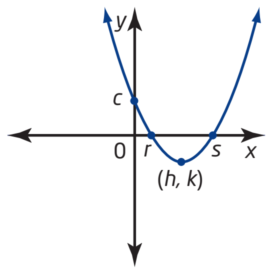
1.3

Maximum or Minimum of a Quadratic Function

Some bridge arches are defined by quadratic functions. Engineers use these quadratic functions to determine the maximum height or the minimum clearance under the support of the bridge at a variety of points. They can give this information to the bridge builders.

A quadratic function can be written in a number of forms. Each form has different advantages. In all forms, a determines the direction of opening and the shape.



- From the standard form, $f(x) = ax^2 + bx + c$, the y-intercept can be identified as c.
- From the factored form, $f(x) = a(x - r)(x - s)$, the x-intercepts can be identified as r and s.
- From the vertex form, $y = a(x - h)^2 + k$, the coordinates of the vertex can be identified as (h, k). If a is positive, the minimum value is k. If a is negative, the maximum value is k.

Tools

- graphing calculator

or

- grid paper

Investigate A

How can you connect different forms of the same quadratic function?

1. Graph each pair of functions.

a) $f(x) = (x + 2)^2 + 3$ and $f(x) = x^2 + 4x + 7$

b) $f(x) = (x + 3)^2 - 4$ and $f(x) = x^2 + 6x + 5$

c) $f(x) = 2(x - 3)^2 + 4$ and $f(x) = 2x^2 - 12x + 22$

d) $f(x) = 3(x - 1)^2 - 7$ and $f(x) = 3x^2 - 6x - 4$

2. Why are the graphs of the functions in each pair the same?

3. How can you rewrite the first equation in each pair in the form of the second equation?

4. How can you rewrite the second equation in each pair in the form of the first equation?

5. **Reflect** How can you use a graph to verify that two quadratic functions in different forms represent the same function? If you are using a graphing calculator, is it enough to observe that the graphs look the same on your screen? Explain.

Connections

Completing the square is part of a process by which a quadratic function in standard form can be arranged into vertex form, $y = a(x - h)^2 + k$. You learned this technique in grade 10.

To convert a quadratic function from standard form to vertex form, you can use the technique of completing the square.

Example 1

Find the Vertex by Completing the Square

Find the vertex of each function by completing the square. Is the vertex a minimum or a maximum? Explain.

a) $f(x) = x^2 + 5x + 7$

b) $f(x) = -\frac{2}{3}x^2 + 8x + 5$

Solution

a)

$$\begin{aligned} f(x) &= x^2 + 5x + 7 \\ &= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7 \\ &= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{28}{4} \\ &= \left(x + \frac{5}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

Add half the coefficient of x, squared, to make the first three terms a perfect square trinomial. Subtract the same amount, $\left(\frac{5}{2}\right)^2$, so the value of the function does not change.

The vertex is at $\left(-\frac{5}{2}, \frac{3}{4}\right)$. This is a minimum because a is 1, a positive value, so the parabola opens upward.

b)

$$\begin{aligned} f(x) &= -\frac{2}{3}x^2 + 8x + 5 \\ &= -\frac{2}{3}(x^2 - 12x) + 5 \end{aligned}$$

Factor out the coefficient of x^2.

$8 \div \left(-\frac{2}{3}\right) = 8 \times \left(-\frac{3}{2}\right)$

$= -12$

$$= -\frac{2}{3}(x^2 - 12x + 36 - 36) + 5$$

Add and subtract $6^2 = 36$ to make a perfect square trinomial.

$$\begin{aligned} &= -\frac{2}{3}[(x - 6)^2 - 36] + 5 \\ &= -\frac{2}{3}(x - 6)^2 + 24 + 5 \\ &= -\frac{2}{3}(x - 6)^2 + 29 \end{aligned}$$

$-\frac{2}{3} \times (-36) = 24$

The vertex is at (6, 29). This is a maximum because the value of a is negative, indicating that the parabola opens downward.

Investigate B

How can you use partial factoring to find a minimum or a maximum?

