Partial fractions

An algebraic fraction such as $\frac{3x+5}{2x^2-5x-3}$ can often be broken down into simpler parts called partial fractions. Specifically

$$\frac{3x+5}{2x^2-5x-3} = \frac{2}{x-3} - \frac{1}{2x+1}$$

In this unit we explain how this process is carried out.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain the meaning of the terms 'proper fraction' and 'improper fraction'
- express an algebraic fraction as the sum of its partial fractions

Contents

1. Introduction

An **algebraic fraction** is a fraction in which the numerator and denominator are both polynomial expressions. A **polynomial expression** is one where every term is a multiple of a power of x, such as

$$5x^4 + 6x^3 + 7x + 4$$

The **degree** of a polynomial is the power of the highest term in x. So in this case the degree is 4.

The number in front of x in each term is called its **coefficient**. So, the coefficient of x^4 is 5. The coefficient of x^3 is 6.

Now consider the following algebraic fractions:

$$\frac{x}{x^2+2} \qquad \frac{x^3+3}{x^4+x^2+1}$$

In both cases the numerator is a polynomial of lower degree than the denominator. We call these **proper fractions**

With other fractions the polynomial may be of higher degree in the numerator or it may be of the same degree, for example

$$\frac{x^4+x^2+x}{x^3+x+2} \qquad \frac{x+4}{x+3}$$

and these are called **improper fractions**.

Key Point

If the degree of the numerator is less than the degree of the denominator the fraction is said to be a **proper fraction**

If the degree of the numerator is greater than or equal to the degree of the denominator the fraction is said to be an **improper fraction**

2. Revision of adding and subtracting fractions

We now revise the process for adding and subtracting fractions. Consider

$$\frac{2}{x-3} - \frac{1}{2x+1}$$

In order to add these two fractions together, we need to find the lowest common denominator. In this particular case, it is $(x-3)(2x+1)$.

We write each fraction with this denominator.

$$\frac{2}{x-3} = \frac{2(2x+1)}{(x-3)(2x+1)} \qquad \text{and} \qquad \frac{1}{2x+1} = \frac{x-3}{(x-3)(2x+1)}$$

So

$$\frac{2}{x-3} - \frac{1}{2x+1} \quad = \quad \frac{2(2x+1)}{(x-3)(2x+1)} - \frac{x-3}{(x-3)(2x+1)}$$

The denominators are now the same so we can simply subtract the numerators and divide the result by the lowest common denominator to give

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{4x+2-x+3}{(x-3)(2x+1)} = \frac{3x+5}{(x-3)(2x+1)}$$

Sometimes in mathematics we need to do this operation in reverse. In calculus, for instance, or when dealing with the binomial theorem, we sometimes need to split a fraction up into its component parts which are called **partial fractions**. We discuss how to do this in the following section.

Exercises 1

Use the rules for the addition and subtraction of fractions to simplify

a) $\frac{3}{x+1} + \frac{2}{x+3}$ b) $\frac{5}{x-2} - \frac{3}{x+2}$ c) $\frac{4}{2x+1} - \frac{2}{x+3}$ d) $\frac{1}{3x-1} - \frac{2}{6x+9}$

3. Expressing a fraction as the sum of its partial fractions

In the previous section we saw that

$$\frac{2}{x-3} - \frac{1}{2x+1} \quad = \quad \frac{3x+5}{(x-3)(2x+1)}$$

Suppose we start with $\frac{3x+5}{(x-3)(2x+1)}$. How can we get this back to its component parts ?

By inspection of the denominator we see that the component parts must have denominators of $x-3$ and $2x+1$ so we can write

$$\frac{3x+5}{(x-3)(2x+1)} \quad = \quad \frac{A}{x-3} + \frac{B}{2x+1}$$

where A and B are numbers. A and B cannot involve x or powers of x because otherwise the terms on the right would be improper fractions.

