geometric series if possible. $\sum_{n=1}^{\infty} 100\left(\frac{8}{9}\right)^{n-1}$.

Solution: Using the formula with $a_1 = 100$, $r = \frac{8}{9}$, we get $S_\infty = \frac{100}{1-\frac{8}{9}} = \frac{100}{\frac{1}{9}} = 900$.

Example B

Find the sum of the geometric series if possible. $\sum_{n=1}^{\infty} 9\left(\frac{4}{3}\right)^{n-1}$.

Solution: In this case, $|r| = \frac{4}{3} > 1$, therefore the sum is infinite and cannot be determined.

Example C

Find the sum of the geometric series if possible. $\sum_{n=1}^{\infty} 5(0.99)^{n-1}$

Solution: In this case $a_1 = 5$ and $r = 0.99$, so $S_\infty = \frac{5}{1-0.99} = \frac{5}{0.01} = 500$.

Guided Practice

Find the sums of the following infinite geometric series, if possible.

1. $\sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^{n-1}$
2. $\sum_{n=1}^{\infty} 4\left(\frac{7}{8}\right)^{n-1}$
3. $\sum_{n=1}^{\infty} 3(-1)^{n-1}$

Answers

1. $|r| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$ so the infinite sum does not exist.

2. $a_1 = 4$ and $r = \frac{7}{8}$ so $S_\infty = \frac{4}{1-\frac{7}{8}} = \frac{4}{\frac{1}{8}} = 32$.

3. $|r| = |-1| = 1 \geq 1$, therefore the infinite sum does not converge. If we observe the behavior of the first few partial sums we can see that they oscillate between 0 and 3.

$$S_1 = 3$$
$$S_2 = 0$$
$$S_3 = 3$$
$$S_4 = 0$$

This pattern will continue so there is no determinable sum for the infinite series.

Problem Set

Find the sums of the infinite geometric series, if possible.

1. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$
2. $\sum_{n=1}^{\infty} \frac{1}{10}\left(-\frac{4}{3}\right)^{n-1}$
3. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1}$
4. $\sum_{n=1}^{\infty} 8(1.1)^{n-1}$

5. $\sum_{n=1}^{\infty} 6(0.4)^{n-1}$
6. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{7}\right)^{n-1}$
7. $\sum_{n=1}^{\infty} \frac{5}{3}\left(\frac{1}{6}\right)^{n-1}$
8. $\sum_{n=1}^{\infty} \frac{1}{5}(1.05)^{n-1}$
9. $\sum_{n=1}^{\infty} \frac{4}{7}\left(\frac{6}{7}\right)^{n-1}$
10. $\sum_{n=1}^{\infty} 15\left(\frac{11}{12}\right)^{n-1}$
11. $\sum_{n=1}^{\infty} 0.01\left(\frac{3}{2}\right)^{n-1}$
12. $\sum_{n=1}^{\infty} 100\left(\frac{1}{5}\right)^{n-1}$

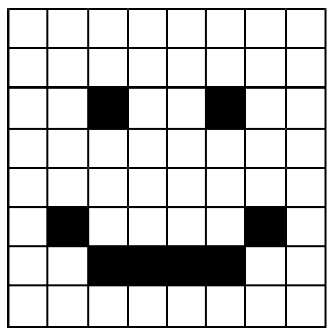
6.9 Matrices to Represent Data

Learning Objectives

Here you will learn what a matrix is and how to use one to represent data.

A **matrix** is a rectangular array of numbers representing data in a variety of forms. Computers work very heavily with matrices because operations with matrices are efficient with memory. Matrices can represent statistical data with numbers, but also graphical data with pictures.

How might you use a matrix to write the following image as something a computer could recognize and work with?



Introduction to Matrices

A matrix is a means of storing information effectively and efficiently. The rows and columns each mean something very specific and the location of a number is just as important as its value. The following are all examples of matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The entries in a matrix can be written out using brackets like [ ], but they can also be described individually using a set of 2 subscript indices i and j that stand for the row number and the column number. Alternatively, the matrix can be named with just a capital letter like A.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Square matrices have the same number of rows as columns. The **order of a matrix** describes the number of rows and the number of columns in the matrix. The following matrix is said to have order 2×3 because it has two rows and three columns. A 1×1 matrix is just a regular number.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

A **symmetric matrix** is a special type of square matrix that has reflection symmetry across the main diagonal. The identity matrix is an example of a symmetric matrix.

The **identity matrix** of order $n \times n$ has zeros everywhere except along the main diagonal where it has ones. Just like the number 1 has an important property with numbers, the identity matrix of any order has special properties as well.

$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

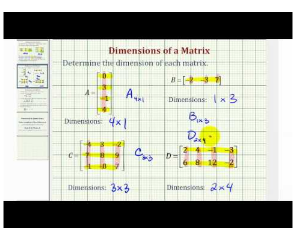
When you turn the rows of a matrix into the columns of a new matrix, the two matrices are **transpositions** of one another. The superscript T stands for transpose. Sometimes using the transpose of a matrix is more useful than using the matrix itself.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A **triangular matrix** is not a matrix in the shape of a triangle. Rather, a **lower triangular matrix** is a square matrix where every entry below the diagonal is zero. An **upper triangular matrix** is a square matrix where every entry above the diagonal is zero. The following is a lower triangular matrix. When you work with solving matrices, look for triangular matrices because they are much easier to solve.

$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

A **diagonal matrix** is both upper and lower triangular which means all the entries except those along the diagonal are zero. The identity matrix is a special case of a diagonal matrix.



MEDIA

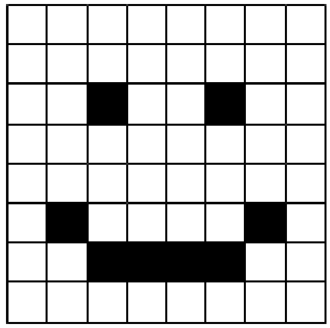
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61418

Examples

Example 1

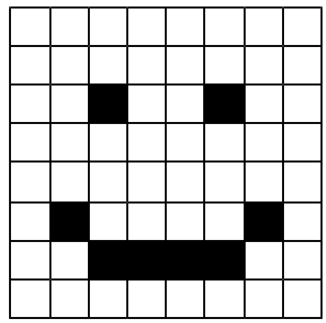
Earlier, you were asked how you might use a matrix to write the following image as something a computer could recognize and work with.



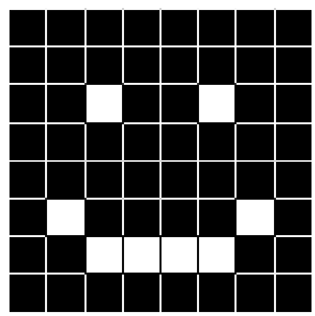
By writing every hollow square as a 0 and a blank square as a 1 a computer could read the picture:

When you use computers to manipulate images, the computer just manipulates the numbers. In this case, if you swap zeros and ones, you get the negative image.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Real photos and computer images have matrices that are much larger and include more numbers than just zero and one to account for more colors.

Example 2

Kate runs three bakeries and each bakery sells bagels and muffins. The rows represent the bakeries and the columns represent bagels (left) and muffins (right) sold. Answer the following questions about Kate's sales.

$$K = \begin{bmatrix} 144 & 192 \\ 115 & 127 \\ 27 & 34 \end{bmatrix}$$

1. What does 127 represent?

    1. 127 represents the number of muffins that Kate sold in her second location. You know this because it is in the muffin column and the second row.
2. How many muffins did Kate sell in total?
    1. The total muffins sold is equal to the sum of the right hand column. $192+127+34=353$
3. How many bagels did Kate sell in her first location?
    1. Kate sold 144 bagels at her first location.
4. Which location is doing poorly?
    1. The third location is doing much worse than the other two locations.

Example 3

Identify the order of the following matrices

$$A=\begin{bmatrix}1 & 3 & 4 & 7\end{bmatrix}, \quad B=\begin{bmatrix}21 & 45 & 1\\ 34 & 1 & 5\end{bmatrix}, \quad C=\begin{bmatrix}25 & 235\\ 562 & 562\\ 4 & 413\\ 454 & 33\\ 1 & 141\end{bmatrix}$$

A is 1×4, B is 2×3, C is 5×2. Note that $4\times 1, 3\times 2, 2\times 5$ are not the same orders and would be incorrect.

Example 4

Write out the 5×4 matrix whose entries are $a_{ij}=\frac{i+j}{j}$.

$$\begin{bmatrix}2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \frac{6}{5}\\ 3 & 2 & \frac{5}{3} & \frac{3}{2} & \frac{7}{5}\\ 4 & \frac{5}{2} & 2 & \frac{7}{4} & \frac{8}{5}\\ 5 & 3 & \frac{7}{3} & 2 & \frac{9}{5}\end{bmatrix}$$

Example 5

Create a 3×3 matrix for each of the following:

a. Diagonal Matrix

Possible answer:

$$\begin{bmatrix}4 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}$$

b. Lower Triangular

Possible answer:

$$\begin{bmatrix}4 & 1 & 1\\ 0 & 3 & 14\\ 0 & 0 & 5\end{bmatrix}$$

c. Symmetric

Possible answer:

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 14 \\ 1 & 14 & 5 \end{bmatrix}$$

d. Identity:

Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that while the identity matrix does technically work for all the parts of this problem, it does not highlight the differences between each definition.

Review

State the order of each of the following matrices:

1. $A = \begin{bmatrix} 4 & 2 & 4 & 7 \\ 5 & 2 & 1 & 0 \end{bmatrix}$

2. $B = \begin{bmatrix} 0 & 1 \\ 34 & 1 \end{bmatrix}$

3. $C = \begin{bmatrix} 2 & 62 \\ 14 & 3 \\ 4 & 3 \\ 1 & 11 \end{bmatrix}$

4. $D = \begin{bmatrix} 12 & 0 & 2 \\ 0 & 3 & 3 \\ 4 & 0 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

5. $E = \begin{bmatrix} 1 & 11 \end{bmatrix}$

6. Give an example of a 1×1 matrix.

7. Give an example of a 3×2 matrix.

8. If a symmetric matrix is also lower triangular, what type of matrix is it?

9. Write out the 2×3 matrix whose entries are $a_{ij} = i - j$.

Morgan worked for three weeks during the summer earning money on Mondays, Tuesdays, Wednesdays, Thursdays, and Fridays. The following matrix represents his earnings.

$$\begin{bmatrix} 24 & 22 & 32 \\ 25 & 28 & 30 \\ 30 & 28 & 32 \\ 10 & 15 & 19 \\ 35 & 32 & 30 \end{bmatrix}$$

10. What do the rows and columns represent?

11. How much money did Morgan make in the first week?

12. How much money did Morgan make on Tuesdays?

13. What day of the week was most profitable?

14. What day of the week was least profitable?
15. Is the following a matrix? Explain.

$$\begin{bmatrix} dogs & 0 \\ cats & 3 \\ sheep & 0 \\ ducks & 4 \end{bmatrix}$$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.3.

6.10 Matrix Operations

Learning Objectives

Here you will add, subtract and multiply matrices. As a result you will discover the algebraic properties of matrices. Algebra refers to your ability to manipulate variables and unknowns based on rules and properties. Matrix algebra is extremely similar to the algebra you already know for numbers with a few important differences. What are these differences?

Algebra with Matrices

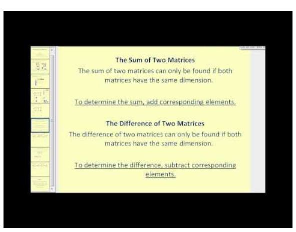
Addition and Subtraction

Two matrices of the **same order** can be added by summing the entries in the corresponding positions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

Two matrices of the **same order** can be subtracted by subtracting the entries in the corresponding positions.

$$\begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$



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Multiplication

You can find the product of matrix A and matrix B if the number of columns in matrix A matches the number of rows in matrix B. Another way to remember this is when you write the orders of matrix A and matrix B next to each other they must be connected by the same number. The resulting matrix has the number of rows from the first matrix and the number of columns from the second matrix.

$(2\times 3)\cdot(3\times 5) = (2\times 5)$

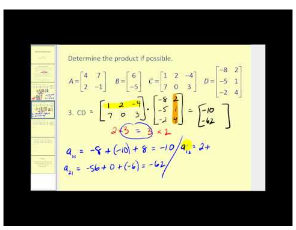
To compute the first entry of the resulting 2×5 matrix you should match the first row from the first matrix and the first column of the second matrix. The arithmetic operation to combine these numbers is identical to taking the dot product between two vectors.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & ? & ? & ? & ? \\ 21 & ? & ? & ? & ? \end{bmatrix}$$

- The entry in the first row first column of the new matrix is computed as $1 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$.
- The entry in the second row first column of the new matrix is computed as $5 \cdot 0 + 6 \cdot 2 + 9 \cdot 1 = 21$.
- The entry in the first row second column of the new matrix is computed as $1 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 = 4$
- The entry in the second row second column of the new matrix is computed as $5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 = 14$

Continue this pattern and you will find that the solution to this multiplication is:

$$C = \begin{bmatrix} 11 & 4 & 12 & 9 & 7 \\ 21 & 14 & 42 & 17 & 15 \end{bmatrix}$$



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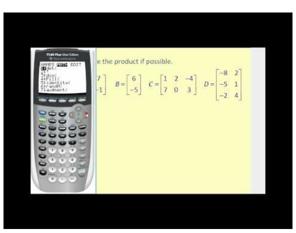

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Other Properties of Matrix Algebra

- Commutativity holds for matrix addition. This means that when matrices A and B can be added (when they have matching orders), then: $A + B = B + A$
- Commutativity does not hold in general for matrix multiplication.
- Associativity holds for both multiplication and addition. $(AB)C = A(BC), (A + B) + C = A + (B + C)$
- Distribution over addition and subtraction holds. $A(B \pm C) = AB \pm AC$

Examples

Example 1

Earlier, you were asked what the differences between matrix and regular algebra are. The main difference between matrix algebra and regular algebra with numbers is that matrices do not have the commutative property for multiplication. There are other complexities that matrices have, but many of them stem from the fact that for most matrices $AB \neq BA$.

Example 2

Show the commutative property does not hold by demonstrating $AB \neq BA$

$$A = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 2 & 0 \\ 4 & 3 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 22 & -1 \\ 5 & 9 & 3 \\ 58 & 62 & 7 \end{bmatrix}$$
$$BA = \begin{bmatrix} 9 & 12 & 20 \\ 6 & 5 & 28 \\ 3 & 2 & 32 \end{bmatrix}$$

Example 3

Compute the following matrix arithmetic: $10 \cdot (2A - 3C) \cdot B$.

$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 \\ 1 & 3 \end{bmatrix}$

When a matrix is multiplied by a scalar (such as with $2A$), multiply each entry in the matrix by the scalar.

$$2A = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix}$$
$$-3C = \begin{bmatrix} -36 & 0 \\ -3 & -9 \end{bmatrix}$$
$$2A - 3C = \begin{bmatrix} -34 & 4 \\ 5 & 1 \end{bmatrix}$$

Since the associative property holds, you can either distribute the ten or multiply by matrix B next.

$$(2A - 3C) \cdot B = \begin{bmatrix} 16 & -22 & -60 \\ 4 & 8 & 12 \end{bmatrix}$$
$$10 \cdot (2A - 3C) \cdot B = \begin{bmatrix} 160 & -220 & -600 \\ 40 & 80 & 120 \end{bmatrix}$$

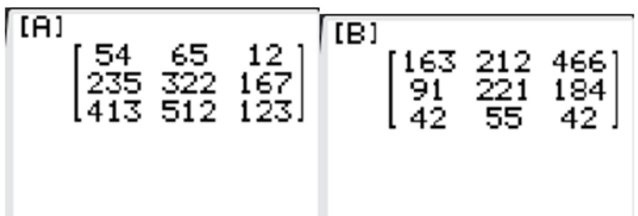
Example 4

Use your calculator to input and compute the following matrix operations.

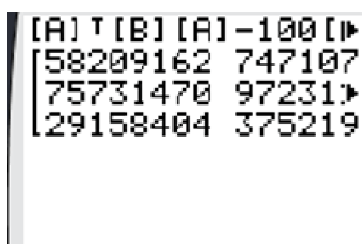
$$A = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}, \quad B = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$$

$$A^T \cdot B \cdot A - 100A$$

Most graphing calculators like the TI-84 can do operations on matrices. Find where you can enter matrices and enter the two matrices.