Tools
- graphing calculator or grid paper

1. Graph the function $f(x) = 2x^2 + 4x$.
2. How many x-intercepts does this function have?
3. Use the x-intercepts to find the vertex of the parabola.
4. Graph the functions $g(x) = 2x^2 + 4x + 2$ and $h(x) = 2x^2 + 4x + 5$ on the same set of axes as $f(x)$.
5. How many x-intercepts does $g(x)$ have? $h(x)$?
6. Describe how to find the vertex of the new parabolas, $g(x)$ and $h(x)$, based on the vertex of the original parabola, $f(x)$.
7. **Reflect** Using your answer from step 6, suggest a method that can be used to find the maximum or minimum of a parabola of the form $f(x) = 2x^2 + 4x + k$ for any value of k.

Example 2

Use Partial Factoring to Find the Vertex of a Quadratic Function

Find the vertex of the function $y = 4x^2 - 12x + 3$ by partial factoring. Is the vertex a minimum or a maximum value? Explain.

Solution

Work with the function $y = 4x^2 - 12x$ to find the x-coordinate of the vertex, since the x-coordinate of the vertex of $y = 4x^2 - 12x + 3$ will be the same.

$y = 4x(x - 3)$

For $y = 0$:

$0 = 4x(x - 3)$

$4x = 0$ or $x - 3 = 0$ — Use the zero principle. If $AB = 0$, then either $A = 0$ or $B = 0$.

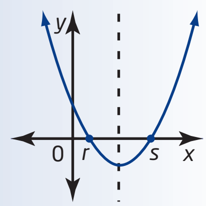
$x = 0$ or $x = 3$ — These give the x-intercepts of the function $y = 4x^2 - 12x$.

The average of these two x-intercepts will give the x-coordinate of the vertex for $y = 4x^2 - 12x$ and $y = 4x^2 - 12x + 3$:

$$\frac{0 + 3}{2} = \frac{3}{2}$$

Connections

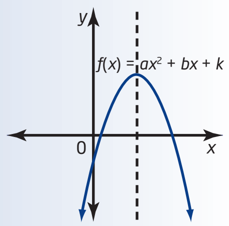
The vertex of a quadratic function is on the line of symmetry, which is halfway between the x-intercepts.



Connections

By partial factoring, $f(x) = ax^2 + bx + k$ can be expressed as $f(x) = ax\left(x + \frac{b}{a}\right) + k$.

This is a family of quadratic functions with axis of symmetry $x = -\frac{b}{2a}$.



To find the y-coordinate of the vertex, substitute $x = \frac{3}{2}$ into $y = 4x^2 - 12x + 3$.

$$
\begin{aligned}
y &= 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 3 \\
&= 4\left(\frac{9}{4}\right) - 18 + 3 \\
&= 9 - 18 + 3 \\
&= -6
\end{aligned}
$$

The vertex of the function $y = 4x^2 - 12x + 3$ is at $\left(\frac{3}{2}, -6\right)$. It is a minimum, because the value of a is positive.

Example 3

Solve a Problem Involving a Minimum or a Maximum

Rachel and Ken are knitting scarves to sell at the craft show. The wool for each scarf costs \$6. They were planning to sell the scarves for \$10 each, the same as last year when they sold 40 scarves. However, they know that if they raise the price, they will be able to make more profit, even if they end up selling fewer scarves. They have been told that for every 50¢ increase in the price, they can expect to sell four fewer scarves. What selling price will maximize their profit and what will the profit be?

Solution

Let x represent the number of 50¢ price changes.

Since each scarf cost \$6 and was sold for \$10, the profit was \$4 per scarf. As they raise the price, their profit per scarf will be $(4 + 0.5x)$ for x changes to the price. They will sell $40 - 4x$ scarves when they make the price change.

$$
\begin{aligned}
\text{Profit} &= \text{profit per scarf} \times \text{number sold} \\
P(x) &= (4 + 0.5x)(40 - 4x) \\
&= -2x^2 + 4x + 160
\end{aligned}
$$

Method 1: Complete the Square to Determine the Vertex

$$
\begin{aligned}
P(x) &= -2(x^2 - 2x) + 160 \\
&= -2(x^2 - 2x + 1 - 1) + 160 \\
&= -2(x - 1)^2 + 2 + 160 \\
&= -2(x - 1)^2 + 162
\end{aligned}
$$

The maximum value of this quadratic function is 162 when $x = 1$. This means that they will make a maximum profit of \$162 if they increase the price once. The selling price is $10 + 0.5(1)$ or \$10.50.

Method 2: Use Partial Factoring to Determine the Vertex

Find the x-coordinate of the vertex of the function $Q(x) = -2x^2 + 4x$, knowing that the vertex of $P(x) = -2x^2 + 4x + 160$ has the same x-coordinate.