The next thing to do is to multiply both sides by the common denominator $(x-3)(2x+1)$. This gives

$$\frac{(3x+5)(x-3)(2x+1)}{(x-3)(2x+1)} = \frac{A(x-3)(2x+1)}{x-3} + \frac{B(x-3)(2x+1)}{2x+1}$$

Then cancelling the common factors from the numerators and denominators of each term gives

$$3x+5 = A(2x+1) + B(x-3)$$

Now this is an **identity**. This means that it is true for any values of x, and because of this we can substitute any values of x we choose into it. Observe that if we let $x = -\frac{1}{2}$ the first term on the right will become zero and hence A will disappear. If we let $x = 3$ the second term on the right will become zero and hence B will disappear.

If $x = -\frac{1}{2}$

$$\begin{aligned} -\frac{3}{2} + 5 &= B\left(-\frac{1}{2} - 3\right) \\ \frac{7}{2} &= -\frac{7}{2}B \end{aligned}$$

from which

$$B = -1$$

Now we want to try to find A.

If $x = 3$

$$14 = 7A$$

so that $A = 2$.
Putting these results together we have

$$\begin{aligned} \frac{3x+5}{(x-3)(2x+1)} &= \frac{A}{x-3} + \frac{B}{2x+1} \\ &= \frac{2}{x-3} - \frac{1}{2x+1} \end{aligned}$$

which is the sum that we started with, and we have now broken the fraction back into its component parts called **partial fractions**.

Example

Suppose we want to express $\frac{3x}{(x-1)(x+2)}$ as the sum of its partial fractions.

Observe that the factors in the denominator are $x - 1$ and $x + 2$ so we write

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

where A and B are numbers.

We multiply both sides by the common denominator $(x - 1)(x + 2)$:

$$3x = A(x + 2) + B(x - 1)$$

This time the special values that we shall choose are $x = -2$ because then the first term on the right will become zero and A will disappear, and $x = 1$ because then the second term on the right will become zero and B will disappear.

If $x = -2$

$$\begin{aligned} -6 &= -3B \\ B &= \frac{-6}{-3} \\ B &= 2 \end{aligned}$$

If $x = 1$

$$\begin{aligned} 3 &= 3A \\ A &= 1 \end{aligned}$$

Putting these results together we have

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}$$

and we have expressed the given fraction in partial fractions.

Sometimes the denominator is more awkward as we shall see in the following section.

Exercises 2

Express the following as a sum of partial fractions

a) $\dfrac{2x-1}{(x+2)(x-3)}$ b) $\dfrac{2x+5}{(x-2)(x+1)}$ c) $\dfrac{3}{(x-1)(2x-1)}$ d) $\dfrac{1}{(x+4)(x-2)}$

4. Fractions where the denominator has a repeated factor

Consider the following example in which the denominator has a repeated factor $(x-1)^2$.

Example

Suppose we want to express $\dfrac{3x+1}{(x-1)^2(x+2)}$ as the sum of its partial fractions.

There are actually three possibilities for a denominator in the partial fractions: $x-1$, $x+2$ and also the possibility of $(x-1)^2$, so in this case we write

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

where A, B and C are numbers.

As before we multiply both sides by the denominator $(x-1)^2(x+2)$ to give

$$3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \qquad (1)$$

Again we look for special values to substitute into this identity. If we let $x = 1$ then the first and last terms on the right will be zero and A and C will disappear. If we let $x = -2$ the first and second terms will be zero and A and B will disappear.

If $x = 1$

$$4 = 3B \qquad \text{so that} \qquad B = \frac{4}{3}$$

If $x = -2$

$$-5 = 9C \qquad \text{so that} \qquad C = -\frac{5}{9}$$

We now need to find A. There is no special value of x that will eliminate B and C to give us A. We could use any value. We could use $x = 0$. This will give us an equation in A, B and C. Since we already know B and C, this would give us A.