Then type in the appropriate operation and see the result. The TI-84 has a built in Transpose button.

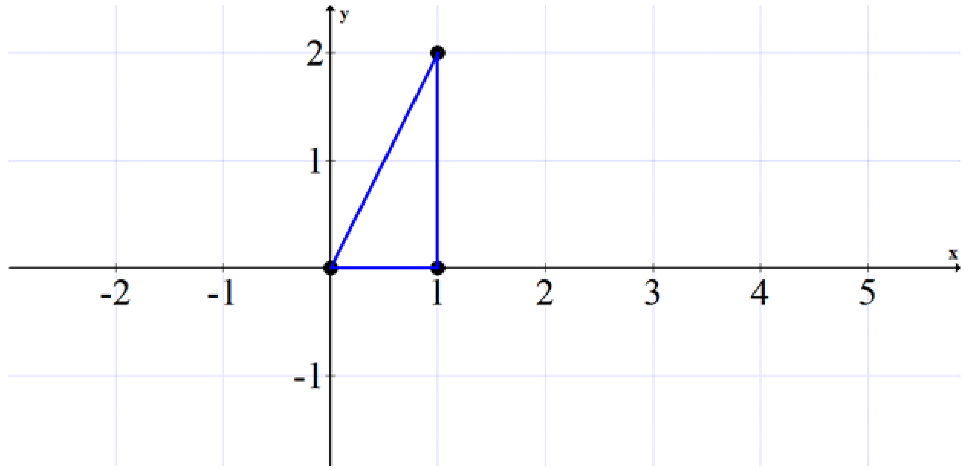


The actual numbers on this guided practice are less important than the knowledge that your calculator can perform all of the matrix algebra demonstrated in this concept. It is useful to fully know the capabilities of the tools at your disposal, but it should not replace knowing why the calculator does what it does.

Example 5

Matrix multiplication can be used as a transformation in the coordinate system. Consider the triangle with coordinates (0, 0) (1, 2) and (1, 0) the following matrix:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix}$$



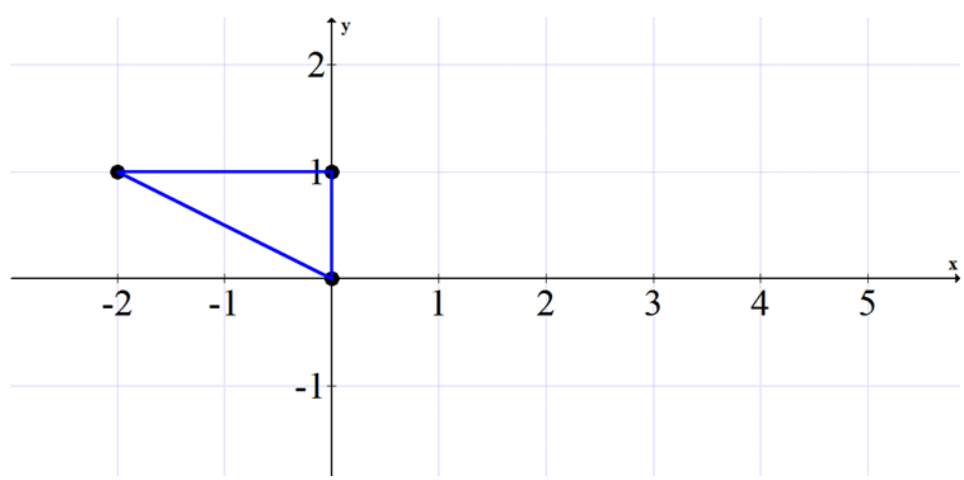
What does the new picture look like?

The matrix simplifies to become:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

When applied to each point as a transformation, a new point is produced. Note that $\begin{bmatrix} x & y \end{bmatrix}$ is a matrix representing each original point and $\begin{bmatrix} x' & y' \end{bmatrix}$ is the new point. The x' is read as "x prime" and is a common way to refer to a result after a transformation.

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$



Notice how the matrix transformation rotates graphs in a counterclockwise direction 90°.

$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} -y & x \end{bmatrix}$

The matrix transformation applied in the following order will rotate a graph clockwise 90°.

$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$

Review

Do #1-#11 without your calculator.

$A = \begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}, C = \begin{bmatrix} 14 & 6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$

1. Find AC. If not possible, explain.
2. Find BA. If not possible, explain.
3. Find CA. If not possible, explain.
4. Find $4B^T$. If not possible, explain.
5. Find $A+C$. If not possible, explain.
6. Find $D-A$. If not possible, explain.
7. Find $2(A+C-D)$. If not possible, explain.
8. Find $(A+C)B$. If not possible, explain.
9. Find $B(A+C)$. If not possible, explain.
10. Show that $A(C+D) = AC+AD$.
11. Show that $A(C-D) = AC-AD$.

Practice using your calculator for #12-#15.

$E = \begin{bmatrix} 312 & 59 & 34 \\ 342 & 156 & 189 \\ 783 & 23 & 133 \end{bmatrix}, F = \begin{bmatrix} 33 & 72 & 21 \\ 93 & 41 & 94 \\ 62 & 75 & 72 \end{bmatrix}, G = \begin{bmatrix} 11 & 735 & 67 \\ 93 & 456 & 2 \\ 94 & 34 & 0 \end{bmatrix}$

12. Find $E+F+G$.
13. Find $2E$.
14. Find $4F$.
15. Find $(E+F)G$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.4.

6.11 Row Operations and Row Echelon Forms

Learning Objectives

Here you will manipulate matrices using row operations into row echelon form and reduced row echelon form.

Applying row operations to reduce a matrix is a procedural skill that takes lots of writing, rewriting and careful arithmetic. The payoff for being able to transform a matrix into a simplified form will become clear later. For now, what does the simplified form mean for a matrix?

Row Operations and Row Echelon Forms

There are only three operations that are permitted to act on matrices. They are the exact same operations that are permitted when solving a system of equations.

1. Add a multiple of one row to another row.
2. Scale a row by multiplying through by a non-zero constant.
3. Swap two rows.

Using these three operations, your job is to simplify matrices into row echelon form. **Row echelon form** must meet three requirements.

1. The leading coefficient of each row must be a one.
2. All entries in a column below a leading one must be zero.
3. All rows that just contain zeros are at the bottom of the matrix.

Here are some examples of matrices in row echelon form:

$$\begin{bmatrix} 1 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form also has one extra stipulation compared with row echelon form.

4. Every leading coefficient of 1 must be the only non-zero element in that column.

Here are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Putting a matrix into reduced row echelon form is a result of performing Gauss-Jordan elimination. The process illustrated in this concept is named after those two mathematicians.

To put the a matrix into reduced row echelon form, use the row operations to change the matrix. Take the following matrix:

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

In each step of reducing the matrix, only one of the three row operations will be used. Specific shorthand will be introduced.

$$\begin{bmatrix}3&7\\2&5\end{bmatrix} \to \begin{matrix}\to\\ \cdot 3\end{matrix} \to \begin{bmatrix}3&7\\6&15\end{bmatrix} \to \begin{matrix}\to\\ -2\cdot I\end{matrix} \to \begin{bmatrix}3&7\\0&1\end{bmatrix}$$

Note that the $\cdot 3$ in between the first two matrices indicates that the second row is scaled by a factor of 3. The $-2\cdot I$ between the next two matrices indicates that the second row has two times the first row subtracted from it. The I is a roman numeral referring to the row number.

$$\begin{bmatrix}3&7\\0&1\end{bmatrix} \to \begin{matrix}-7II\\ \to\end{matrix} \to \begin{bmatrix}3&0\\0&1\end{bmatrix} \to \begin{matrix}\cdot\frac{1}{3}\\ \to\end{matrix} \to \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Row reducing a 2×2 matrix to become the identity matrix illustrates the fact that the rows of the original matrix are linearly independent.

Examples

Example 1

Earlier, you were asked what it means for a matrix to be simplified. There are two forms of a matrix that are most simplified. The most important is reduced row echelon form that follows the four stipulations from the guidance section. An example of a matrix in reduced row echelon form is:

$$\begin{bmatrix}1&0&0&2&43\\0&1&0&2&3\\0&0&1&98&5\end{bmatrix}$$

Example 2

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix}2&4&0\\0&3&1\\1&2&4\end{bmatrix}$$

$$\begin{bmatrix}2&4&0\\0&3&1\\1&2&4\end{bmatrix} \to \begin{matrix}\to\\ \to\\ -\frac{I}{2}\end{matrix} \to \begin{bmatrix}2&4&0\\0&3&1\\0&0&4\end{bmatrix} \to \begin{matrix}\to\\ \to\\ \div 4\end{matrix} \to \begin{bmatrix}2&4&0\\0&3&1\\0&0&1\end{bmatrix} \to \begin{matrix}\div 2\\ \div 3\\ \to\end{matrix} \to \begin{bmatrix}1&2&0\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix}$$

Note that in the preceding step, two operations were used. This is acceptable when the operations do not interfere or interact with each other.

$$\begin{bmatrix}1&2&0\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix} \to \begin{matrix}\to\\ -\frac{III}{3}\\ \to\end{matrix} \to \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix} \to \begin{matrix}-2II\\ \to\\ \to\end{matrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Again, row reducing a 3×3 matrix to become the identity matrix is just an exercise that illustrates the fact that the rows were linearly independent.

Example 3

Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix}0&4&5\\2&6&8\end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix} \begin{array}{c} \rightarrow \; II \; \rightarrow \\ \rightarrow \; I \; \rightarrow \end{array} \begin{bmatrix} 2 & 6 & 8 \\ 0 & 4 & 5 \end{bmatrix} \begin{array}{c} \rightarrow \; \div 2 \; \rightarrow \\ \rightarrow \end{array} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\begin{array}{c} \rightarrow \\ \rightarrow \; -I \; \rightarrow \end{array} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{c} \rightarrow \; -3II \; \rightarrow \\ \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example 4

Reduce the following matrix to row echelon form.

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix} \begin{array}{c} \rightarrow \; \div 3 \; \rightarrow \\ \rightarrow \; \div 2 \; \rightarrow \\ \rightarrow \end{array} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 5 & 17 \end{bmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \; -I \; \rightarrow \\ \rightarrow \; -5I \; \rightarrow \end{array} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 7 \end{bmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \; III \; \rightarrow \\ \rightarrow \; II \; \rightarrow \end{array} \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \; \div 7 \; \rightarrow \\ \rightarrow \end{array} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Example 5

Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix} \begin{array}{c} \rightarrow \; \cdot 5 \; \rightarrow \\ \rightarrow \; \cdot 3 \; \rightarrow \end{array} \begin{bmatrix} 15 & 20 & 5 & 0 \\ 15 & -3 & 0 & 3 \end{bmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \; -I \; \rightarrow \end{array} \begin{bmatrix} 15 & 20 & 5 & 0 \\ 0 & -23 & -5 & 3 \end{bmatrix}$$

$$\begin{array}{c} \rightarrow \; \cdot 23 \; \rightarrow \\ \rightarrow \; \cdot 20 \; \rightarrow \end{array} \begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix} \begin{array}{c} \rightarrow \; +II \; \rightarrow \\ \rightarrow \end{array} \begin{bmatrix} 345 & 0 & 0 & 60 \\ 0 & -460 & -115 & 60 \end{bmatrix}$$

$$\begin{array}{c} \rightarrow \; \div 345 \; \rightarrow \\ \rightarrow \; \div -460 \; \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & \frac{60}{345} \\ 0 & 1 & \frac{115}{460} & -\frac{60}{460} \end{bmatrix}$$

Notice how fractions were avoided until the final step. Adding and subtracting large numbers in a matrix is easier to handle than adding and subtracting small numbers because then you don't need to find a common denominator.

Review

1. Give an example of a matrix in row echelon form.

2. Give an example of a matrix in reduced row echelon form.

3. What are the three row operations you are allowed to perform when reducing a matrix?

4. If a square matrix reduces to the identity matrix, what does that mean about the rows of the original matrix?

Use the following matrix for 5-6.

$$A = \begin{bmatrix} -3 & -4 & -12 \\ 4 & 4 & 12 \\ -11 & -12 & -35 \end{bmatrix}$$

5. Reduce matrix A to row echelon form.

6. Reduce matrix A to reduced row echelon form. Are the rows of matrix A linearly independent?

Use the following matrix for 7-8.

$$B = \begin{bmatrix} 3 & -4 & 8 \\ 9 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

7. Reduce matrix B to row echelon form.

8. Reduce matrix B to reduced row echelon form. Are the rows of matrix B linearly independent?

Use the following matrix for 9-10.

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 3 & 6 & -3 & 1 \\ 6 & 12 & -7 & 0 \end{bmatrix}$$

9. Reduce matrix C to row echelon form.

10. Reduce matrix C to reduced row echelon form. Are the rows of matrix C linearly independent?

Use the following matrix for 11-12.

$$D = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{bmatrix}$$

11. Reduce matrix D to row echelon form.

12. Reduce matrix D to reduced row echelon form. Are the rows of matrix D linearly independent?

Use the following matrix for 13-14.

$$E = \begin{bmatrix} -5 & -6 & -12 \\ -1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$$

13. Reduce matrix E to row echelon form.

14. Reduce matrix E to reduced row echelon form. Are the rows of matrix E linearly independent?

Use the following matrix for 15-16.

$$F = \begin{bmatrix} -23 & 6 & 3 \\ 2 & -\frac{1}{2} & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

15. Reduce matrix F to row echelon form.

16. Reduce matrix F to reduced row echelon form. Are the rows of matrix F linearly independent?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.5.

6.12 Augmented Matrices

Learning Objectives

Here you will solve systems of equations using augmented matrices.

The reason why the rules for row reducing matrices are the same as the rules for eliminating coefficients when solving a system of equations is because you are essentially doing the same thing in each case. When you write and rewrite the equation every time you end up writing down lots of extra information. Matrices take care of this information by embedding it in the location of each entry. How would you use matrices to write the following system of equations?

$$\begin{aligned} 5x+y&=6 \\ x+y&=10 \end{aligned}$$

Solving Systems of Equations with Augmented Matrices

In order to represent a system as a matrix equation, first write all the equations in standard form so that the coefficients of the variables line up in columns. Then copy down just the coefficients in a **coefficient matrix array**. Next copy the variables in a **variable matrix** and the constants into a **constant matrix.**

$$\begin{aligned} x+y+z&=9 \\ x+2y+3z&=22 \\ 2x+3y+4z&=31 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

The reason why this works is because of the way matrix multiplication is defined.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x+1y+1z \\ 1x+2y+3z \\ 2z+3y+4z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

Notice how putting brackets around the two matrices on the right does very little to hide the fact that this is just a regular system of 3 equations and 3 variables.

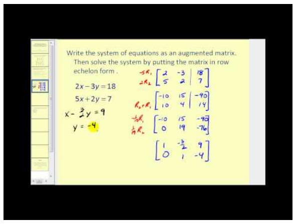
Once you have your system represented as a matrix you can solve it using an augmented matrix. An **augmented matrix** is two matrices that are joined together and operated on as if they were a single matrix. In the case of solving a system, you need to augment the coefficient matrix and the constant matrix. The vertical line indicates the separation between the coefficient matrix and the constant matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

To solve, reduce the matrix to reduced row echelon form.

$$\left[\begin{array}{ccc|c}1&1&1&9\\1&2&3&22\\2&3&4&31\end{array}\right]\begin{array}{c} \\ \rightarrow\\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ -I\\ -2I\end{array}\begin{array}{c} \\ \rightarrow\\ \rightarrow\end{array}\left[\begin{array}{ccc|c}1&1&1&9\\0&1&2&13\\0&1&2&13\end{array}\right]\begin{array}{c} \\ \\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\\ -II\end{array}\begin{array}{c} \\ \\ \rightarrow\end{array}\left[\begin{array}{ccc|c}1&1&1&9\\0&1&2&13\\0&0&0&0\end{array}\right]$$

Because the last row is all 0's, this system is dependent. Therefore, there are an infinite number of solutions.