$Q(x) = -2x(x - 2)$

Substitute $Q(x) = 0$ to find the x-intercepts.

$0 = -2x(x - 2)$
$-2x = 0$ or $x - 2 = 0$
$x = 0$ or $x = 2$

The x-coordinate of the vertex is $x = 1$ (the average of 0 and 2).

$P(1) = -2(1)^2 + 4(1) + 160$
$= 162$

The vertex of this function is at (1, 162). This means that they will make a maximum profit of $162 if they increase the price once. The selling price is 10 + 0.5(1) or $10.50.

Example 4

Connect Projectiles to Quadratic Functions

Jamie throws a ball that will move through the air in a parabolic path due to gravity. The height, h, in metres, of the ball above the ground after t seconds can be modelled by the function $h(t) = -4.9t^2 + 40t + 1.5$.

a) Find the zeros of the function and interpret their meaning.

b) Determine the time needed for the ball to reach its maximum height.

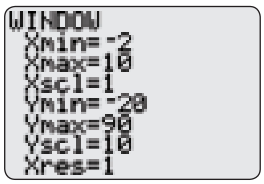
c) What is the maximum height of the ball?

Connections

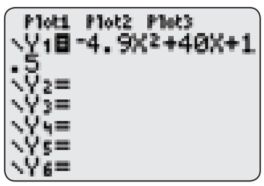
The zeros of a function are the values of the independent variable for which the function has value zero. They correspond to the x-intercepts of the graph of the function.

Solution

a) • Use the window settings shown.



• Graph **Y1** $= -4.9x^2 + 40x + 1.5$.

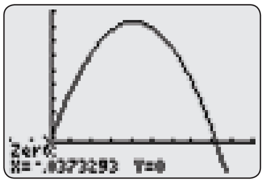


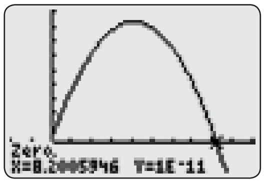
• Press (2nd) [CALC] to access the **CALCULATE** menu.

Technology Tip

See the Use Technology feature at the end of this section for a TI-Nspire™ CAS graphing calculator solution.

• Select **2:zero** to find the x-intercepts of the function.





The zeros are approximately -0.037 and 8.2.

The solution $t = -0.037$ indicates when, in the past, the ball would have been thrown from ground level in order for it to follow the given path. The solution $t = 8.2$ indicates when the ball will return to the ground. The ball returns to the ground 8.2 s after Jamie threw it.

b) The maximum is midway between the two zeros. So, find the average of the two solutions from part a).

$$\frac{-0.037 + 8.2}{2} = 4.0815$$

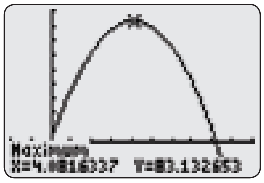
The ball will take approximately 4.1 s to reach its maximum height.

c) The maximum height can be found by substituting $t = 4.1$ into the function.

$$h(t) = -4.9t^2 + 40t + 1.5$$
$$h(4.1) = -4.9(4.1)^2 + 40(4.1) + 1.5$$
$$\doteq 83.13$$

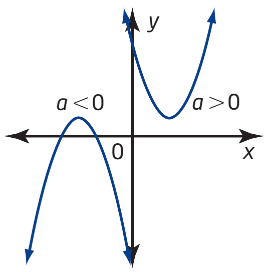
The ball will reach a maximum height of approximately 83.1 m.

This solution can be verified using the maximum function on the graphing calculator.



Key Concepts

- The minimum or maximum value of a quadratic function occurs at the vertex of the parabola.
- The vertex of a quadratic function can be found by
  – graphing
  – completing the square: for $f(x) = a(x - h)^2 + k$, the vertex is (h, k)
  – partial factoring: for $f(x) = ax\left(x + \frac{b}{a}\right) + k$, the x-coordinate of the vertex is $-\frac{b}{2a}$
- The sign of the coefficient a in the quadratic function $f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ determines whether the vertex is a minimum or a maximum.
  If $a > 0$, then the parabola opens upward and has a minimum.
  If $a < 0$, then the parabola opens downward and has a maximum.



Communicate Your Understanding

C1 In one step of completing the square, you divide the coefficient of x by 2 and square the result. Why?