But here we shall demonstrate a different technique - one called *equating coefficients*. We take equation 1 and multiply-out the right-hand side, and then collect up like terms.

$$\begin{aligned} 3x+1 &= A(x-1)(x+2) + B(x+2) + C(x-1)^2 \\ &= A(x^2+x-2) + B(x+2) + C(x^2-2x+1) \\ &= (A+C)x^2 + (A+B-2C)x + (-2A+2B+C) \end{aligned}$$

This is an identity which is true for all values of x. On the left-hand side there are no terms involving x^2 whereas on the right we have $(A+C)x^2$. The only way this can be true is if

$$A + C = 0$$

This is called **equating coefficients** of x^2. We already know that $C = -\frac{5}{9}$ so this means that $A = \frac{5}{9}$. We also already know that $B = \frac{4}{3}$. Putting these results together we have

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$$

and the problem is solved.

Exercises 3

Express the following as a sum of partial fractions

a) $\frac{5x^2+17x+15}{(x+2)^2(x+1)}$ b) $\frac{x}{(x-3)^2(2x+1)}$ c) $\frac{x^2+1}{(x-1)^2(x+1)}$

5. Fractions in which the denominator has a quadratic term

Sometimes we come across fractions in which the denominator has a quadratic term which cannot be factorised. We will now learn how to deal with cases like this.

Example

Suppose we want to express

$$\frac{5x}{(x^2+x+1)(x-2)}$$

as the sum of its partial fractions.

Note that the two denominators of the partial fractions will be (x^2+x+1) and $(x-2)$. When the denominator contains a quadratic factor we have to consider the possibility that the numerator can contain a term in x. This is because if it did, the numerator would still be of lower degree than the denominator - this would still be a proper fraction. So we write

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

As before we multiply both sides by the denominator $(x^2+x+1)(x-2)$ to give

$$5x = (Ax+B)(x-2) + C(x^2+x+1)$$

One special value we could use is $x = 2$ because this will make the first term on the right-hand side zero and so A and B will disappear.

If $x = 2$

$$10 = 7C \qquad \text{and so} \qquad C = \frac{10}{7}$$

Unfortunately there is no value we can substitute which will enable us to get rid of C so instead we use the technique of equating coefficients. We have

$$\begin{aligned} 5x &= (Ax + B)(x - 2) + C(x^2 + x + 1) \\ &= Ax^2 - 2Ax + Bx - 2B + Cx^2 + Cx + C \\ &= (A + C)x^2 + (-2A + B + C)x + (-2B + C) \end{aligned}$$

We still need to find A and B. There is no term involving x^2 on the left and so we can state that

$$A + C = 0$$

Since $C = \frac{10}{7}$ we have $A = -\frac{10}{7}$.
The left-hand side has no constant term and so

$$-2B + C = 0 \qquad \text{so that} \qquad B = \frac{C}{2}$$

But since $C = \frac{10}{7}$ then $B = \frac{5}{7}$. Putting all these results together we have

$$\begin{aligned} \frac{5x}{(x^2 + x + 1)(x - 2)} &= \frac{-\frac{10}{7}x + \frac{5}{7}}{x^2 + x + 1} + \frac{\frac{10}{7}}{x - 2} \\ &= \frac{-10x + 5}{7(x^2 + x + 1)} + \frac{10}{7(x - 2)} \\ &= \frac{5(-2x + 1)}{7(x^2 + x + 1)} + \frac{10}{7(x - 2)} \end{aligned}$$

Exercises 4

Express the following as a sum of partial fractions

a) $\dfrac{x^2 - 3x - 7}{(x^2 + x + 2)(2x - 1)}$ b) $\dfrac{13}{(2x + 3)(x^2 + 1)}$ c) $\dfrac{x}{(x^2 - x + 1)(3x - 2)}$

6. Dealing with improper fractions

So far we have only dealt with proper fractions, for which the numerator is of lower degree than the denominator. We now look at how to deal with improper fractions.

Consider the following example.

Example

Suppose we wish to express $\dfrac{4x^3 + 10x + 4}{x(2x + 1)}$ in partial fractions.