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Examples

Example 1

Earlier, you were asked how to write a system of equations as a matrix equation. If you were to write the system as a matrix equation, you could write:

$$5x+y=6$$
$$x+y=10$$
$$\begin{bmatrix}5&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}6\\10\end{bmatrix}$$

Example 2

Solve the following system using an augmented matrix.

$$x+y+z=6$$
$$x-y-z=-4$$
$$x+2y+3z=14$$

$$\left[\begin{array}{ccc|c}1&1&1&6\\1&-1&-1&-4\\1&2&3&14\end{array}\right]\begin{array}{c} \\ \rightarrow\\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ -I\\ -I\end{array}\begin{array}{c} \\ \\ \rightarrow\end{array}\left[\begin{array}{ccc|c}1&1&1&6\\0&-2&-2&-10\\0&1&2&8\end{array}\right]\begin{array}{c}\rightarrow\\ \rightarrow\\ \end{array}\begin{array}{c}-III\\ +3III\\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\\ \end{array}$$

$$\left[\begin{array}{ccc|c}1&0&-1&-2\\0&1&4&14\\0&1&2&8\end{array}\right]\begin{array}{c} \\ \\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\\ -II\end{array}\begin{array}{c} \\ \\ \rightarrow\end{array}\left[\begin{array}{ccc|c}1&0&-1&-2\\0&1&4&14\\0&0&-2&-6\end{array}\right]\begin{array}{c} \\ \\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\\ \div-2\end{array}\begin{array}{c} \\ \\ \rightarrow\end{array}$$

$$\left[\begin{array}{ccc|c}1&0&-1&-2\\0&1&4&14\\0&0&1&3\end{array}\right]\begin{array}{c}\rightarrow\\ \rightarrow\\ \end{array}\begin{array}{c}+III\\ -4III\\ \rightarrow\end{array}\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right]$$

Every matrix can be interpreted as its own linear system. The final augmented matrix can be interpreted as:

$$\begin{aligned}1x+0y+0z&=1\\0x+1y+0z&=2\\0x+0y+1z&=3\end{aligned}$$

Which means $x=1,y=2,z=3$.

Example 3

Solve the following system using augmented Matrices.

$$\begin{aligned}w+x+z&=11\\w+x&=9\\x+y&=7\\y+z&=5\end{aligned}$$

While substitution would work in this problem, the idea is to demonstrate how augmented matrices will work even with larger matrices.

$$\left[\begin{array}{cccc|c}1&1&0&1&11\\1&1&0&0&9\\0&1&1&0&7\\0&0&1&1&5\end{array}\right]\begin{array}{ccc}&\rightarrow&\\\rightarrow&IV&\\\rightarrow&II&\rightarrow\\\rightarrow&III&\rightarrow\end{array}\left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\1&1&0&0&9\end{array}\right]\begin{array}{ccc}&\rightarrow&\\&\rightarrow&\\&\rightarrow&\\\rightarrow&-I&\rightarrow\end{array}$$

$$\left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\0&0&0&-1&-2\end{array}\right]\begin{array}{ccc}&\rightarrow&\\&\rightarrow&\\&\rightarrow&\\\rightarrow&\cdot(-1)&\rightarrow\end{array}\left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\0&0&0&1&2\end{array}\right]\begin{array}{ccc}\rightarrow&-IV&\rightarrow\\&\rightarrow&\\\rightarrow&-IV&\rightarrow\\&\rightarrow&\end{array}$$

$$\left[\begin{array}{cccc|c}1&1&0&0&9\\0&1&1&0&7\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]\begin{array}{ccc}&\rightarrow&\\\rightarrow&-III&\rightarrow\\&\rightarrow&\\&\rightarrow&\end{array}\left[\begin{array}{cccc|c}1&1&0&0&9\\0&1&0&0&4\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]\begin{array}{ccc}\rightarrow&-II&\rightarrow\\&\rightarrow&\\&\rightarrow&\\&\rightarrow&\end{array}\left[\begin{array}{cccc|c}1&0&0&0&5\\0&1&0&0&4\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]$$

$w=5,x=4,y=3,z=2$

Example 4

Use an augmented matrix to solve the following system.

$$\begin{aligned}3x+y&=-15\\x+2y&=15\end{aligned}$$

The row reduction steps are not shown, only the initial and final augmented matrices.

$$\left[\begin{array}{cc|c}3&1&-15\\1&2&15\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1&0&-9\\0&1&12\end{array}\right]$$

Example 5

Use an augmented matrix to solve the following system.

$$\begin{aligned}-a+b-c&=0\\2a-2b-3c&=25\\3a-4b+3c&=2\end{aligned}$$

The row reduction steps are not shown, only the initial and final augmented matrices.

$$\left[\begin{array}{ccc|c}-1&1&-1&0\\2&-2&-3&25\\3&-4&3&2\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-2\\0&0&1&-5\end{array}\right]$$

Review

Solve the following systems of equations using augmented matrices. If one solution does not exist, explain why not.

1.

$$\begin{aligned}4x-2y&=-20\\x-3y&=-15\end{aligned}$$

2.

$$\begin{aligned}3x+5y&=33\\-x-2y&=-13\end{aligned}$$

3.

$$\begin{aligned}x+4y&=11\\3x+12y&=33\end{aligned}$$

4.

$$\begin{aligned}-3x+y&=-7\\-x+4y&=5\end{aligned}$$

5.

$$\begin{aligned}3x+y&=6\\-6x-2y&=10\end{aligned}$$

6.

$$\begin{aligned} 2x-y+z&=4\\ 4x+7y-z&=38\\ -x+3y+2z&=23 \end{aligned}$$

7.

$$\begin{aligned} 4x+y-z&=-16\\ -3x+4y+z&=18\\ x+y-3z&=-17 \end{aligned}$$

8.

$$\begin{aligned} 3x+2y-3z&=7\\ -x+5y+2z&=29\\ x+2y+z&=15 \end{aligned}$$

9.

$$\begin{aligned} 2x+y-2z&=4\\ -4x-2y+4z&=-8\\ 3x+y-z&=5 \end{aligned}$$

10.

$$\begin{aligned} -x+3y+z&=11\\ 3x+y+2z&=27\\ 5x-y-z&=5 \end{aligned}$$

11.

$$\begin{aligned} 3x+2y+4z&=21\\ -2x+3y+z&=-11\\ x+2y-3z&=-3 \end{aligned}$$

12.

$$\begin{aligned} -x+2y-6z&=4\\ 8x+5y+3z&=-8\\ 2x-4y+12z&=5 \end{aligned}$$

13.

$$
\begin{aligned}
3x+5y+8z &= 37 \\
-6x+3y+z &= 42 \\
x+3y-2z &= 5
\end{aligned}
$$

14.

$$
\begin{aligned}
4x+y-6z &= -38 \\
2x+7y+8z &= 108 \\
-3x+2y-3z &= -15
\end{aligned}
$$

15.

$$
\begin{aligned}
6x+3y-2z &= -22 \\
-4x-2y+4z &= 28 \\
3x+3y+2z &= 7
\end{aligned}
$$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.6.

6.13 Determinant of Matrices

Learning Objectives

Here you will find the determinants of 2×2 and higher order matrices.

A **determinant** is a number computed from the entries in a square matrix. It has many properties and interpretations that you will explore in linear algebra. This concept is focused on the procedure of calculating determinants. Once you know how to calculate the determinant of a 2×2 matrix, then you will be able to calculate the determinant of a 3×3 matrix. Once you know how to calculate the determinant of a 3×3 matrix you can calculate the determinant of a 4×4 and so on.

A logical question about determinants is where does the procedure come from? Why are determinants defined in the way that they are?

The Determinant

The determinant of a matrix A is written as $|A|$. For a 2×2 matrix A, the value is calculated as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If you substitute numbers for the letters and try to calculate $\det A$ for $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$, you get:

$\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 1 = 15 - 2 = 13$

Notice how the diagonals are multiplied and then subtracted.

The determinant of a 3×3 matrix is more involved.

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Usually you will start by looking at the top row, although any row or column will work. Then use the checkerboard pattern for signs (shown below) and create smaller 2×2 matrices.

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

The smaller 2×2 matrices are the entries that remain when the row and column of the coefficient you are working with are ignored.

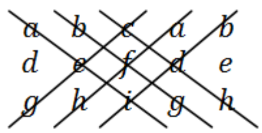
$\det B = |B| = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Next take the determinant of the smaller 2×2 matrices and you get a long string of computations.

$$\begin{aligned}&= +a(ei-fh)-b(di-fg)+c(dh-eg)\\&= aei-afh-bdi+bfg+cdh-ceg\\&= aei+bfg+cdh-ceg-afh-bdi\end{aligned}$$

Most people do not remember this sequence. A French mathematician named Sarrus demonstrated a great device to memorize the computation of the determinant for 3×3 matrices. The first step is simply to copy the first two columns to the right of the matrix. Then draw three diagonal lines going down and to the right.

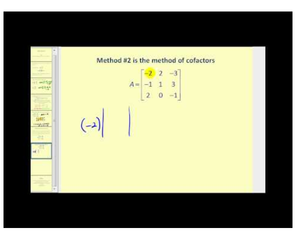
$B=\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$



Notice that they correspond exactly to the three positive terms of the determinant demonstrated above. Next draw three diagonals going up and to the right. These diagonals correspond exactly to the three negative terms.

$\det B = aei+bfg+cdh-ceg-afh-bdi$

Sarrus's Rule does not work for the determinants of matrices that are not of order 3×3.



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Examples

Example 1

Earlier, you were asked where the procedure for finding the determinants came from. Determinants for 2×2 matrices are defined the way they are because of the general solution to a system of 2 variables and 2 equations.

$$\begin{aligned}ax+by&=e\\cx+dy&=f\end{aligned}$$

To eliminate the x, scale the first equation by c and the second equation by a.

$$\begin{aligned} acx + bcy &= ec \\ acx + ady &= af \end{aligned}$$

Subtract the second equation from the first and solve for y.

$$\begin{aligned} ady - bcy &= af - ec \\ y(ad - bc) &= af - ec \\ y &= \frac{af - ec}{ad - bc} \end{aligned}$$

When you solve for x you also get $ad - bc$ in the denominator of the general solution. This pattern led people to start using this strategy in solving systems of equations. The determinant is defined in this way so it will always be the denominator of the general solution of either variable.

Example 2

Find the determinant of the following matrix.

$C = \begin{bmatrix} -4 & 12 \\ 1 & -3 \end{bmatrix}$

$\det C = \begin{vmatrix} -4 & 12 \\ 1 & -3 \end{vmatrix} = 12 - 12 = 0$

Example 3

Find $\det B$ for $B = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix}$

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{vmatrix} &= 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 3(0 \cdot 5 - 2 \cdot 1) - 2(5 \cdot 5 - 2 \cdot 2) + 1(5 \cdot 1 - 2 \cdot 0) \\ &= -6 - 42 + 5 = -43 \end{aligned}$$

Example 4

Find the determinant of B from example B using Sarrus's Rule.

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ 5 & 0 & 2 & 5 & 0 \\ 2 & 1 & 5 & 2 & 1 \end{matrix}$$

det $B = 0 + 8 + 5 - 0 - 6 - 50 = -43$

As you can see, Sarrus's Rule is efficient and much of the calculations can be done mentally. Additionally, zero values make much of the multiplication easier.

Example 5

Find the determinant of the following 4×4 matrix by carefully choosing the row or column to work with.

$$E=\begin{bmatrix}4 & 5 & 0 & 2\\ -1 & -3 & 0 & 3\\ 4 & 8 & 1 & 5\\ -3 & 2 & 0 & 9\end{bmatrix}$$

Notice that the third column is made up with zeros and a one. Choose this column to make up the coefficients because then instead of having to evaluate the determinant of four individual 3×3 matrices, you only need to do one.

$$\begin{aligned}\begin{vmatrix}4 & 5 & 0 & 2\\ -1 & -3 & 0 & 3\\ 4 & 8 & 1 & 5\\ -3 & 2 & 0 & 9\end{vmatrix} &= 0\cdot\begin{vmatrix}-1 & -3 & 3\\ 4 & 8 & 5\\ -3 & 2 & 9\end{vmatrix} - 0\cdot\begin{vmatrix}4 & 5 & 2\\ 4 & 8 & 5\\ -3 & 2 & 9\end{vmatrix} + 1\cdot\begin{vmatrix}4 & 5 & 2\\ -1 & -3 & 3\\ -3 & 2 & 9\end{vmatrix} - 0\cdot\begin{vmatrix}4 & 5 & 2\\ -1 & -3 & 3\\ 4 & 8 & 5\end{vmatrix}\\ &= \begin{vmatrix}4 & 5 & 2\\ -1 & -3 & 3\\ -3 & 2 & 9\end{vmatrix}\\ &= 4\cdot(-3)\cdot 9+5\cdot 3\cdot(-3)+2\cdot(-1)\cdot 2-18-24-(-45)\\ &= -154\end{aligned}$$

Review

Find the determinants of each of the following matrices.

1. $\begin{bmatrix}4 & 5\\ 2 & 3\end{bmatrix}$
2. $\begin{bmatrix}-3 & 6\\ 2 & 5\end{bmatrix}$
3. $\begin{bmatrix}-1 & 2\\ 2 & 0\end{bmatrix}$
4. $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
5. $\begin{bmatrix}6 & 5\\ 2 & -2\end{bmatrix}$
6. $\begin{bmatrix}1 & 2\\ 6 & 3\end{bmatrix}$
7. $\begin{bmatrix}-1 & 3 & -4\\ 4 & 2 & 1\\ 1 & 2 & 5\end{bmatrix}$
8. $\begin{bmatrix}4 & 5 & 8\\ 9 & 0 & 1\\ 0 & 3 & -2\end{bmatrix}$
9. $\begin{bmatrix}0 & 7 & -1\\ 2 & -3 & 1\\ 6 & 8 & 0\end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 & 4 & 6 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ -6 & 2 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 0 & 0 & 1 \\ 2 & 1 & 8 & 3 \\ 9 & 3 & 2 & 6 \\ -4 & 2 & 5 & 1 \end{bmatrix}$

15. Can you find the determinant for any matrix? Explain.

16. The following matrix has a determinant of zero: $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. If the determinant of a matrix is zero, what does that say about the rows of the matrix?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.7.

6.14 Cramer's Rule

Learning Objectives

Here you will solve systems of equations using Cramer's Rule.

A system of equations can be represented and solved in general using matrices and determinants. This method can be significantly more efficient than eliminating variables in equations. What does it mean for a solution method to be more efficient? Is Cramer's Rule the most efficient means of solving a system of equations?

Using Cramer's Rule

The determinant is defined in a seemingly arbitrary way; however, when you look at the general solution for a 2×2 matrix, the reasoning why it is defined this way is apparent.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

When you solve the system above for y and x, you get the following:

$$\begin{aligned} y &= \frac{af - ce}{ad - bc} \\ x &= \frac{bf - de}{ad - bc} \end{aligned}$$

Note that the system can be represented by the matrix and the solutions can be written as ratios of two determinants. The determinant in the denominator is of the coefficient matrix. **Cramer's Rule** states that for two equations, the numerator of the x solution is the determinant of the new matrix whose columns are made up of the y coefficients and the solution coefficients. The numerator of the y solution is the determinant of the new matrix made up of the x coefficients and the solution coefficients.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

This is a fantastic improvement over solving systems using substitution or elimination. Cramer's Rule also works with larger order matrices. For a system of 3 variables and 3 equations the reasoning is identical.

$$\begin{aligned} ax+by+cz&=j\\ dx+ey+fz&=k\\ gx+hy+iz&=l \end{aligned}$$

The system can be represented as a matrix.

$$\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} j\\ k\\ l \end{bmatrix}$$

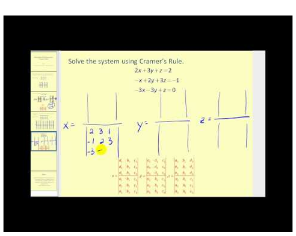
The three solutions can be represented as a ratio of determinants.

$$x=\frac{\begin{vmatrix} j & b & c\\ k & e & f\\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i \end{vmatrix}}$$

$$y=\frac{\begin{vmatrix} a & j & c\\ d & k & f\\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i \end{vmatrix}}$$

$$z=\frac{\begin{vmatrix} a & b & j\\ d & e & k\\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i \end{vmatrix}}$$

Remember that evaluating the determinants of 3×3 matrices using Sarrus's rule is very efficient.



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Examples

Example 1

Earlier, you were asked about efficient solutions. You have seen that using traditional row reduction to solve a system of equations can take a while and use up a lot of paper. Efficiency partly means requiring less time and space. If this was all that efficiency meant then it would not make sense to solve systems of two equations with two unknowns using matrices because the solution could be found more quickly using substitution. However, the other part of efficiency is minimizing the number of decisions that have to be made. A computer is very good at adding, subtracting and multiplying numbers, but not very good at deciding whether eliminating x or eliminating y would be better. This is why a definite algorithm using matrices and Cramer's Rule is more efficient.