C2 How are the functions $f(x) = 4x(x - 3)$, $g(x) = 4x(x - 3) + 2$, and $h(x) = 4x(x - 3) - 1$ related? Explain using words and diagrams.

C3 Ryan does not understand the concept of partial factoring to determine the vertex. Use the function $y = 3x^2 - 9x - 17$ to outline the technique for him.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Complete the square for each function.

a) $y = x^2 + 4x$

b) $f(x) = x^2 + 7x + 11$

c) $g(x) = x^2 - 3x + 1$

d) $y = x^2 - 11x - 4$

e) $f(x) = x^2 + 13x + 2$

f) $y = x^2 - 9x - 9$

2. Determine the vertex of each quadratic function by completing the square. State if the vertex is a minimum or a maximum.

a) $f(x) = x^2 + 10x + 6$

b) $f(x) = 2x^2 + 12x + 16$

c) $f(x) = -3x^2 + 6x + 1$

d) $f(x) = -x^2 + 12x - 5$

e) $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$

f) $f(x) = \frac{2}{3}x^2 + \frac{16}{3}x + \frac{25}{3}$

For help with question 3, refer to Example 2.

3. Use partial factoring to determine the vertex of each function. State if the vertex is a minimum or a maximum.

a) $f(x) = 3x^2 - 6x + 11$

b) $f(x) = -2x^2 + 8x - 3$

c) $f(x) = \frac{1}{2}x^2 - 3x + 8$

d) $f(x) = -\frac{5}{3}x^2 + 5x - 10$

e) $f(x) = 0.3x^2 - 3x + 6$

f) $f(x) = -0.2x^2 - 2.8x - 5.4$

4. Use Technology Use a graphing calculator to verify your answers to questions 2 and 3.

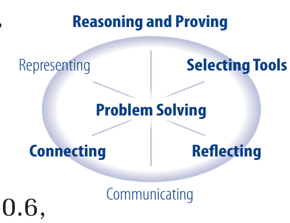
B Connect and Apply

For help with questions 5 and 6, refer to Example 3.

5. An electronics store sells an average of 60 entertainment systems per month at an average of \$800 more than the cost price. For every \$20 increase in the selling price, the store sells one fewer system. What amount over the cost price will maximize profit?

6. Last year, a banquet hall charged \$30 per person, and 60 people attended the hockey banquet dinner. This year, the hall's manager has said that for every 10 extra people that attend the banquet, they will decrease the price by \$1.50 per person. What size group would maximize the profit for the hall this year?

For help with question 7, refer to Example 4.

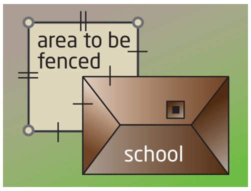


7. A ball is kicked into the air and follows a path described by $h(t) = -4.9t^2 + 6t + 0.6$, where t is the time, in seconds, and h is the height, in metres, above the ground. Determine the maximum height of the ball, to the nearest tenth of a metre.

8. The cost, C, in dollars, of fuel per month for Sanjay to operate his truck is given by $C(v) = 0.0029v^2 - 0.48v + 142$, where v represents his average driving speed, in kilometres per hour. Find the most efficient speed at which Sanjay should drive his truck.

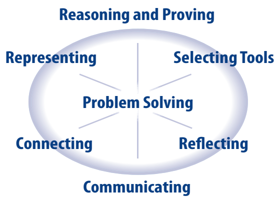
9. Arnold has 24 m of fencing to surround a garden, bounded on one side by the wall of his house. What are the dimensions of the largest rectangular garden that he can enclose?

10. The area shown is to be enclosed by 30 m of fencing. Find the dimensions that will maximize the enclosed area.



11. The sum of two numbers is 10. What is the maximum product of these numbers?

12. A function models the effectiveness of a TV commercial. After n viewings, the effectiveness, e, is $e = -\frac{1}{90}n^2 + \frac{2}{3}n$.



a) Determine the range for the effectiveness and the domain of the number of viewings. Explain your answers for the domain and range.

b) Use either completing the square or partial factoring to find the vertex. Is it a minimum or a maximum? Explain.

c) What conclusions can you make from this function?

d) Graph the function on a graphing calculator to verify your conclusions from part c).

13. All quadratic functions of the form $y = 2x^2 + bx$ have some similar properties.

a) Choose five different values of b and graph each function.