The numerator is of degree 3. The denominator is of degree 2. So this fraction is improper. This means that if we are going to divide the numerator by the denominator we are going to divide a term in x^3 by one in x^2, which gives rise to a term in x. Consequently we express the partial fractions in the form:

$$\frac{4x^3+10x+4}{x(2x+1)} = Ax + B + \frac{C}{x} + \frac{D}{2x+1}$$

Multiplying both sides by the denominator $x(2x+1)$ gives

$$4x^3+10x+4 = Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx$$

Note that by substituting the special value $x = 0$, all terms on the right except the third will be zero. If we use the special value $x = -\frac{1}{2}$ all terms on the right except the last one will be zero.

<u>If $x = 0$</u>

$$4 = C$$

<u>If $x = -\frac{1}{2}$</u>

$$\begin{aligned} -\frac{4}{8} - \frac{10}{2} + 4 &= -\frac{1}{2}D \\ -\frac{1}{2} - 5 + 4 &= -\frac{1}{2}D \\ -1\frac{1}{2} &= -\frac{1}{2}D \\ D &= 3 \end{aligned}$$

Special values will not give A or B so we shall have to equate coefficients.

$$\begin{aligned} 4x^3+10x+4 &= Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx \\ &= 2Ax^3 + Ax^2 + 2Bx^2 + Bx + 2Cx + C + Dx \\ &= 2Ax^3 + (A+2B)x^2 + (B+2C+D)x + C \end{aligned}$$

Now look at the term in x^3.

$$2A = 4 \qquad \text{so that} \qquad A = 2$$

Now look at the term in x^2. There is no such term on the left. So

$$A + 2B = 0 \qquad \text{so that} \qquad A = -2B \qquad \text{so that} \qquad B = -\frac{2}{2} = -1$$

Putting all these results together gives

$$\frac{4x^3+10x+4}{x(2x+1)} = 2x - 1 + \frac{4}{x} + \frac{3}{2x+1}$$

and the problem is solved.

Exercise 5

Express the following as a sum of powers of x and partial fractions

a) $\frac{x^3+1}{x^2+1}$ b) $\frac{2x^4+3x^2+1}{x^2+3x+2}$ c) $\frac{7x^2-1}{x+3}$

Answers

Exercise 1

a) $\dfrac{5x+11}{(x+1)(x+3)}$ b) $\dfrac{2x+16}{(x-2)(x+2)}$ c) $\dfrac{10}{(2x+1)(x+3)}$ d) $\dfrac{11}{(3x-1)(6x+9)}$

Exercise 2

a) $\dfrac{1}{x+2}+\dfrac{1}{x-3}$ b) $\dfrac{3}{x-2}-\dfrac{1}{x+1}$ c) $\dfrac{3}{x-1}-\dfrac{6}{2x-1}$ d) $\dfrac{1}{6(x-2)}-\dfrac{1}{6(x+4)}$

Exercise 3

a) $\dfrac{2}{x+2}-\dfrac{1}{(x+2)^2}+\dfrac{3}{x+1}$ b) $\dfrac{1}{49(x-3)}+\dfrac{3}{7(x-3)^2}-\dfrac{2}{49(2x+1)}$

c) $\dfrac{1}{2(x-1)}+\dfrac{1}{(x-1)^2}+\dfrac{1}{2(x+1)}$

Exercise 4

a) $\dfrac{2x+1}{x^2+x+2}-\dfrac{3}{2x-1}$ b) $\dfrac{4}{2x+3}-\dfrac{2x-3}{x^2+1}$ c) $\dfrac{-2x+3}{7(x^2-x+1)}+\dfrac{6}{7(3x-2)}$

Exercise 5

a) $x+\dfrac{-x+1}{x^2+1}$ b) $2x^2-6x+17+\dfrac{6}{x+1}-\dfrac{45}{x+2}$ c) $7x-21+\dfrac{62}{x+3}$