Example 2

Represent the following system of equations as a matrix equation and solve using Cramer's Rule.

$$\begin{aligned} y-13&=-3x \\ x&=19-4y \end{aligned}$$

First write each equation in standard form.

$$\begin{aligned} 3x+y&=13 \\ x+4y&=19 \end{aligned}$$

Then write as a coefficient matrix times a variable matrix equal to a solution matrix.

$\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix}$

$$x=\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}=\frac{\begin{vmatrix} 13 & 1 \\ 19 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}}=\frac{13\cdot4-19\cdot1}{3\cdot4-1\cdot1}=\frac{33}{11}=3$$

$$y=\frac{\begin{vmatrix} 3 & 13 \\ 1 & 19 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}}=\frac{3\cdot19-13}{11}=\frac{44}{11}=4$$

Example 3

What is y equal to in the following system?

$$\begin{aligned} x+2y-z&=0 \\ 7x-0y+z&=14 \\ 0x+y+z&=10 \end{aligned}$$

If you attempted to solve this using elimination, it would take over a page of writing and rewriting to solve. Cramer's Rule speeds up the solving process.

$$\begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 10 \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 7 & 14 & 1 \\ 0 & 10 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{14+0+(-70)-0-10-0}{0+0+(-7)-0-1-14} = \frac{-66}{-22} = 3$$

Example 4

Solve the following system using Cramer's Rule.

$$\begin{aligned} 5x+12y &= 72 \\ 18x-12y &= 108 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 72 & 12 \\ 108 & -12 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{72\cdot(-12)-12\cdot 108}{5\cdot(-12)-12\cdot 18} = \frac{-2160}{276} = \frac{180}{23}$$

$$y = \frac{\begin{vmatrix} 5 & 72 \\ 18 & 108 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{5\cdot 108-72\cdot 18}{276} = \frac{-756}{276} = -\frac{63}{23}$$

Example 5

What is the value of z in the following system?

$$\begin{aligned} 3x+2y+z &= 7 \\ 4x+0y+z &= 6 \\ 6x-y+0z &= 5 \end{aligned}$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 7 \\ 4 & 0 & 6 \\ 6 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 6 & -1 & 0 \end{vmatrix}} = \frac{0+2\cdot 6\cdot 6+7\cdot 4\cdot(-1)-0-(-1)\cdot 6\cdot 3-5\cdot 4\cdot 2}{0+2\cdot 1\cdot 6+1\cdot 4\cdot(-1)-0-(-1)\cdot 1\cdot 3-0} = \frac{22}{11} = 2$$

Review

Solve the following systems of equations using Cramer's Rule. If one solution does not exist, explain.

1.

$$\begin{aligned} 4x - 2y &= -20 \\ x - 3y &= -15 \end{aligned}$$

2.

$$\begin{aligned} 3x + 5y &= 33 \\ -x - 2y &= -13 \end{aligned}$$

3.

$$\begin{aligned} x + 4y &= 11 \\ 3x + 12y &= 33 \end{aligned}$$

4.

$$\begin{aligned} -3x + y &= -7 \\ -x + 4y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 3x + y &= 6 \\ -6x - 2y &= 10 \end{aligned}$$

6. Use Cramer's Rule to solve for x in the following system:

$$\begin{aligned} 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned}$$

7. Use Cramer's Rule to solve for y in the following system:

$$\begin{aligned} 4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17 \end{aligned}$$

8. Use Cramer's Rule to solve for z in the following system:

$$\begin{aligned}3x+2y-3z&=7\\-x+5y+2z&=29\\x+2y+z&=15\end{aligned}$$

9. Use Cramer's Rule to solve for x in the following system:

$$\begin{aligned}2x+y-2z&=-5\\-4x-2y+3z&=2\\3x+y-z&=3\end{aligned}$$

10. Use Cramer's Rule to solve for y in the following system:

$$\begin{aligned}-x+3y+z&=11\\3x+y+2z&=27\\5x-y-z&=5\end{aligned}$$

11. Use Cramer's Rule to solve for z in the following system:

$$\begin{aligned}3x+2y+4z&=21\\-2x+3y+z&=-11\\x+2y-3z&=-3\end{aligned}$$

Solve the following systems of equations using Cramer's Rule. Practice using your calculator to help with at least one problem. If one solution does not exist, explain.

12.

$$\begin{aligned}-x+2y-6z&=4\\8x+5y+3z&=-8\\2x-4y+12z&=5\end{aligned}$$

13.

$$\begin{aligned}3x+5y+8z&=37\\-6x+3y+z&=42\\x+3y-2z&=5\end{aligned}$$

14.

$$\begin{aligned} 4x+y-6z &= -38 \\ 2x+7y+8z &= 108 \\ -3x+2y-3z &= -15 \end{aligned}$$

15.

$$\begin{aligned} 6x+3y-2z &= -22 \\ -4x-2y+4z &= 28 \\ 3x+3y+2z &= 7 \end{aligned}$$

16. When using Cramer's Rule to solve a system of equations you will occasionally find that the determinant of the coefficient matrix is zero. When this happens, how can you tell whether your system has no solution or infinite solutions?

Review (Answer)

To see the Review answers, open this PDF file and look for section 8.8.

6.15 Inverse Matrices

Learning Objectives

Here you will learn how to find the inverse of a matrix and how to solve a system of equations using an inverse matrix.

Two numbers are multiplicative inverses if their product is 1. Every number besides the number 0 has a multiplicative inverse. For matrices, two matrices are inverses of each other if they multiply to be the identity matrix.

What kinds of matrices do not have inverses?

Inverses of Matrices

Multiplicative inverses are two numbers or matrices whose product is one or the identity matrix. Consider a matrix A that has inverse A^{-1}. How do you find matrix A^{-1} if you just have matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = ?$$

The answer is that you augment matrix A with the identity matrix and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \rightarrow \\ \\ \end{array} \begin{array}{c} \rightarrow \\ -I \\ \rightarrow \end{array} \begin{array}{c} \\ \rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ +II \end{array} \begin{array}{c} \\ \\ \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \div(-2) \\ \div(-3) \end{array} \begin{array}{c} \\ \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \end{array} \begin{array}{c} -3III \\ -III \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{c} \rightarrow \\ \\ \\ \end{array} \begin{array}{c} -2II \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

The matrix on the right is the inverse matrix A^{-1}.

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Fractions are usually unavoidable when computing inverses.

One reason why inverses are so powerful is because they allow you to solve systems of equations with the same logic as you would solve a single linear equation. Consider the following system based on the coefficients of matrix A from above.

$$\begin{aligned} x+2y+3z &= 96 \\ x+0y+z &= 36 \\ 0x+2y-z &= -12 \end{aligned}$$

By writing this system as a matrix equation you get:

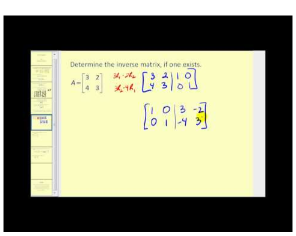
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$
$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

If this were a normal linear equation where you had a constant times the variable equals a constant, you would multiply both sides by the multiplicative inverse of the coefficient. Do the same in this case.

$$A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

All that is left is for you to substitute in and to perform the matrix multiplication to get the solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \cdot 96 + \frac{4}{3} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{6} \cdot 96 - \frac{1}{6} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{3} \cdot 96 - \frac{1}{3} \cdot 36 - \frac{1}{3} \cdot (-12) \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix}$$



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Examples

Example 1

Earlier, you were asked what types of matrices do not have inverses. Non-square matrices do not generally have inverses. Square matrices that have determinants equal to zero do not have inverses.

Example 2

Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 6 \\ 4 & 24 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 4 & 24 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{c} \rightarrow \\ -4I \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{array}\right]$$

This matrix is not invertible because its rows are not linearly independent. To test to see if a square matrix is invertible, check whether or not the determinant is zero. If the determinant is zero then the matrix is not invertible because the rows are not linearly independent.

Example 3

Confirm matrix A and A^{-1} are inverses by computing $A^{-1} \cdot A$ and $A \cdot A^{-1}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$A^{-1} \cdot A = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} =$$

$$a_{11} = -\frac{1}{3} \cdot 1 + \frac{4}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 1$$

$$a_{22} = \frac{1}{6} \cdot 2 - \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 2 = 1$$

$$a_{33} = \frac{1}{3} \cdot 3 - \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = 1$$

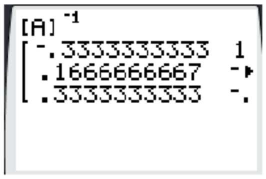
Note that the rest of the entries turn out to be zero. This is left for you to confirm.

Example 4

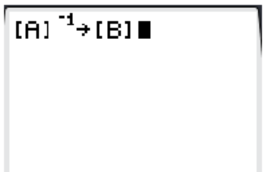
Use a calculator to compute A^{-1}, compute $A^{-1} \cdot A$, compute $A \cdot A^{-1}$ and compute $A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$.

Start by entering just matrix A into the calculator.

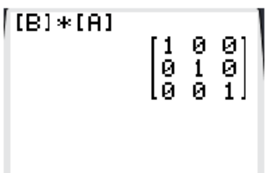
To compute matrix A^{-1} use the inverse button programmed into the calculator. Do not try to raise the matrix to the negative one exponent. This will not work.



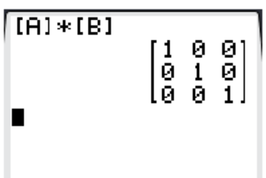
Note that the calculator may return decimal versions of the fractions and will not show the entire matrix on its limited display. You will have to scroll to the right to confirm that A^{-1} matches what you have already found. Once you have found A^{-1} go ahead and store it as matrix B so you do not need to type in the entries.



$A^{-1} \cdot A = B \cdot A$

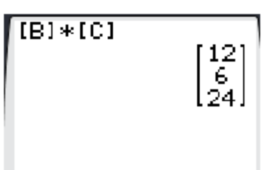


$A \cdot A^{-1} = A \cdot B$



$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot C$

You need to create matrix $C = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$



Being able to effectively use a calculator should improve your understanding of matrices and allow you to check all the work you do by hand.

Example 5

The identity matrix happens to be its own inverse. Find another matrix that is its own inverse.

Helmert came up with a very clever matrix that happens to be its own inverse. Here are the 2×2 and the 3×3 versions.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

Review

Find the inverse of each of the following matrices, if possible. Make sure to do some by hand and some with your calculator.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. Show that Helmert's 2×2 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

14. Show that Helmert's 3×3 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$.

15. Non-square matrices sometimes have left inverses, where $A^{-1} \cdot A = I$, or right inverses, where $A \cdot A^{-1} = I$. Why can't non-square matrices have "regular" inverses?

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.9.

6.16 Partial Fractions

Learning Objectives

Here you will apply what you know about systems and matrices to decompose rational expressions into the sum of several partial fractions.

When given a rational expression like $\frac{4x-9}{x^2-3x}$ it is very helpful in calculus to be able to write it as the sum of two simpler fractions like $\frac{3}{x}+\frac{1}{x-3}$. The challenging part is trying to get from the initial rational expression to the simpler fractions.

You may know how to add fractions and go from two or more separate fractions to a single fraction, but how do you go the other way around?

Partial Fraction Decomposition

Partial fraction decomposition is a procedure that reverses adding fractions with unlike denominators. The most challenging part is coming up with the denominators of each individual partial fraction. See if you can spot the pattern.

$\frac{6x-1}{x^2(x-1)(x^2+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{Dx+E}{x^2+2}$

In this example each individual factor of the denominator must be represented. Linear factors that are raised to a power greater than one must have each successive power included as a separate denominator. Quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of x. Take a look at the examples to see partial fraction decomposition put into practice.

Examples

Example 1

Earlier, you were asked how to go from one fraction to multiple simpler fractions. To decompose the rational expression into the sum of two simpler fractions you need to use partial fraction decomposition.

$$\begin{aligned}
\frac{4x-9}{x^2-3x} &= \frac{A}{x}+\frac{B}{x-3}\\
4x-9 &= A(x-3)+Bx\\
4x-9 &= Ax-3A+Bx
\end{aligned}$$

Notice that the constant term -9 must be equal to the constant term $-3A$ and that the terms with x must be equal as well.

$$\begin{aligned}
-9 &= -3A\\
4 &= A+B
\end{aligned}$$

Solving this system yields:

$A = 3, \quad B = 1$

Therefore,

$\frac{4x-9}{x^2-3x} = \frac{3}{x} + \frac{1}{x-3}$

Example 2

Use partial fractions to decompose the following rational expression.

$\frac{7x^2+x+6}{x^3+3x}$

First factor the denominator and identify the denominators of the partial fractions.

$\frac{7x^2+x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$

When the fractions are eliminated by multiplying through by the LCD the equation becomes:

$$7x^2 + x + 6 = A(x^2+3) + x(Bx+C)$$
$$7x^2 + x + 6 = Ax^2 + 3A + Bx^2 + Cx$$

Notice the squared term, linear term and constant term form a system of three equations with three variables.

$$A + B = 7$$
$$C = 1$$
$$3A = 6$$

In this case it is easy to see that $A = 2, B = 5, C = 1$. Often, the resulting system of equations is more complex and would benefit from your knowledge of solving systems using matrices.

$\frac{7x^2+x+6}{x(x^2+3)} = \frac{2}{x} + \frac{5x+1}{x^2+3}$

Example 3

Decompose the following rational expression.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}$

First identify the denominators of the partial fractions.

$$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x} + \frac{E}{x^2}$$

When the entire fraction is multiplied through by $(x-1)^3x^2$ the equation results to:

$$5x^4 - 3x^3 - x^2 + 4x - 1$$
$$= A(x-1)^2x^2 + B(x-1)x^2 + Cx^2 + D(x-1)^3x + E(x-1)^3$$

Multiplication of each term can be done separately to be extra careful.

$$\begin{aligned}
&Ax^4 - 2Ax^3 + Ax^2 \\
&Bx^3 - Bx^2 \\
&Cx^2 \\
&Dx^4 - 3Dx^3 + 3Dx^2 - Dx \\
&Ex^3 - 3Ex^2 + 3Ex - E
\end{aligned}$$

Group terms with the same power of x and set equal to the corresponding term.

$$\begin{aligned}
5x^4 &= Ax^4 + Dx^4 \\
-3x^3 &= -2Ax^3 + Bx^3 - 3D^3 + Ex^3 \\
-x^2 &= Ax^2 - Bx^2 + Cx^2 + 3Dx^2 - 3Ex^2 \\
4x &= -Dx + 3Ex \\
-1 &= -E
\end{aligned}$$

From these 5 equations, every x can be divided out. Assume that $x \neq 0$ because if it were, then the original expression would be undefined.

$$\begin{aligned}
5 &= A + D \\
-3 &= -2A + B - 3D + E \\
-1 &= A - B + C + 3D - 3E \\
4 &= -D + 3E \\
-1 &= E
\end{aligned}$$

This is a system of equations of five variables and 5 equations. Some of the equations can be solved using logic and substitution like $E = -1,\ D = -7,\ A = 12$. You can use any method involving determinants or matrices. In this case it is easiest to substitute known values into equations with one unknown value to get more known values and repeat.

$$\begin{aligned}
B &= 1 \\
C &= 6 \\
\frac{5x^4 - 3x^3 - x^2 + 4x - 1}{(x-1)^3 x^2} &= \frac{12}{x-1} + \frac{1}{(x-1)^2} + \frac{6}{(x-1)^3} + \frac{-7}{x} + \frac{-1}{x^2}
\end{aligned}$$

Example 4

Use matrices to complete the partial fraction decomposition of the following rational expression and confirm the solution.

$$\frac{2x+4}{(x-1)(x+3)}$$

$$\frac{2x+4}{(x-1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$
$$2x+4 = Ax+3A+Bx+B$$

$$2 = A+B$$
$$4 = 3A+B$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 1 & 4 \end{array}\right] \rightarrow \begin{array}{c} \rightarrow \\ -3 \cdot I \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right] \rightarrow \begin{array}{c} \rightarrow \\ \div -2 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \begin{array}{c} -II \\ \rightarrow \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$A = 1, B = 1$$

$$\frac{2x+4}{(x-1)(x+3)} = \frac{1}{x+1} + \frac{1}{x+3}$$

To confirm this answer, add the fractions.