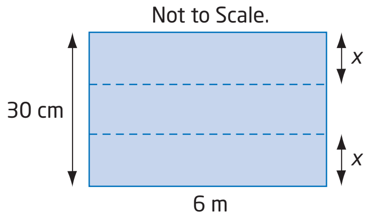
b) What are the similar properties?

c) Determine the vertex of each parabola.

d) Find the relationship between the vertices of these parabolas.

C Extend

14. A sheet of metal that is 30 cm wide and 6 m long is to be used to make a rectangular eavestrough by bending the sheet along the dotted lines.



What value of x maximizes the capacity of the eavestrough?

15. A ball is thrown vertically upward with an initial velocity of v metres per second and is affected by gravity, g. The height, h, in metres, of the ball after t seconds is given by $h(t) = -\frac{1}{2}gt^2 + vt$.

a) Show that the ball will reach its maximum height at $t = \frac{v}{g}$.

b) Show that the maximum height of the ball will be $\frac{v^2}{2g}$.

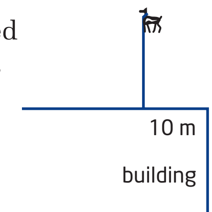
16. **Math Contest** Given that $x^2 = y^3 = z$, where x, y, and z are integers, how many different values of z are there for $z < 1001$?

A 0 **B** 3 **C** 4 **D** 10

17. **Math Contest** A function of two variables is defined as $f(x, y) = x^2 + y^2 + 4x - 6y + 7$. What is the minimum value of this function?

A 7 **B** -13 **C** -6 **D** 0

18. **Math Contest** A dog's 15-m-long leash is attached to a building. The leash is attached 10 m from one corner of the building. Assume that the sides of the building are long enough that the dog cannot go around any of the other corners. The greatest area that the dog can cover, in square metres, is



A 250π **B** $\frac{475\pi}{4}$ **C** 112.5π **D** 125π

Use a TI-Nspire™ CAS Graphing Calculator to Find the Maximum or Minimum and the Zeros of a Quadratic Function

Jamie throws a ball that will move through the air in a parabolic path due to gravity. The height, h, in metres, of the ball above the ground after t seconds can be modelled by the function $h(t) = -4.9t^2 + 40t + 1.5$.

a) Find the zeros of the function and interpret their meaning.

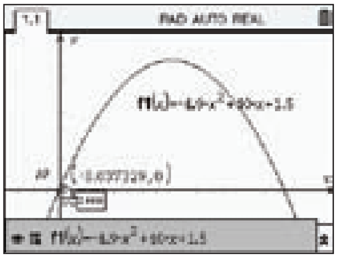
b) Determine the time needed for the ball to reach its maximum height.

c) What is the maximum height of the ball?

Solution

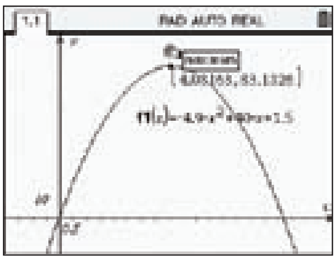
a) Turn on the TI-Nspire™ CAS graphing calculator.

- Press (home) and select **6:New Document**.
- Select **2:Add Graphs & Geometry**.
- Type $-4.9x^2 + 40x + 1.5$ for function f1 and press (enter).
- Press (menu). Select **4:Window**.
- Select **1:Window Settings**. Set **XMin** to −2, **XMax** to 10, **Ymin** to −40, and **YMax** to 100. Tab down to **OK** and press (enter).
- Press (menu) and select **6:Points & Lines**.
- Select **2:Point On**. Move the cursor to the graph and press (enter).
- Press (esc).
- Press (ctrl) and then (grab) to grab the point. Use the cursor keys (the arrows on the NavPad) to move the point along the graph toward the left zero. When you reach the zero, "zero" will appear in a box. Read the coordinates of the zero. It occurs at a time of approximately −0.037 s.



Similarly, you can find the right zero at a time of about 8.20 s.

b) To find the maximum height of the ball, move the point toward the maximum on the graph. When you reach the maximum, "maximum" will appear inside a box. Read the coordinates of the maximum. It occurs at a time of approximately 4.08 s.



c) The maximum height of the ball is approximately 83.13 m.

Tools

- TI-Nspire™ CAS graphing calculator

Connections

Example 4 on page 29 is used to model the steps needed to find the maximum or minimum and the zeros of a quadratic function using a TI-Nspire™ CAS graphing calculator.