$\frac{1}{x+1} + \frac{1}{x+3} = \frac{x+3}{(x+1)(x+3)} + \frac{x+1}{(x+1)(x+3)} = \frac{2x+4}{(x+1)(x+3)}$

Example 5

Use matrices to help you decompose the following rational expression. Confirm the solution by adding the partial fractions.

$\frac{5x-2}{(2x-1)(3x+4)}$

$$\begin{aligned}
\frac{5x-2}{(2x-1)(3x+4)} &= \frac{A}{2x-1}+\frac{B}{3x+4}\\
5x-2 &= A(3x+4)+B(2x-1)\\
5x-2 &= 3Ax+4A+2Bx-B\\
5 &= 3A+2B\\
-2 &= 4A-B
\end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 5\\ 4 & -1 & -2\end{array}\right]\begin{array}{c}\rightarrow \quad \cdot 4 \quad \rightarrow\\ \rightarrow \quad \cdot 3 \quad \rightarrow\end{array}\left[\begin{array}{cc|c} 12 & 8 & 20\\ 12 & -3 & -6\end{array}\right]\begin{array}{c}\rightarrow\\ \rightarrow \quad -I \quad \rightarrow\end{array}\left[\begin{array}{cc|c} 12 & 8 & 20\\ 0 & -11 & -26\end{array}\right]\begin{array}{c}\rightarrow \quad \cdot 11 \quad \rightarrow\\ \rightarrow \quad \cdot 8 \quad \rightarrow\end{array}\left[\begin{array}{cc|c} 132 & 88 & 220\\ 0 & -88 & -208\end{array}\right]$$

$$\begin{array}{c}\rightarrow \quad +II \quad \rightarrow\\ \rightarrow\end{array}\left[\begin{array}{cc|c} 132 & 0 & 12\\ 0 & -88 & -208\end{array}\right]\begin{array}{c}\rightarrow \quad \div 132 \quad \rightarrow\\ \rightarrow \quad \div -88 \quad \rightarrow\end{array}\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{11}\\ 0 & 1 & \frac{26}{11}\end{array}\right]$$

$$\begin{aligned}
A = \frac{1}{11}&, B = -\frac{26}{11}\\
\frac{5x-2}{(2x-1)(3x+4)} &= \frac{\frac{1}{11}}{2x-1}+\frac{\frac{26}{11}}{3x+4}
\end{aligned}$$

To confirm, add the fractions.

$$\begin{aligned}
\frac{5x-2}{(2x-1)(3x+4)} &= \frac{\frac{1}{11}}{2x-1}+\frac{\frac{26}{11}}{3x+4}\\
5x-2 &= \frac{1}{11}(3x+4)+\frac{26}{11}(2x-1)\\
55x-22 &= 3x+4+26(2x-1)\\
55x-22 &= 3x+4+52x-26\\
55x-22 &= 55x-22
\end{aligned}$$

Review

Decompose the following rational expressions. Practice using matrices with at least one of the problems.

1. $\frac{3x-4}{(x-1)(x+4)}$
2. $\frac{2x+1}{x^2(x-3)}$
3. $\frac{x+1}{x(x-5)}$
4. $\frac{x^2+3x+1}{x(x-3)(x+6)}$
5. $\frac{3x^2+2x-1}{x^2(x+2)}$
6. $\frac{x^2+1}{x(x-1)(x+1)}$
7. $\frac{4x^2-9}{x^2(x-4)}$
8. $\frac{2x-4}{(x+7)(x-3)}$
9. $\frac{3x-4}{x^2(x^2+1)}$
10. $\frac{2x+5}{(x-3)(x^2+4)}$

11. $\frac{3x^2+2x-5}{x^2(x-3)(x^2+1)}$
12. Confirm your answer to #1 by adding the partial fractions.
13. Confirm your answer to #3 by adding the partial fractions.
14. Confirm your answer to #6 by adding the partial fractions.
15. Confirm your answer to #9 by adding the partial fractions.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.10.

6.17 General Form of a Conic

Learning Objectives

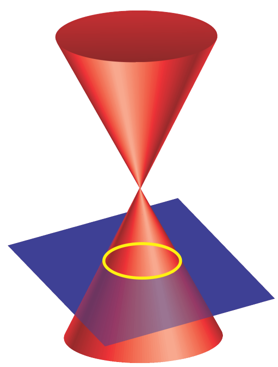
Here you will see how each conic section is the intersection of a plane and a cone, review completing the square and start working with the general equation of a conic.

Conics are a family of graphs that include parabolas, circles, ellipses and hyperbolas. All of these graphs come from the same general equation and by looking and manipulating a specific equation you can learn to tell which conic it is and how it can be graphed.

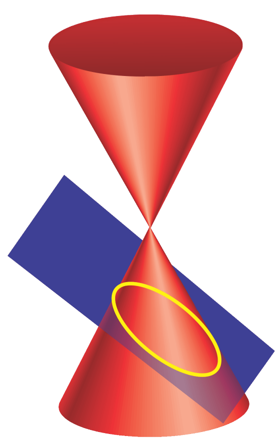
What is the one essential skill that enables you to manipulate the equation of a conic in order to sketch its graph?

Introduction to Conics

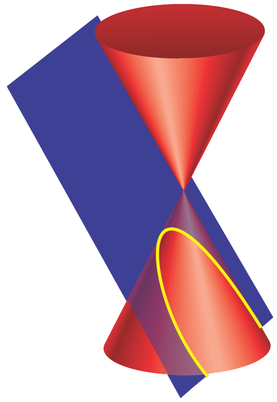
The word conic comes from the word cone which is where the shapes of parabolas, circles, ellipses and hyperbolas originate. Consider two cones that open up in opposite directions and a plane that intersects it horizontally. A flat intersection would produce a perfect circle.



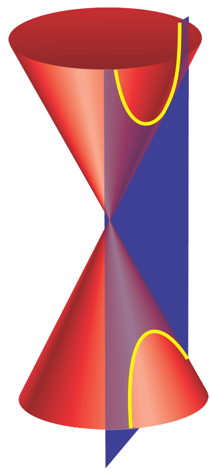
To produce an ellipse, tilt the plane so that the circle becomes elongated and oval shaped. Notice that the angle that the plane is tilted is still less steep than the slope of the side of the cone.



As you tilt the plane even further and the slope of the plane equals the slope of the cone edge you produce a parabola. Since the slopes are equal, a parabola only intersects one of the cones.



Lastly, if you make the plane steeper still, the plane ends up intersecting both the lower cone and the upper cone creating the two parts of a hyperbola.

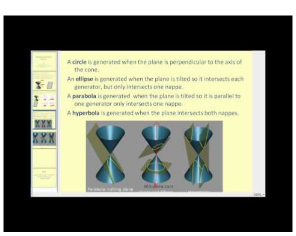


The intersection of three dimensional objects in three dimensional space to produce two dimensional graphs is quite challenging. In practice, the knowledge of where conics come from is not widely used. It will be more important for you to be able to manipulate an equation into standard form and graph it in a regular coordinate plane. The regular form of a conic is:

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

Before you start manipulating the general form of a conic equation you should be able to recognize whether it is a circle, ellipse, parabola or hyperbola. In standard form, the two coefficients to examine are A and C.

- For **circles,** the coefficients of x^2 and y^2 are the same sign and the same value: $A=C$
- For **ellipses,** the coefficients of x^2 and y^2 are the same sign and different values: $A,C>0,\ A\neq C$
- For **hyperbolas,** the coefficients of x^2 and y^2 are opposite signs: $C<0<A$ or $A<0<C$
- For **parabolas,** either the coefficient of x^2 or y^2 must be zero: $A=0$ or $C=0$



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Each specific type of conic has its own graphing form, but in all cases the technique of completing the square is essential.

For review, let's complete the square in the expression x^2+6x. and demonstrate graphically what completing the square represents.

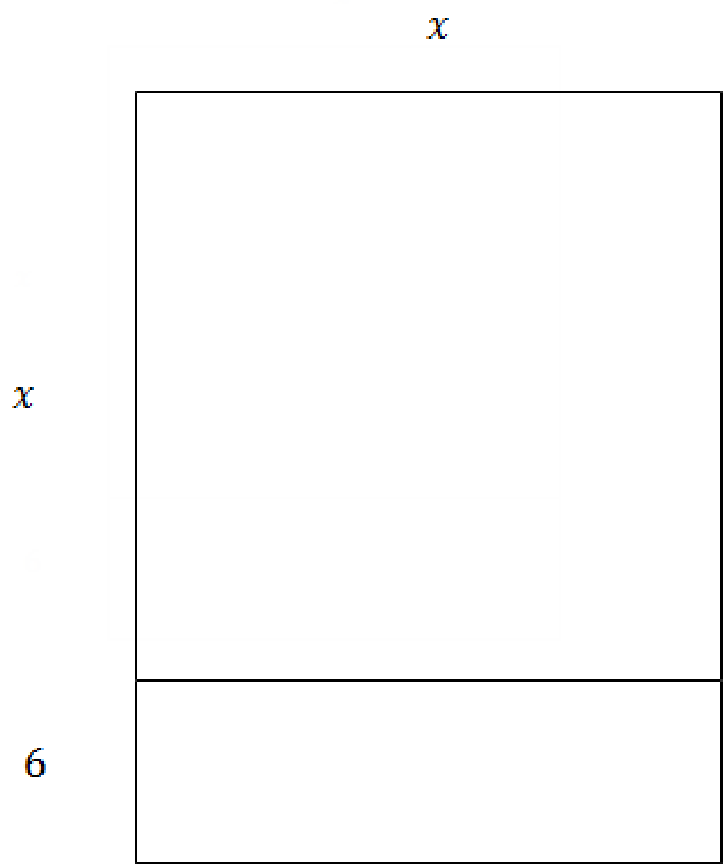
Algebraically, completing the square just requires you to divide the coefficient of x by 2 and square the result. In this case $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Since you cannot add nine to an expression without changing its value, you must simultaneously add nine and subtract nine so the net change will be zero.

$x^2+6x+9-9$

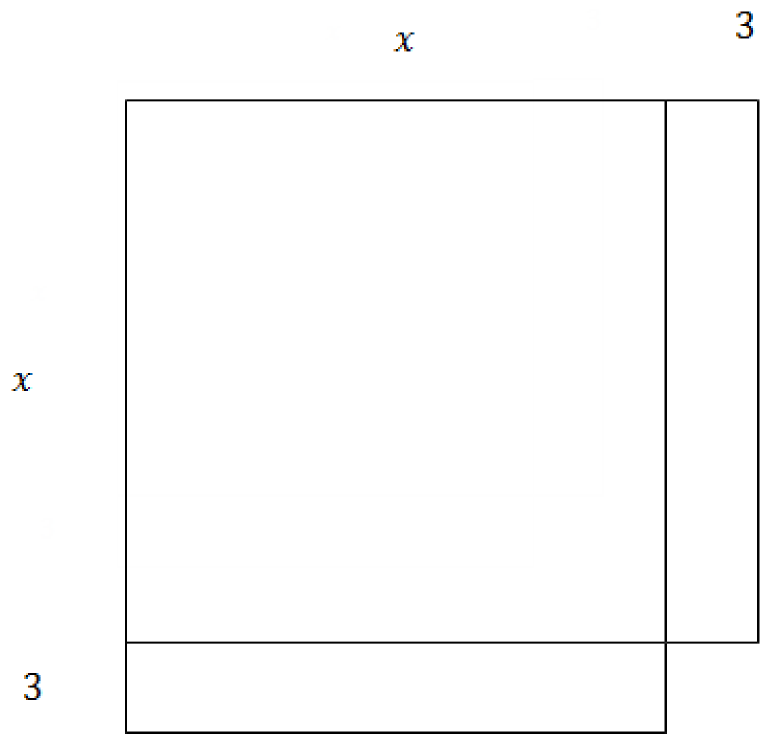
Now you can factor by recognizing a perfect square.

$(x+3)^2-9$

Graphically the original expression x^2+6x can be represented by the area of a rectangle with sides x and $(x+6)$.



The term "complete the square" has visual meaning as well algebraic meaning. The rectangle can be rearranged to be more square-like so that instead of small rectangle of area $6x$ at the bottom, there is a rectangle of area $3x$ on two sides of the x^2 square.



Notice what is missing to make this shape a perfect square? A little corner square of 9 is missing which is why the 9 should be added to make the perfect square of $(x+3)(x+3)$.

Examples

Example 1

Earlier, you were asked what skills you need for conics. The one essential skill that you need for conics is completing the square. If you can complete the square with two variables then you will be able to graph every type of conic.

Example 2

What type of conic is each of the following relations?

1. $5y^2-2x^2=-25$
   1. Hyperbola because the x^2 and y^2 coefficients are different signs.
2. $x=-\frac{1}{2}y^2-3$
   1. Parabola (sideways) because the x^2 term is missing.
3. $4x^2+6y^2=36$
   1. Ellipse because the x^2 and y^2 coefficients are different values but the same sign.
4. $x^2-\frac{1}{4}y=1$
   1. Parabola (upright) because the y^2 term is missing.
5. $-\frac{x^2}{8}+\frac{y^2}{4}=1$
   1. Hyperbola because the x^2 and y^2 coefficients are different signs.

6. $-x^2+99y^2=12$
    1. Hyperbola because the x^2 and y^2 coefficients are different signs.

Example 3

Complete the square for both the x and y terms in the following equation.

$x^2+6x+2y^2+16y=0$

First write out the equation with space so that there is room for the terms to be added to both sides. Since this is an equation, it is appropriate to add the values to both sides instead of adding and subtracting the same value simultaneously. As you rewrite with spaces, factor out any coefficient of the x^2 or y^2 terms since your algorithm for completing the square only works when this coefficient is one.

$x^2+6x+\_\_+2(y^2+8y+\_\_)=0$

Next complete the square by adding a nine and what looks like a 16 on the left (it is actually a 32 since it is inside the parentheses).

$x^2+6x+9+2(y^2+8y+16)=9+32$

Factor.

$(x+3)^2+2(y+4)^2=41$

Example 4

Identify the type of conic in each of the following relations.

1. $3x^2=3y^2+18$
    1. The relation is a hyperbola because when you move the $3y^2$ to the left hand side of the equation, it becomes negative and then the coefficients of x^2 and y^2 have opposite signs.
2. $y=4(x-3)^2+2$
    1. Parabola
3. $x^2+y^2=4$
    1. Circle
4. $y^2+2y+x^2-6x=12$
    1. Circle
5. $\frac{x^2}{6}+\frac{y^2}{12}=1$
    1. Ellipse
6. $x^2-y^2+4=0$
    1. Hyperbola

Example 5

Complete the square for both x and y in the following equation.

$$-3x^2-24x+4y^2-32y=8$$

$$\begin{aligned}
-3x^2-24x+4y^2-32y&=8\\
-3(x^2+8x+__)+4(y^2-8y+__)&=8\\
-3(x^2+8x+16)+4(y^2-8y+16)&=8-48+64\\
-3(x+4)^2+4(y-4)^2&=24
\end{aligned}$$

Review

Identify the type of conic in each of the following relations.

1. $3x^2+4y^2=12$
2. $x^2+y^2=9$
3. $\frac{x^2}{4}+\frac{y^2}{9}=1$
4. $y^2+x=11$
5. $x^2+2x-y^2+6y=15$
6. $x^2=y-1$

Complete the square for x and/or y in each of the following expressions.

7. x^2+4x
8. y^2-8y
9. $3x^2+6x+4$
10. $3y^2+9y+15$
11. $2x^2-12x+1$

Complete the square for x and/or y in each of the following equations.

12. $4x^2-16x+y^2+2y=-1$
13. $9x^2-54x+y^2-2y=-81$
14. $3x^2-6x-4y^2=9$
15. $y=x^2+4x+1$

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.1.

6.18 Parabolas

Learning Objectives

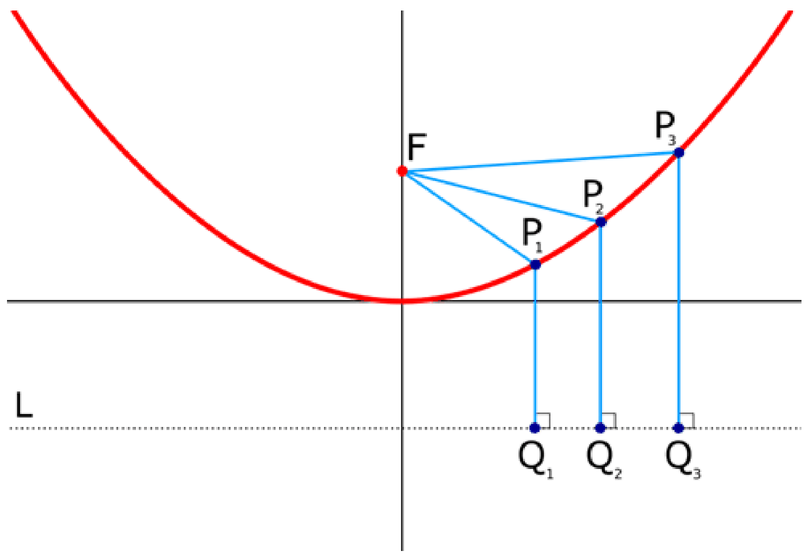
Here you will define a parabola in terms of its directrix and focus, graph parabolas vertically and horizontally, and use a new graphing form of the parabola equation.

When working with parabolas in the past you probably used vertex form and analyzed the graph by finding its roots and intercepts. There is another way of defining a parabola that turns out to be more useful in the real world. One of the many uses of parabolic shapes in the real world is satellite dishes. In these shapes it is vital to know where the receptor point should be placed so that it can absorb all the signals being reflected from the dish.

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

Graphing Parabolas

The definition of a **parabola** is the collection of points equidistant from a point called the **focus** and a line called the **directrix**.

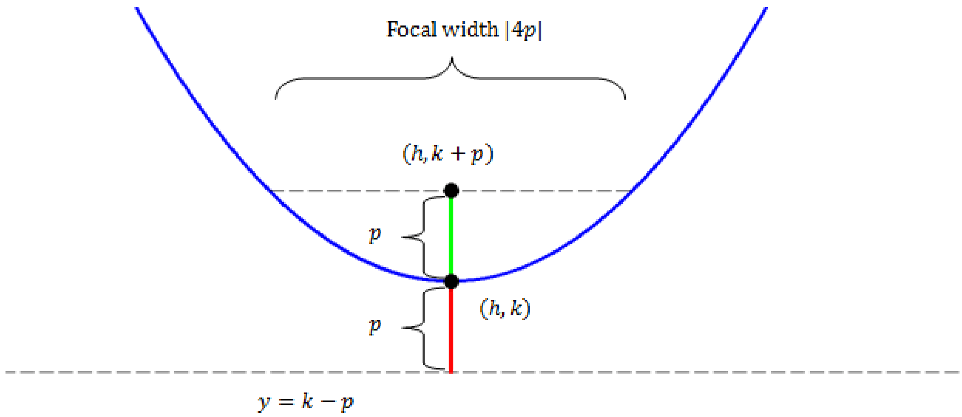


Notice how the three points P_1, P_2, P_3 are each connected by a blue line to the focus point F and the directrix line L.

$$\overline{FP_1} = \overline{P_1Q_1}$$
$$\overline{FP_2} = \overline{P_2Q_2}$$
$$\overline{FP_3} = \overline{P_3Q_3}$$

There are two graphing equations for parabolas that will be used in this concept. The only difference is one equation graphs parabolas opening vertically and one equation graphs parabolas opening horizontally. You can recognize the parabolas opening vertically because they have an x^2 term. Likewise, parabolas opening horizontally have a y^2 term.

The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$. The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.



Note that the vertex is still (h,k). The parabola opens upwards or to the right if the $4p$ is positive. The parabola opens down or to the left if the $4p$ is negative. The focus is just a point that is distance p away from the vertex. The directrix is just a line that is distance p away from the vertex in the opposite direction. You can sketch how wide the parabola is by noting the focal width is $|4p|$.

Once you put the parabola into this graphing form you can sketch the parabola by plotting the vertex, identifying p and plotting the focus and directrix and lastly determining the focal width and sketching the curve.

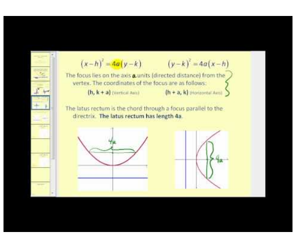
Take the conic:

$2x^2 + 16x + y = 0$

This is a parabola because the y^2 coefficient is zero.

$$\begin{aligned} x^2 + 8x &= -\frac{1}{2}y \\ x^2 + 8x + 16 &= -\frac{1}{2}y + 16 \\ (x+4)^2 &= -\frac{1}{2}(y-32) \\ (x+4)^2 &= -4 \cdot \frac{1}{8}(y-32) \end{aligned}$$

The vertex is (-4, 32). The focal length is $p = \frac{1}{8}$. This parabola opens down which means that the focus is at $\left(-4, 32 - \frac{1}{8}\right)$ and the directrix is horizontal at $y = 32 + \frac{1}{8}$. The focal width is $\frac{1}{2}$.



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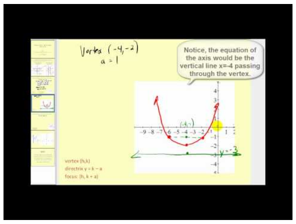

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Examples

Example 1

Earlier, you were asked where the receptor should be located on a satellite dish that is four feet wide and nine inches deep.

Since real world problems do not come with a predetermined coordinate system, you can choose to make the vertex of the parabola at (0, 0). Then, if everything is done in inches, another point on the parabola will be (24, 9). (Many people might mistakenly believe the point (48, 9) is on the parabola but remember that half this width stretches to (-24, 9) as well.) Using these two points, the focal width can be found.

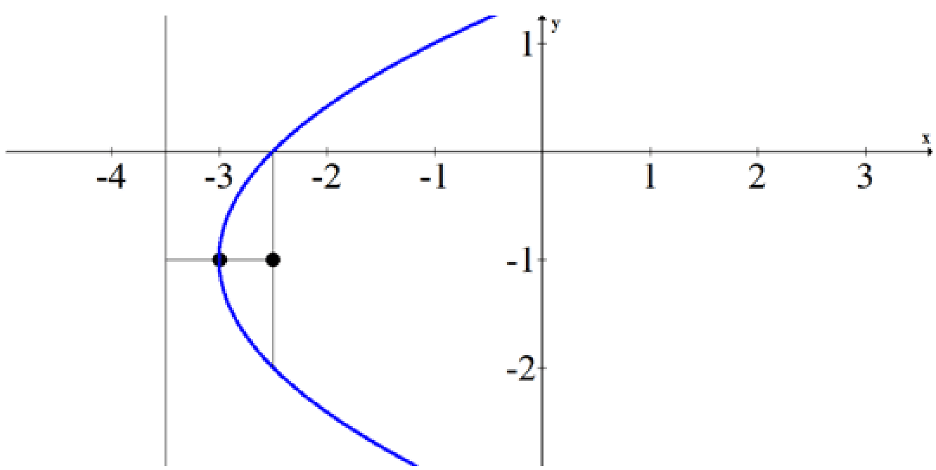
$$(x-0)^2 = 4p(y-0)$$
$$(24-0)^2 = 4p(9-0)$$
$$\frac{24^2}{4 \cdot 9} = p$$
$$16 = p$$

The receptor should be sixteen inches away from the vertex of the parabolic dish.

Example 2

Sketch the following parabola and identify the important pieces of information.

$(y+1)^2 = 4 \cdot \frac{1}{2} \cdot (x+3)$



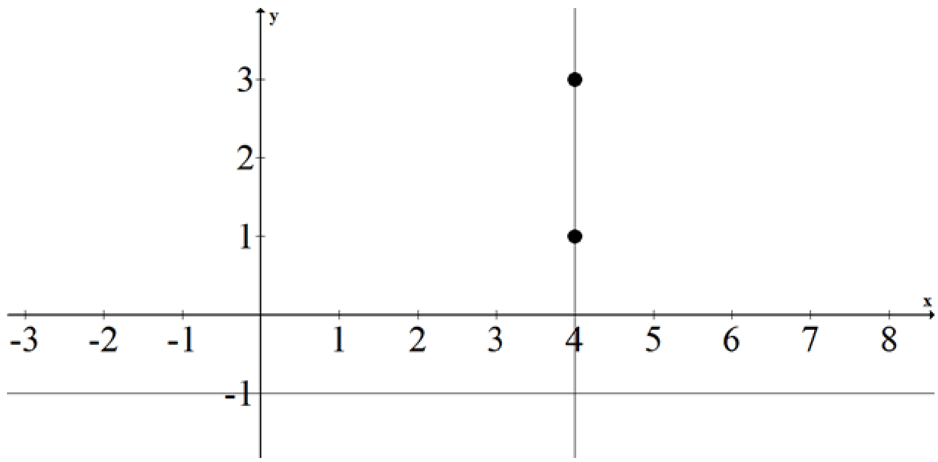
The vertex is at (-3, -1). The parabola is sideways because there is a y^2 term. The parabola opens to the right because the $4p$ is positive. The focal length is $p = \frac{1}{2}$ which means the focus is $\frac{1}{2}$ to the right of the vertex at (-2.5,

-1) and the directrix is $\frac{1}{2}$ to the left of the vertex at $x = -3.5$. The focal width is 2 which is why the width of the parabola stretches from (-2.5, 0) to (-2.5, -2).

Example 3

What is the equation of a parabola that has a focus at (4, 3) and a directrix of $y = -1$?

It would probably be useful to graph the information that you have in order to reason about where the vertex is.



The vertex must be halfway between the focus and the directrix. This places it at (4, 1). The focal length is 2. The parabola opens upwards. This is all the information you need to create the equation.

$(x-4)^2 = 4\cdot 2\cdot (y-1)$

OR $(x-4)^2 = 8(y-1)$

Example 4

What is the equation of a parabola that opens to the right with focal width from (6, -7) to (6, 12)?

The focus is in the middle of the focal width. The focus is $\left(6, \frac{5}{2}\right)$. The focal width is 19 which is four times the focal length so the focal length must be $\frac{19}{4}$. The vertex must be a focal length to the left of the focus, so the vertex is at $\left(6-\frac{19}{4}, \frac{5}{2}\right)$. This is enough information to write the equation of the parabola.

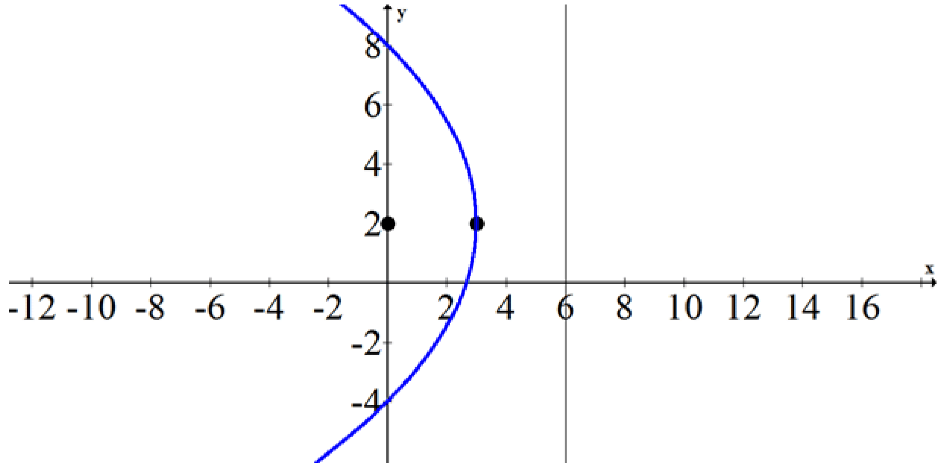
$$\left(y-\frac{5}{2}\right)^2 = 4\cdot \frac{19}{4}\cdot \left(x-6+\frac{19}{4}\right)$$

Example 5

Sketch the following conic by putting it into graphing form and identifying important information.

$$y^2 - 4y + 12x - 32 = 0$$

$$\begin{aligned}
y^2 - 4y &= -12x + 32 \\
y^2 - 4y + 4 &= -12x + 32 + 4 \\
(y-2)^2 &= -12(x-3) \\
(y-2)^2 &= -4\cdot 3\cdot (x-3)
\end{aligned}$$



The vertex is at (3, 2). The focus is at (0, 2). The directrix is at $x = 6$.

Review

1. What is the equation of a parabola with focus at (1, 4) and directrix at $y = -2$?
2. What is the equation of a parabola that opens to the left with focal width from (-2, 5) to (-2, -7)?
3. What is the equation of a parabola that opens to the right with vertex at (5, 4) and focal width of 12?
4. What is the equation of a parabola with vertex at (1, 8) and directrix at $y = 12$?
5. What is the equation of a parabola with focus at (-2, 4) and directrix at $x = 4$?
6. What is the equation of a parabola that opens downward with a focal width from (-4, 9) to (16, 9)?
7. What is the equation of a parabola that opens upward with vertex at (1, 11) and focal width of 4?

Sketch the following parabolas by putting them into graphing form and identifying important information.

8. $y^2 + 2y - 8x + 33 = 0$
9. $x^2 - 8x + 20y + 36 = 0$
10. $x^2 + 6x - 12y - 15 = 0$
11. $y^2 - 12y + 8x + 4 = 0$
12. $x^2 + 6x - 4y + 21 = 0$
13. $y^2 + 14y - 2x + 59 = 0$
14. $x^2 + 12x - \frac{8}{3}y + \frac{92}{3} = 0$
15. $x^2 + 2x - \frac{4}{5}y + 1 = 0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.2.

6.19 Circles

Learning Objectives

Here you will formalize the definition of a circle, translate a conic from standard form into graphing form, and graph circles. A **circle** is the collection of points that are the same distance from a single point. What is the connection between the Pythagorean Theorem and a circle?

Graphing Circles

The single point that all the points on a circle are equidistant from is called the **center** of the circle. A circle does not have a focus or a directrix, instead it simply has a center. Circles can be recognized immediately from the general equation of a conic when the coefficients of x^2 and y^2 are the same sign and the same value. Circles are not functions because they do not pass the vertical line test. The distance from the center of a circle to the edge of the circle is called the **radius** of the circle. The distance from one end of the circle through the center to the other end of the circle is called the **diameter**. The diameter is equal to twice the radius.

The graphing form of a circle is:

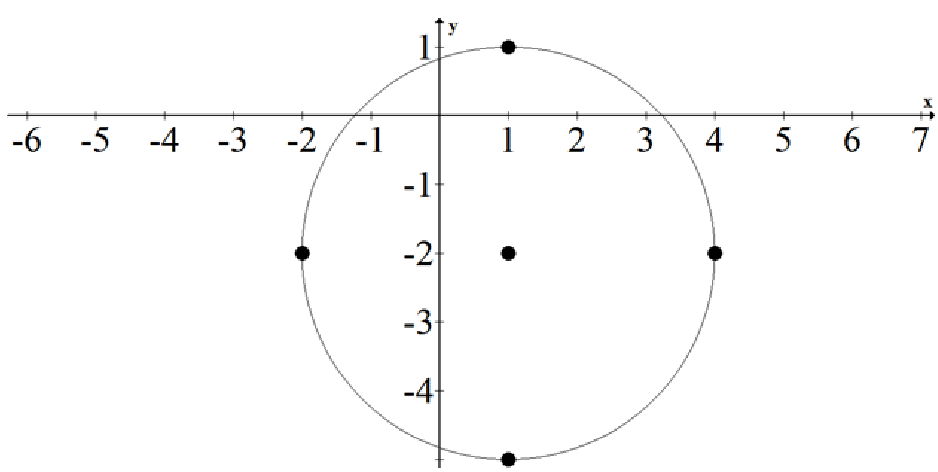
$(x-h)^2+(y-k)^2=r^2$

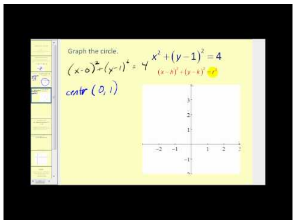
The center of the circle is at (h,k) and the radius of the circle is r. Note that this looks remarkably like the Pythagorean Theorem.

To graph a circle, first plot the center and then apply the radius. Take the following equation for a circle:

$(x-1)^2+(y+2)^2=9$

The center is at $(1,-2)$. Plot that point and the four points that are exactly 3 units from the center.





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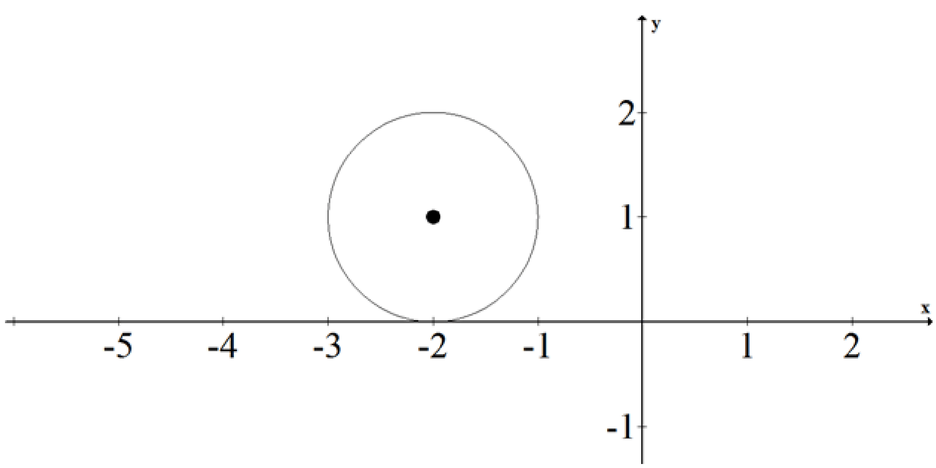
Examples

Example 1

Earlier, you were asked about the connection between circles and the Pythagorean Theorem. The reason why the graphing form of a circle looks like the Pythagorean Theorem is because each x and y coordinate along the outside of the circle forms a perfect right triangle with the radius as the hypotenuse.

Example 2

Graph the following conic: $(x+2)^2+(y-1)^2=1$



Example 3

Turn the following equation into graphing form for a circle. Identify the center and the radius.

$36x^2+36y^2-24x+36y-275=0$

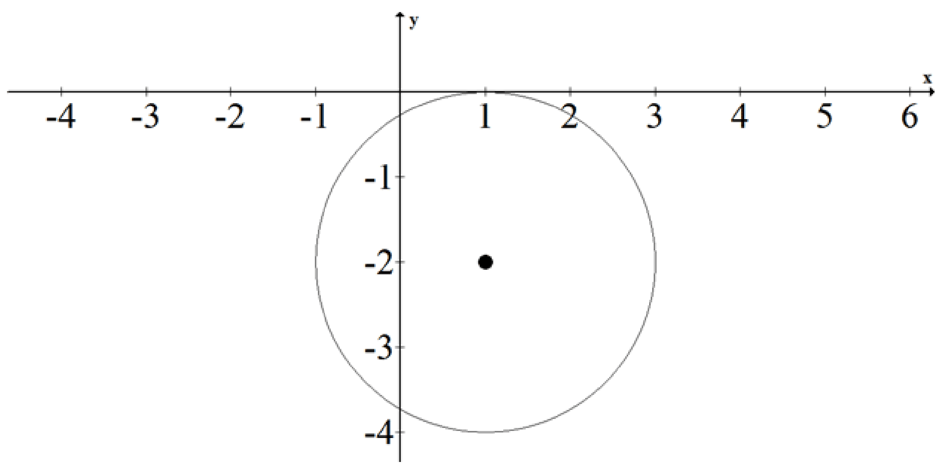
Complete the square and then divide by the coefficient of x^2 and y^2

$$36x^2-24x+36y^2+36y=275$$
$$36\left(x^2-\frac{2}{3}x+\_\_\right)+36(y^2+y+\_\_)=275$$
$$36\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+36\left(y^2+y+\frac{1}{4}\right)=275+4+9$$
$$36\left(x-\frac{1}{3}\right)^2+36\left(y+\frac{1}{2}\right)^2=288$$
$$\left(x-\frac{1}{3}\right)^2+\left(y+\frac{1}{2}\right)^2=8$$

The center is $\left(\frac{1}{3},-\frac{1}{2}\right)$. The radius is $\sqrt{8}=2\sqrt{2}$.

Example 4

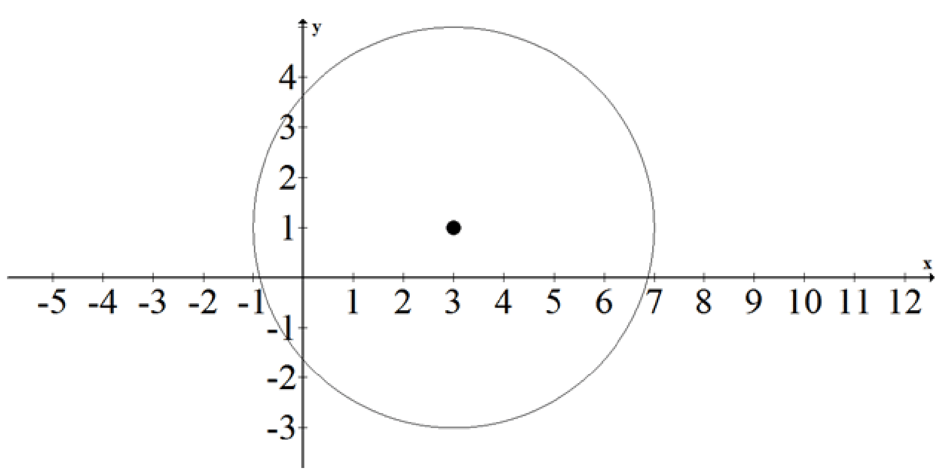
Write the equation for the following circle.



$(x-1)^2+(y+2)^2=4$

Example 5

Write the equation of the following circle.



The center of the circle is at (3, 1) and the radius of the circle is $r = 4$. The equation is $(x-3)^2 + (y-1)^2 = 16$.

Review

Graph the following conics:

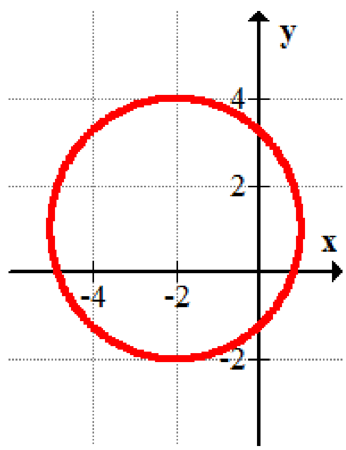
1. $(x+4)^2 + (y-3)^2 = 1$
2. $(x-7)^2 + (y+1)^2 = 4$
3. $(y+2)^2 + (x-1)^2 = 9$
4. $x^2 + (y-5)^2 = 8$
5. $(x-2)^2 + y^2 = 16$

Translate the following conics from standard form to graphing form.

6. $x^2 - 4x + y^2 + 10y + 18 = 0$
7. $x^2 + 2x + y^2 - 8y + 1 = 0$
8. $x^2 - 6x + y^2 - 4y + 12 = 0$
9. $x^2 + 2x + y^2 + 14y + 25 = 0$
10. $x^2 - 2x + y^2 - 2y = 0$

Write the equations for the following circles.

11.



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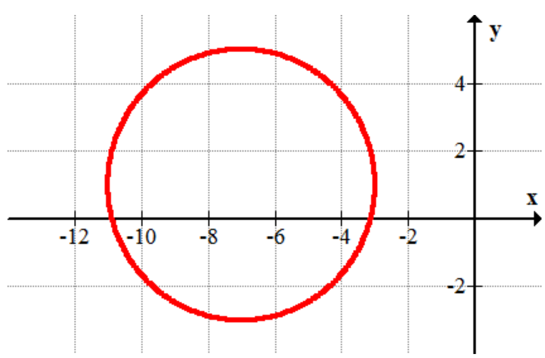

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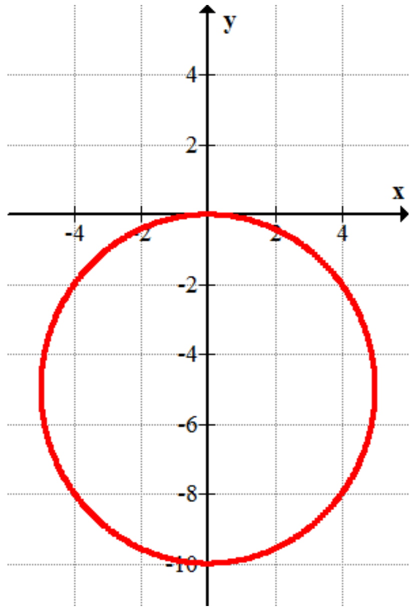

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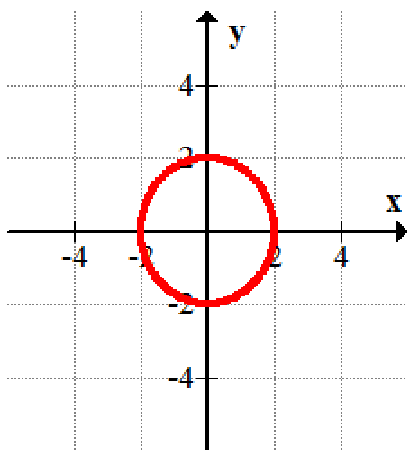

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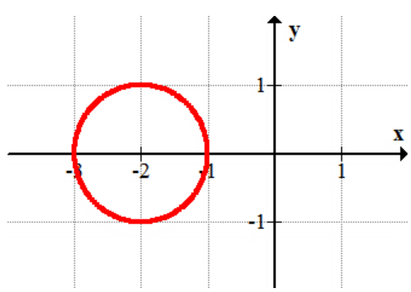


Review (Answers)

To see the Review answers, open this PDF file and look for section 9.3.

6.20 Ellipses

Learning Objectives

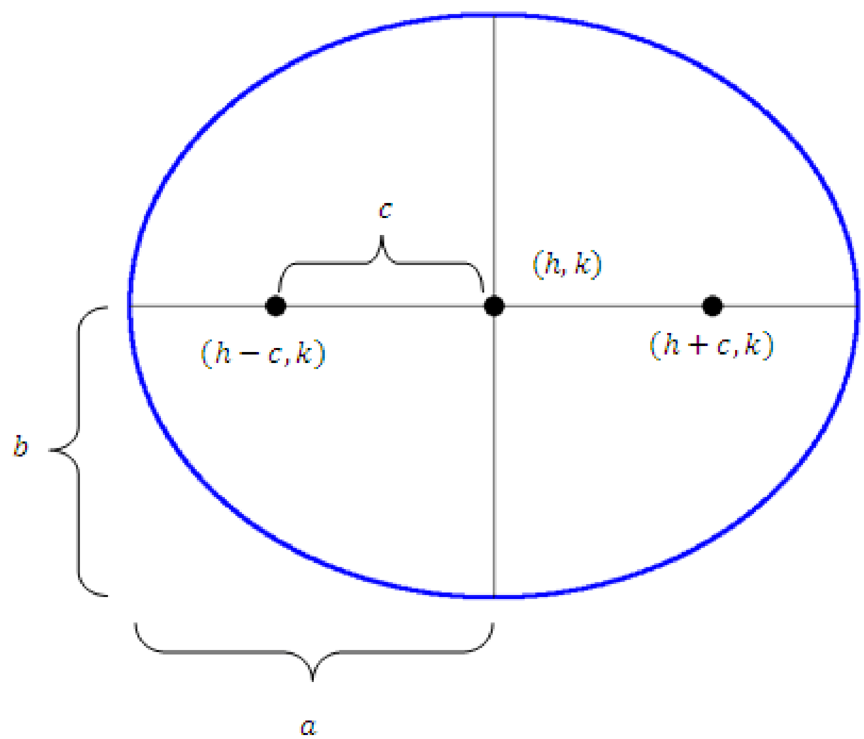
Here you will translate ellipse equations from standard conic form to graphing form, graph ellipses and identify the different axes. You will also identify eccentricity and solve word problems involving ellipses.

An **ellipse** is commonly known as an oval. Ellipses are just as common as parabolas in the real world with their own uses. Rooms that have elliptical shaped ceilings are called whisper rooms because if you stand at one focus point and whisper, someone standing at the other focus point will be able to hear you.

Ellipses look similar to circles, but there are a few key differences between these shapes. Ellipses have both an x-radius and a y-radius while circles have only one radius. Another difference between circles and ellipses is that an ellipse is defined as the collection of points that are a set distance from two focus points while circles are defined as the collection of points that are a set distance from one center point. A third difference between ellipses and circles is that not all ellipses are similar to each other while all circles are similar to each other. Some ellipses are narrow and some are almost circular. How do you measure how strangely shaped an ellipse is?

Graphing Ellipses

An ellipse has two **foci**. For every point on the ellipse, the sum of the distances to each foci is constant. This is what defines an ellipse. Another way of thinking about the definition of an ellipse is to allocate a set amount of string and fix the two ends of the string so that there is some slack between them. Then use a pencil to pull the string taught and trace the curve all the way around both fixed points. You will trace an ellipse and the fixed end points of the string will be the foci. Foci is the plural form of focus. In the picture below, (h,k) is the center of the ellipse and the other two marked points are the foci.



The **major axis** is the longest distance from end to end of an ellipse and is twice as long as the semi-major axis. The **semi-major axis** is the distance from the center of the ellipse to the furthest point on the ellipse and the **semi-minor axis** is the distance from the center to the edge of the ellipse on the axis that is perpendicular to the semi-major axis.

The general equation for an ellipse is:

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

In this case the major axis is horizontal because a, the x-radius, is larger. If the y-radius were larger, then a and b would reverse. In other words, the coefficient a always comes from the length of the semi major axis (the longer one) and the coefficient b always comes from the length of the semi minor axis (the shorter one).

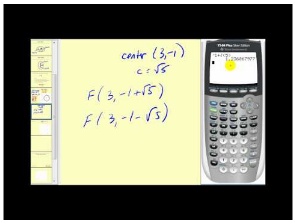
In order to find the locations of the two foci, you will need to find the focal radius represented as c using the following relationship:

$a^2-b^2=c^2$

Once you have the focal radius, measure from the center along the major axis to locate the foci. The general shape of an ellipse is measured using eccentricity. **Eccentricity** is a measure of how oval or how circular the shape is. Ellipses can have an eccentricity between 0 and 1 where a number close to 0 is extremely circular and a number close to 1 is less circular. Eccentricity is calculated by:

$e=\frac{c}{a}$

Ellipses also have two directrix lines that correspond to each focus but on the outside of the ellipse. The distance from the center of the ellipse to each directrix line is $\frac{a^2}{c}$.



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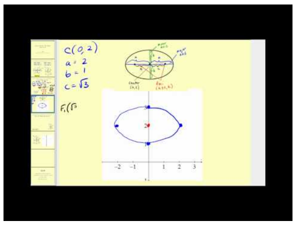

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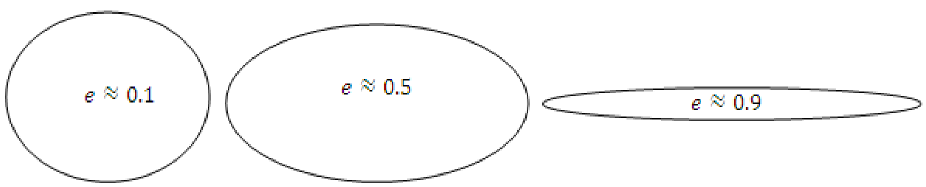
Examples

Example 1

Earlier, you were asked how you measure how strangely shaped an ellipse is. Ellipses are measured using their eccentricity. Here are three ellipses with estimated eccentricity for you to compare.



Eccentricity is the ratio of the focal radius to the semi major axis: $e = \frac{c}{a}$.

Example 2

Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{16} = 1$

The center of the ellipse is at (2, -1). The major axis is vertical which means the semi major axis is $a = 4$. The vertices are (2, 3) and (2, -5).

$$16^2 - 4^2 = c^2$$
$$4\sqrt{15} = \sqrt{240} = c$$

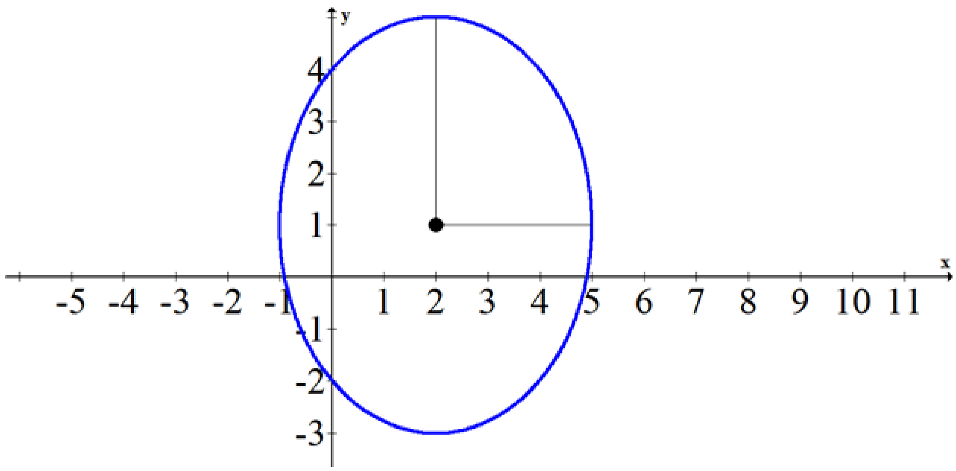
Thus the foci are $(2, -1 + 4\sqrt{15})$ and $(2, -1 - 4\sqrt{15})$

Example 3

Sketch the following ellipse.

$\frac{(y-1)^2}{16}+\frac{(x-2)^2}{9}=1$

Plotting the foci are usually important, but in this case the question simply asks you to sketch the ellipse. All you need is the center, x-radius and y-radius.



Example 4

Put the following conic into graphing form.

$25x^2-150x+36y^2+72y-639=0$

$$\begin{aligned}
25x^2-150x+36y^2+72y-639&=0\\
25(x^2-6x)+36(y^2+2y)&=639\\
25(x^2-6x+9)+36(y^2+2y+1)&=639+225+36\\
25(x-3)^2+36(y+1)^2&=900\\
\frac{25(x-3)^2}{900}+\frac{36(y+1)^2}{900}&=\frac{900}{900}\\
\frac{(x-3)^2}{36}+\frac{(y+1)^2}{25}&=1
\end{aligned}$$

Example 5

Put the following conic into graphing form.

$9x^2-9x+4y^2+12y+\frac{9}{4}=-8$

$$
\begin{aligned}
9x^2-9x+4y^2+12y+\frac{9}{4}&=-8\\
9x^2-9x+\frac{9}{4}+4y^2+12y&=-8\\
9\left(x^2-x-\frac{1}{4}\right)+4(y^2+3y)&=-8\\
9\left(x-\frac{1}{2}\right)^2+4\left(y^2+3y+\frac{9}{4}\right)&=-8+4\cdot\frac{9}{4}\\
9\left(x-\frac{1}{2}\right)^2+4\left(y+\frac{3}{2}\right)^2&=1\\
\frac{\left(x-\frac{1}{2}\right)^2}{\frac{1}{9}}+\frac{\left(y+\frac{3}{2}\right)^2}{\frac{1}{4}}&=1
\end{aligned}
$$

Review

Find the vertices, foci, and eccentricity for each of the following ellipses.

1. $\frac{(x-1)^2}{4}+\frac{(y+5)^2}{16}=1$
2. $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$
3. $(x-2)^2+\frac{(y-1)^2}{4}=1$

Now sketch each of the following ellipses *(note that they are the same as the ellipses in #1 - #3).*

4. $\frac{(x-1)^2}{4}+\frac{(y+5)^2}{16}=1$
5. $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$
6. $(x-2)^2+\frac{(y-1)^2}{4}=1$

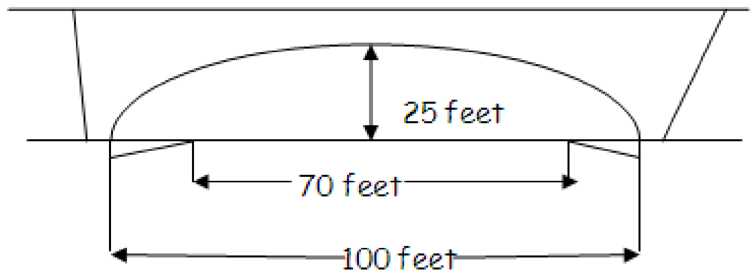
Put each of the following equations into graphing form.

7. $x^2+2x+4y^2+56y+197=16$
8. $x^2-8x+9y^2+18y+25=9$
9. $9x^2-36x+4y^2+16y+52=36$

Find the equation for each ellipse based on the description.

10. An ellipse with vertices (4, -2) and (4, 8) and minor axis of length 6.
11. An ellipse with minor axis from (4, -1) to (4, 3) and major axis of length 12.
12. An ellipse with minor axis from (-2, 1) to (-2, 7) and one focus at (2, 4).
13. An ellipse with one vertex at (6, -15), and foci at (6, 10) and (6, -14).

A bridge over a roadway is to be built with its bottom the shape of a semi-ellipse 100 feet wide and 25 feet high at the center. The roadway is to be 70 feet wide.



14. Find one possible equation of the ellipse that models the bottom of the bridge.
15. What is the clearance between the roadway and the overpass at the edge of the roadway?

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.4.

6.21 Hyperbolas

Learning Objectives

Here you will translate conic equations into graphing form and graph hyperbolas. You will also learn how to measure the eccentricity of a hyperbola and solve word problems.

Hyperbolas are relations that have asymptotes. When graphing rational functions you often produce a hyperbola. In this concept, hyperbolas will not be oriented in the same way as with rational functions, but the basic shape of a hyperbola will still be there.

Hyperbolas can be oriented so that they open side to side or up and down. One of the most common mistakes that you can make is to forget which way a given hyperbola should open. What are some strategies to help?

Graphing Hyperbolas

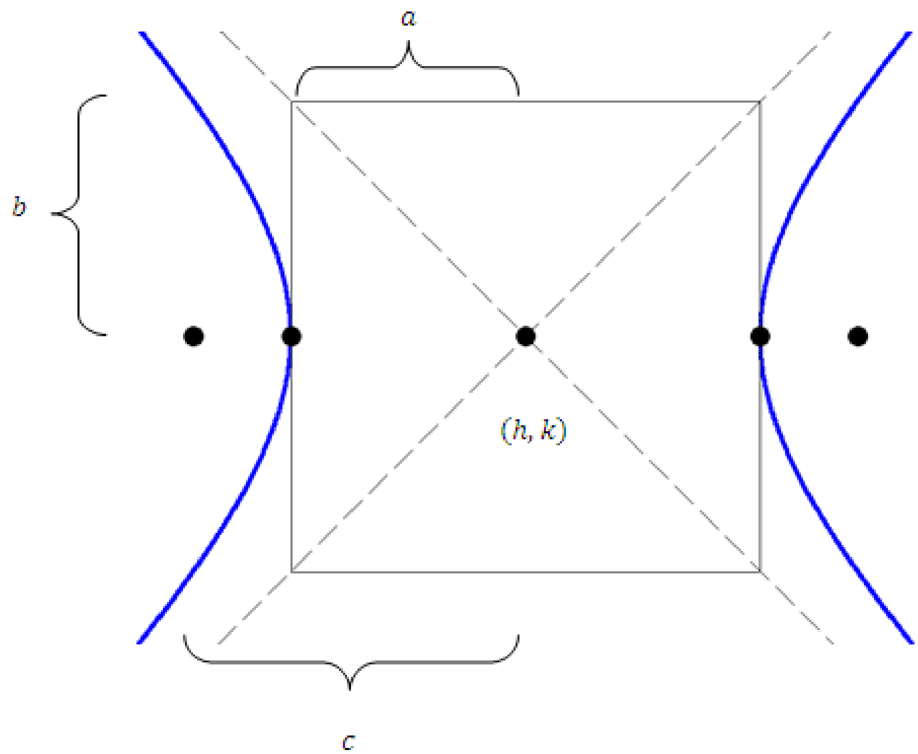
A hyperbola has two foci. For every point on the hyperbola, the difference of the distances to each foci is constant. This is what defines a hyperbola. The graphing form of a hyperbola that opens side to side is:

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

A hyperbola that opens up and down is:

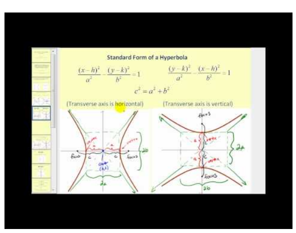
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Notice that for hyperbolas, a goes with the positive term and b goes with the negative term. It does not matter which constant is larger.



When graphing, the constants a and b enable you to draw a rectangle around the center. The t**ransverse axis** travels from vertex to vertex and has length $2a$. The **conjugate axis** travels perpendicular to the transverse axis through the center and has length $2b$. The foci lie beyond the vertices so the eccentricity, which is measured as $e = \frac{c}{a}$, is larger than 1 for all hyperbolas. Hyperbolas also have two directrix lines that are $\frac{a^2}{c}$ away from the center (not shown on the image).

The focal radius is $a^2 + b^2 = c^2$.



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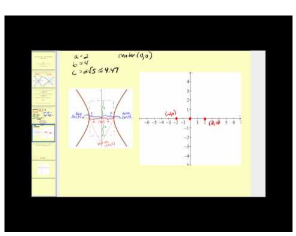

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Examples

Example 1

Earlier, you were asked how to determine the direction that a hyperbola opens. The best strategy to remember which direction the hyperbola opens is often the simplest. Consider the hyperbola $x^2 - y^2 = 1$. This hyperbola opens side to side because x can clearly never be equal to zero. This is a basic case that shows that when the negative is with the y value then the hyperbola opens up side to side.

Example 2

Put the following hyperbola into graphing form, list the components, and sketch it.

$9x^2 - 4y^2 + 36x - 8y - 4 = 0$

$$9(x^2 + 4x) - 4(y^2 + 2y) = 4$$
$$9(x^2 + 4x + 4) - 4(y^2 + 2y + 1) = 4 + 36 - 4$$
$$9(x+2)^2 - 4(y+1)^2 = 36$$
$$\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} = 1$$

Shape: Hyperbola that opens horizontally.

Center: (-2, -1)

$a = 2$

$b = 3$

$c = \sqrt{13}$

$e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

$d = \frac{a^2}{c} = \frac{4}{\sqrt{13}}$

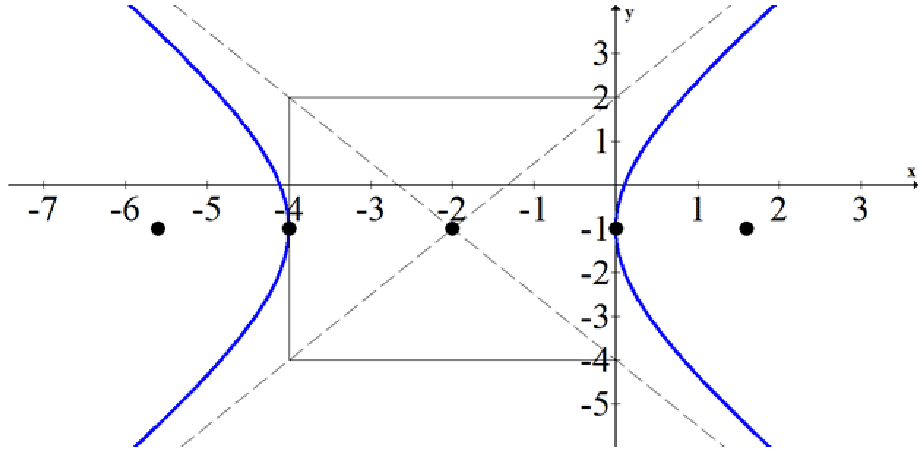
Foci: (-2+$\frac{\sqrt{13}}{2}$, -1), (-2-$\frac{\sqrt{13}}{2}$, -1)

Vertices: (-4, -1), (0, -1)

Equations of asymptotes: $\pm\frac{3}{2}(x+2) = (y+1)$

Note that it is easiest to write the equations of the asymptotes in point-slope form using the center and the slope.

Equations of directrices: $y = -2 \pm \frac{4}{\sqrt{13}}$



Example 3

Find the equation of the hyperbola with foci at (-3, 5) and (9, 5) and asymptotes with slopes of $\pm\frac{4}{3}$.

The center is between the foci at (3, 5). The focal radius is $c = 6$. The slope of the asymptotes is always the rise over run inside the box. In this case since the hyperbola is horizontal and a is in the x direction the slope is $\frac{b}{a}$. This makes a system of equations.

$$\frac{b}{a} = \pm\frac{4}{3}$$
$$a^2 + b^2 = 6^2$$

When you solve,you get $a = \sqrt{13}$, $b = \frac{4}{3}\sqrt{13}$.

$\frac{(x-3)^2}{13} - \frac{(y-5)^2}{\frac{16}{9}\cdot 13} = 1$

Example 4

Find the equation of the conic that has a focus point at (1, 2), a directrix at $x = 5$, and an eccentricity equal to $\frac{3}{2}$. Use the property that the distance from a point on the hyperbola to the focus is equal to the eccentricity times the distance from that same point to the directrix:

$\overline{PF} = e\overline{PD}$

This relationship bridges the gap between ellipses which have eccentricity less than one and hyperbolas which have eccentricity greater than one. When eccentricity is equal to one, the shape is a parabola.

$\sqrt{(x-1)^2 + (y-2)^2} = \frac{3}{2}\sqrt{(x-5)^2}$

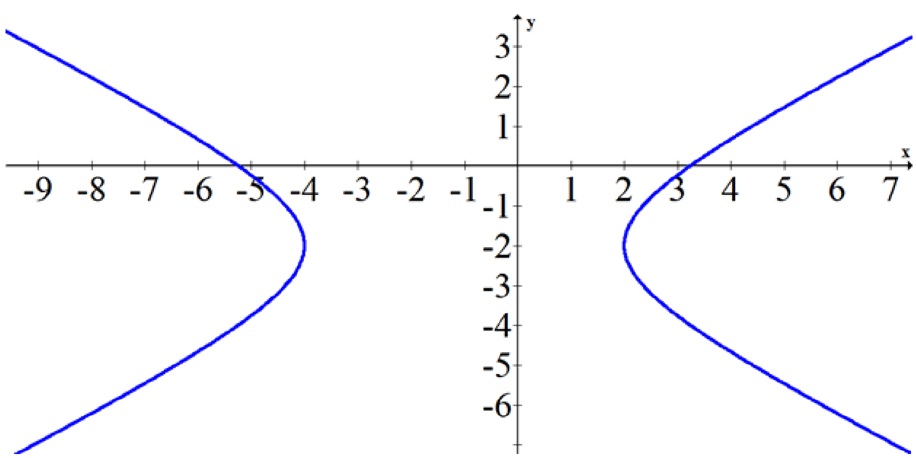
Square both sides and rearrange terms so that it is becomes a hyperbola in graphing form.

$$x^2-2x+1+(y-2)^2=\frac{9}{4}(x^2-10x+25)$$
$$x^2-2x+1-\frac{9}{4}x^2+\frac{90}{4}x-\frac{225}{4}+(y-2)^2=0$$
$$-\frac{5}{4}x^2+\frac{92}{4}x+(y-2)^2=\frac{221}{4}$$
$$-5x^2+92x+4(y-2)^2=221$$
$$-5\left(x^2-\frac{92}{5}x\right)+4(y-2)^2=221$$

$$-5\left(x^2-\frac{92}{5}x+\frac{92^2}{10^2}\right)+4(y-2)^2=221-\frac{2116}{5}$$
$$-5\left(x-\frac{92}{10}\right)^2+4(y-2)^2=-\frac{1011}{5}$$
$$\left(x-\frac{92}{10}\right)^2-(y-2)^2=\frac{1011}{100}$$
$$\frac{\left(x-\frac{92}{10}\right)^2}{\left(\frac{1011}{100}\right)}-\frac{(y-2)^2}{\left(\frac{1011}{100}\right)}=1$$

Example 5

Given the following graph, estimate the equation of the conic.



Since exact points are not marked, you will need to estimate the slope of asymptotes to get an approximation for a and b. The slope seems to be about $\pm\frac{2}{3}$. The center seems to be at (-1, -2). The transverse axis is 6 which means $a=3$.

$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{4}=1$

Review

Use the following equation for #1 - #5: $x^2+2x-4y^2-24y-51=0$

1. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
2. Identify whether the hyperbola opens side to side or up and down.
3. Find the location of the vertices.
4. Find the equations of the asymptotes.
5. Sketch the hyperbola.

Use the following equation for #6 - #10: $-9x^2 - 36x + 16y^2 - 32y - 164 = 0$

6. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
7. Identify whether the hyperbola opens side to side or up and down.
8. Find the location of the vertices.
9. Find the equations of the asymptotes.
10. Sketch the hyperbola.

Use the following equation for #11 - #15: $x^2 - 6x - 9y^2 - 54y - 81 = 0$

11. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
12. Identify whether the hyperbola opens side to side or up and down.
13. Find the location of the vertices.
14. Find the equations of the asymptotes.
15. Sketch the hyperbola.

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.5.